The Foundations of Mathematics History, Concepts, and Applications in Higher Level Mathematics

by

Clifton Henry

A CAPSTONE PROJECT

submitted in partial fulfillment of the requirements for the acknowledgement

Honors Distinction

Mathematics
School of Engineering, Mathematics, and Science

TYLER JUNIOR COLLEGE
Tyler, Texas

2016

When people hear the word mathematics, some think numbers and others think solving for x and y. In all truth, the field of mathematics is more than numbers and equations. Mathematics was founded based off the principle of counting in 50,000 B.C.E. Mathematical contributions and ideas increased from the Babylonian Empire in 2000 BC to the 18th century, where mathematicians and philosophers developed ideas in algebra, geometry and calculus.

It can be argued that mathematical contributions in the 19th century shaped the field of mathematics and set in place the true foundations of mathematics. In the 19th century, mathematical logic was founded, followed by the development of set theory, and then came the concept of mathematical function. Logic, set theory, and function are some of the foundations of mathematics, and the three revolutionized the field of mathematics as a whole. Mathematical fields, such as topology, abstract algebra, and probability rely heavily on the three foundations.

Instead of being seen as numbers and equations, mathematics is the field of logic and proof that is based off the mathematical foundations of logic, set theory, and function. It is important to know how the ideas were developed and put to use in order for mathematics to have its meaning and significance. After understanding how the foundations were developed, it is important to understand how they apply in advanced mathematical fields such as topology, abstract algebra, and probability. After understanding the development of logic, set theory and function, and their applications, mathematics will have more clarity and understanding instead of being seen as numbers and solving for equations.

Logic was first developed by the Greek philosopher Aristotle and he was the first to establish a logical system that is still in effect. Aristotle's approach of logic was systematic in his book, *Prior Analytics*, where some of his ideas and methods are a part of modern logic. In his work, Aristotle addressed logic as formal epistemology, where it was important to reach a conclusion by establishing premises, which are in the form of deduction.

George Boole, an English mathematician and logician, launched the beginning of mathematical logic when he wrote *Laws of Thought* in 1854. According to John Corcoran, philosophy professor at University of Buffalo, Boole's intention was to show that logic was not a form of philosophy, but a form of mathematics called mathematical analysis, where logic applies to mathematical fields such as calculus and algebra. Boole's contribution to logic gave foresight to mathematics as a discipline of reason and inspired other mathematicians and logicians to perfect mathematical logic.

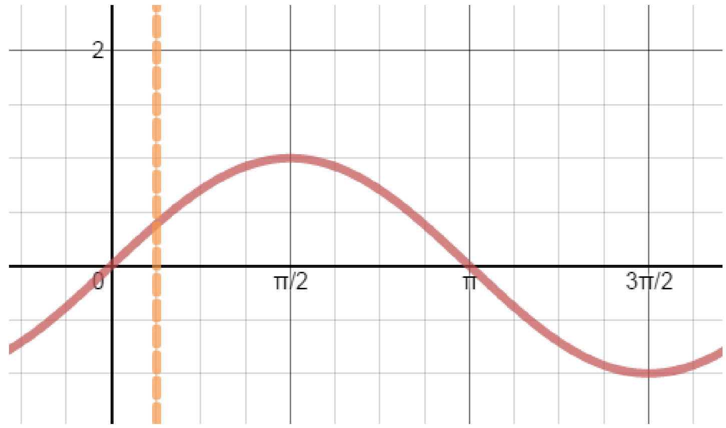
Bertrand Russell contributed to logic and the foundations of mathematics by developing Russell's Paradox (Russell-Zermelo Paradox), which established the theory of types. Russell showed that the logic of mathematics is the form of philosophical logic, and he expounded his theory in two books, *Principles of Mathematics* (1903) and *Principia Mathematica* (1908). The books showed that axioms connected mathematics and analytic philosophy to make forms of tautologies, representing classical and modern logic. Russell showed that mathematical logic is the form of Aristotle's philosophy of logic.

Other logicians such as Gottlob Frege, author of the *Foundations of Arithmetic* (1884) and Augustus De Morgan, founder of De Morgan's Laws, made contributions to logic as well, but the two that stand out and formalized mathematical logic was George Boole and Bertrand Russell. Boole is the father of mathematical logic as he set in place for the theory of logic to be seen in mathematical fields, while Russell connected logic with philosophy and mathematics combining to make the philosophy of mathematics. Russell showed how mathematical logic is a part of modern logic today by showing that is based off classical logic and Aristotle's foundations of logical philosophy.

Some logical concepts are tautologies, quantifiers, conditionals, biconditionals, deductive reasoning, and inductive reasoning. A tautology is a statement that is true by necessity or by virtue of its logical form. For example, the statement "Either I'm watching a football game or I'm not" is a tautology, because its logical form ("excluded middle") requires this sentence to be true. Quantifiers are phrases of the form "for all x" ($\forall x$) or "there exists an x" ($\exists x$). For example, the statement "For all prime numbers, there are exactly two factors." uses a quantifier.

A conditional is a statement of the form "If p, then q", where p is the hypothesis and q is the conclusion. For example, the statement "If $x + 2 = 3$, then $x = 1$" is a conditional statement with a hypothesis of "$x + 2 = 3$" and a conclusion of "$x = 1$." A biconditional is a statement of the form "p if and only if q." If true, then statements p and q must either be true simultaneously or false simultaneously. For example, the statement "y = x+2 is a function if and only if the graph of y = x+2 passes the vertical line test (VLT)" is a biconditional statement. This true statement means that "y = x+2 is a function" and "the graph of y = x+2 passes the vertical line test" are either both true or both false (they are both true).

Deductive reasoning is the process of reasoning from one or more statements (hypotheses) to reach a logically certain conclusion. Most deductive reasoning requires two initial statements to be true. One statement is a conditional of the form "If p, then q." The second statement is p. The logical conclusion is statement q. For example, from the previous biconditional, we know that all functions' graphs pass the VLT. If we are told that y = sin(x) is a function, then we can logically conclude that the graph of sin(x) passes the VLT.



The graph of y = sin(x) passes the VLT

Inductive reasoning is the process of reasoning in which the hypotheses are viewed as supplying strong evidence for the truth of the conclusion. Inductive reasoning usually stems from observing that something is true repeatedly, then concluding that it must always be true. Such reasoning implies the conclusion is probable, but not *certain*, as opposed to deductive reasoning. For example, some prime numbers are 3, 7, 19, 41, and 97. Using inductive reasoning, one might conclude that all prime numbers are odd. This conclusion is actually false, since 2 is prime, but not odd.

Along with logic, set theory is one of the foundations of mathematics because its properties are present in higher level mathematics. Set theory is based off the mathematical foundations of logic, and one of the most known contributors to set theory is Georg Cantor. Cantor discovered the property of set theory starting in 1872 when he was working on number theory.

According to Joseph Dauben of Hebert Lehman College, Cantor discovered set theory by working with German mathematician Richard Dedekind when they were working on number theory and trigonometric series. During his work in number theory, Cantor realized he used a mathematical property called one-to-one correspondence, which was one of the first properties of set theory. He used one-to-one correspondence to show equivalence between a set of rational numbers and natural numbers, and this was the start of set theory.

Cantor's other contribution to set theory was the development of infinite sets and transfinite cardinal numbers. John Lucas, author of *Introduction to Abstract Mathematics* (2^{nd} ed.), explained that Cantor established that infinite sets can be either countable by using the one-to-one correspondence to show equivalence, or infinite sets can be uncountable by not agreeing with the one-to-one correspondence method. Another known contributor to set theory was Augustus De Morgan. De Morgan's contributions were in the algebra of sets, which are set operations of unions, intersections, and complements. De Morgan's algebra of sets is known as mathematical laws that share his name, De Morgan's Laws, which are seen in the mathematical field of probability.

The logician Russell Bertrand also made a contribution to set theory by developing his method known as "Russell's Paradox". Not only showing that mathematics was a form of philosophy, Russell showed a contradiction in Cantor's theory of cardinal numbers and his definition of set theory. Russell believed that set theory could be derived within a paradox. In the foundations of mathematics' history, Cantor and other mathematicians contributed to "naïve set theory", where they loosely defined the real definition of set theory, but instead gave its language. The real definition of set theory is axiomatic, where it is a form of logic that contains a collection of objects (numbers or symbols) that are showing relationship.

Some main ideas in set theory are the union of two sets ($A \cup B$), intersection of two sets ($A \cap B$), subsets ($A \subseteq B$), and complements of a set (A^C). The union of two sets is the set of elements in the universe belonging to set A or set B, or both. The intersection of two sets is the set of elements in the universe belonging to both set A and set B. For example, suppose A =

{vanilla, strawberry} and B = {vanilla, chocolate}. Then $A \cup B$ = {vanilla, strawberry, chocolate}, while $A \cap B$ ={vanilla}.

A subset is a relation between sets that states that the elements of one set are contained in a second set. A complement is the set of all elements in the universe that are not in the set A. For example, suppose you consider all whole numbers as the universe (U = {0, 1, 2, 3…}). If E is the set of even numbers (E = {0, 2, 4, 6, …}) and A = {2, 4, 6, 8, 10}, then every element of A is contained in E. Hence, $A \subseteq E$. Also, the complement of E is $E^C = \{1, 3, 5, 7 \ldots\}$, i.e. the odd numbers.

The concept of the mathematical function goes back to the ancient civilization period. João Pedro Ponte, professor of Mathematics at Lisbon University, explained that the concept of function started in Ancient Greece with the technique of counting numbers, arithmetical operations of adding, subtracting, multiplying, and dividing, and the start of the square root, square powers, reciprocals, and cube roots. While all the listed mathematical properties play a part in mathematical function, the real concept of function started in the late 17th century.

Ponte conveyed that the idea of function started with the beginnings of infinitesimal calculus, which traces back to the philosopher Rene Descartes. Descartes founded the derivative which is a property in calculus, and he established the idea by finding a point tangent to a curve. Famous mathematicians like Isaac Newton and Gottfried Leibniz contributed to more ideas in calculus by relating it to geometric properties and it expounded Descartes' ideas of the derivative and tangency. According to Olov Viirman of Karlstad University, Leonhard Euler and Johann Bernoulli revolutionized the concept of function, as Euler showed how calculus is a study of function and Bernoulli gave the first definition of the function concept as an analytic expression.

By these two setting the course for function concept, calculus began to grow over the 18th and 19th centuries as it became a part of mathematical function.

Cantor's set theory also transformed the notation and concept of function. Set theory shows the relation and order of collection of objects that make a function, and the function concept is also represented by the one-to-one correspondence method, which is also a property of set theory. With the development of calculus and the understanding of notations between variables in the 18th and 19th centuries, function concept became a foundation of mathematics, and it lead to the development of other mathematical fields such as topology and analysis.

Some main ideas of functions are mapping, compositions, and one-to-one correspondence. Mapping is a mathematical relation such that each element of a given set (the domain of the function) is associated with an element of another set (the range of the function). For example, consider the relation $y = x^2$. For any value of x, there is an associated value of y. For example, if $x = 2$, the associated value of y would be 2^2, or 4. A composition is the successive application of one function to the output of a second function. For example, suppose we have $x = t + 4$ and $y = x^2 + 2$. If we apply the first function to $t = 10$, we get an associated value of $x = 10 + 4 = 14$. Applying the second function to $x = 14$ gives an associated value of $y = 14^2 + 2 = 198$.

One-to-one correspondence is a mapping that says for every element of set A, there is exactly one corresponding element in set B, and for every element in B, there is exactly one corresponding element in A. For functions, this translates to saying that if a function produces equal outputs, then the inputs must have also been equal. For example, consider the function $f(x) = 2x + 7$. If two inputs x_1 and x_2 yield equal outputs, then we have $2x_1 + 7 = 2x_2 + 7$.

Using basic algebra, this reduces to $x_1 = x_2$. Since equal outputs resulted in equal inputs, $f(x)$ is a one-to-one correspondence.

After knowing how these foundations were developed, and what some of their main concepts are, it is important to know how those concepts tie in to higher level math. Of the higher level mathematics, probability, topology, and abstract algebra will be discussed. It is imperative to know the significance of each higher level field, some of their concepts, and these foundations apply to some of those concepts.

According to Marcel Finan of Arkansas Tech University, probability is the measure of occurrence of an event. Some basic terms of probability are experiment, outcomes, events, and sample spaces. An experiment is a process that produces a random result, called an outcome. An example of an experiment is tossing a coin twice. The set of all possible outcomes is called the sample space of an experiment. For this experiment, the sample space is {HH, TT, HT, TH}. An event is a collection of outcomes for an experiment. For example, in this experiment, one event could be {HH, TT}. Mutually exclusive events are two events with no common outcome, and the events {HH, TT} and {HT, TH} are examples of mutually exclusive events. Complements are two events that are mutually exclusive, but whose union is the entire sample space. The two events in the previous example represent complements by being mutually exclusive, but its union is the whole sample space.

A key concept in probability is a probability distribution function. A probability function is a function of a random variable that gives the probability that a particular outcome will occur. An example of a probability function is p(x) = Pr(X = x), which is also known as a probability mass function. Conditional probability is the probability of one event occurring, A, given that event B has already happened. For example, the probability of drawing a heart from a deck of cards is P(heart) = 13/52, or 25%. However, if you impose the condition that the card is red, then the conditional probability of drawing a heart is P(heart | red) = 13/26, or 50% (since there are only 26 red cards). Joint probability is the probability that an outcome in the intersection of two events will occur. Joint probability relates to conditional probability with the formula P(A and B) = P(A)*P(B|A).

To make connections between our foundations and probability, we will focus on some specific concepts in the field and link them to these foundations. The first foundation we will link to probability is logic. The specific concept that will be used to link these is Bayes' Theorem and its proof. The formula for Bayes' Theorem is Pr(B|A) = $\frac{\Pr(A|B)\Pr(B)}{\Pr(A|B)\Pr(B)+\Pr(A|B^c)\Pr(B^c)}$, which gives the inverse of the original condition. The proof of Bayes Theorem can be found in most probability texts written by statisticians, such as Marcel Finan. This proof relies on deductive reasoning and axioms known as Kolmogorov Axioms. (Axiom 3, P($\bigcup E_i$) = $\sum P(E_i)$)

The second foundation to link to probability is set theory. The specific concepts will be used to link these are joint probability and the probability of the union of mutually exclusive events. The probability of the union of mutually exclusive events is P(A$\cup B) = P(A) + P(B)$. Joint probability is P(A $\cap B) = P(A) * P(B|A)$. Both probabilities are related to set theory due to their properties of union and complement, which are concepts of set theory.

The third foundation to link to probability is functions. The specific concepts that will be used to link these are probability mass functions and cumulative distribution functions. The probability mass function is p(x) = Pr(X = x), and the cumulative distribution function is F(x) =$\sum_{i \le n} p(i)$. Both of these functions are mappings. Each one takes an element of the sample space of a random variable and maps it to a real number between 0 and 1 (inclusive).

The next higher level math that will connect to these foundations is topology. Topology is the study of geometric properties that relate to relative position, as opposed to lengths and distances. For example, if a sandwich is in a lunch bag and someone sits on it, that sandwich will change its form, but relative to the bag, the sandwich will be in the same position.

Some main concepts in topology are open sets, closed sets, bounded sets, countable sets, and density of rational numbers. An open set (in the real number line) is any set that can be expressed as the union of open intervals of the form (a,b). A closed set (in the real number line) is any set whose complement is open. A bounded set is a set with the property that there exists real numbers b and B such that $b \leq x \leq B$ for all x in that set. A countable set is a set that is either finite, or can be put in one-to-one correspondence with the positive integers. According to some mathematicians, such as Fred Croom, defines the Density of Rational Numbers as a property of rational numbers that states between any two real numbers, there is a rational number (32).

The first foundation to link topology with is logic. The specific concept to use is the Density of Rational Numbers. The statement of the theorem uses logical quantifiers. Specifically, this theorem can be worded as follows.

"For every two real numbers r_1 and r_2, there exists a rational number q such that $r_1 < q < r_2$."

This theorem uses logical quantifiers, specifically, "for every" (∀) and "there exists" (∃). In fact, this theorem can be stated entirely using only variable and logical symbols.

$$\forall r_1, r_2 \in \mathbb{R}, \exists q \in \mathbb{Q} \ni r_1 < q < r_2$$

The second foundation that will be linked to topology is set theory. The concept that be used for the connection will be Cantor's Nested Interval Theorem. Croom stated:

“If $\{[a_n, b_n]\}_{n=1}^{\infty}$ is a nested sequence of closed and bounded intervals, then $\bigcap_{n=1}^{\infty}[a_n, b_n]$ is not empty”(46).

In simpler terms, Cantor’s Nested Interval Theorem says if you have a sequence of closed intervals where one interval is a subset of the previous interval, and none of the intervals have an infinite lower or upper bounds, then there are numbers in common to all such intervals. For example, consider the intervals

[-1.1, 1.1], [-1.01, 1.01], [-1.001, 1.001], [-1.0001, 1.0001], …

where each interval’s endpoints has one extra zero in the decimals that the preceding interval. This sequence of nested intervals has a non-empty intersection. Particularly, all of these intervals contain every value between -1 and 1, inclusive. Clearly, Cantor’s Nested Interval Theorem is related to set theory, since it involves subsets and intersections.

The last foundation that will be tied with topology is functions. The concept that will be connected with the two is the principle of a countable set. A countable set is a set that is either finite, or can be put in one-to-one correspondence with the positive integers. For example, the set of even positive integers {2, 4, 6, 8, …} is countable, because the function $f(x) = \frac{x}{2}$ is a one-to-one correspondences from the even positive integers {2, 4, 6, 8, …} to the positive integers {1, 2, 3, 4…}. It is simple to verify that $f(x)$ is one-to-one: if two inputs x_1 and x_2 yield equal outputs, then it follows that $\frac{x_1}{2} = \frac{x_2}{2}$. Therefore, $x_1 = x_2$. The principle of a countable set clearly involves a one-to-one correspondence, connecting topology to functions.

The next higher level math that will be connected to these foundations is abstract algebra. Abstract algebra is the study of sets and mappings that preserve various characteristics of those sets. Some main concepts in abstract algebra are equivalence relations, isomorphisms, groups, rings, and fields. An equivalence relation is a relation on a set X that is reflexive (for every $a \in X, a \sim a$), symmetric (if $a \sim b$, then $b \sim a$), and transitive (if $a \sim b$ and $b \sim c$, then $a \sim c$). Isomorphism is a one-to-one, onto correspondence between groups that preserves operations. A group is a set G with an operation $+$ on G with the following properties:

1) The operation is associative: for every $a, b, c \in G,\ a + (b + c) = (a + b) + c$
2) There exists an identity 0 such that for every $a \in G, a + 0 = a$ and $0 + a = a$
3) For each $a \in G$, there exists an inverse $b \in G$ such that $a + b = b + a = 0$

A ring is a set R together with two operations (+ and *) with the following properties:

1) (R, +) is a commutative group (i.e. $a + b = b + a$, in addition to other group properties)
2) Multiplication is associative: $a * (b * c) = (a * b) * c$
3) Multiplication is distributive over addition (i.e. $a * (b + c) = a * b + a * c$)

A field is a ring where multiplication is commutative (as well as addition), and there exists a multiplicative identity (usually denoted “1”) and multiplicative inverses for non-zero elements.

The first two foundations that will be linked to abstract algebra are set theory and logic. The concepts connecting these foundations are very intertwined, making it natural to discuss both foundations simultaneously. The concepts that will be used for the connections are equivalence relations and equivalence classes.

As previously noted, an equivalence relation is a relation on a set X that is reflexive (for every $a \in X, a \sim a$), symmetric (if $a \sim b$, then $b \sim a$), and transitive (if $a \sim b$ and $b \sim c$, then $a \sim c$).

Given a set X and an equivalence relation ~, an equivalence class for an element $a \in X$ is the set of all elements $b \in X$ such that $a \sim b$. In symbols, $[a] = \{b \in X | a \sim b\}$. A common theorem in abstract algebra is all equivalence classes for a set X form a partition of X. In other words, the union of all sets of the form $[a]$ will equal X, but those such sets are either equal or disjoint. One can think of the these sets as pieces of a puzzle: no two pieces overlap, but together, they form the entire picture.

To connect equivalence relations and equivalence classes to set theory and logic, a specific example will be used. Consider the set of integers $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. Define $x \sim y$ to mean that the difference $x - y$ is divisible by 4. It can be shown that this is an equivalence through deductive reasoning. If you take any integer, such as 7, the equivalence class [7] is the set of all integers x such that $x - 7$ is divisible by 4. Specifically, $[7] = \{\dots, -5, -1, 3, 7, 11, \dots\}$. If you form the equivalence classes for all integers, the union of those equivalence classes would equal the set of integers. In fact, it only takes four such classes to acquire all of the integers:

$$[7] = \{\dots, -5, -1, 3, 7, 11, \dots\}$$

$$[16] = \{\dots, -4, 0, 4, 8, 12, \dots\}$$

$$[9] = \{\dots, -3, 1, 5, 9, 13 \dots\}$$

$$[14] = \{\dots - 2, 2, 6, 10, 14, \dots\}$$

Notice $[7] \cup [16] \cup [9] \cup [14] = \mathbb{Z}$ (read *downward* through the above sets to see all integers). This is one requirement for the set of (all) equivalence classes to be a partition. Also notice that none of these equivalence classes have any elements in common. This is the second requirement for the set of all equivalence classes to be a partition. John Durbin, author of Modern Algebra: An Introduction (2^{nd} ed.), shows the general proof that an equivalence relation invokes a partition, which relies heavily on deductive reasoning and logical quantifiers (since it must be shown that *for all* equivalence classes, they must be equal or disjoint) (58). Clearly, these concepts also rely on ideas from set theory, thus connecting abstract algebra to both set theory and logic.

The last foundation that will be connected with abstract algebra is functions. The concept of isomorphism will be used to link the higher level math and foundation. An isomorphism is a one-to-one, onto correspondence between groups that preserves operations. Specifically, if G is a group with operation $+$ and H is a group with operation *, then $f: G \to H$ is an isomorphism if:

1) $f(x)$ is one-to-one, meaning $f(x_1) = f(x_2)$ implies $x_1 = x_2$
2) $f(x)$ is onto, meaning for every $y \in H$, there exists some $x \in G$ such that $f(x) = y$
3) $f(x)$ preserves operations, meaning $f(x_1 + x_2) = f(x_1) * f(x_2)$

For example, consider the function $f(x) = e^x$ from the group $(\mathbb{R}, +)$ to the group $(\mathbb{R}^+, *)$ (the set of *positive* real numbers). Since

$$f(x_1 + x_2) = e^{x_1 + x_2} = e^{x_1}e^{x_2} = f(x_1) * f(x_2),$$

$f(x)$ preserves the operations on the groups. It can be easily shown via basic algebra that $f(x)$ is both one-to-one and onto, making $f(x)$ and isomorphism. Thus, this connects abstract algebra to functions.

When people hear the word mathematics, some think numbers, while others think equations involving x and y. In all honesty, the field is based off the form of reason, defined through the foundations of mathematics, which includes logic, set theory, and the concept of function. These three foundations gave mathematics its language and applies to higher level mathematical topics, such as abstract algebra, topology, and probability. After seeing how the foundations were developed and apply to higher level fields, people can see that mathematics is the form reason and thought, not numbers.

Works Cited

Corcoran, John. "Aristotle's Prior Analytics and Boole's Laws of Thought" *Taylor and Francis Group* (2003): 261-268. Print

Croom, Fred. *Principles of Topology*. Florida: Saunders College Publishing, 1989. Print.

Dauben, Joseph. "Georg Cantor and the Battle for Transfinite Set Theory." *ACMS,* 1993. Web. 2004.

Durbin, John. *Modern Algebra: An Introduction.* 2nd Edition. New York: John Wiley & Sons, 1985. Print.

Finan, Marcel. *A Probability Course for the Actuaries a Preparation for Exam P/1*. Russellville: Arkansas Tech University, 2013. Print.

Lucas, John. *Introduction to Abstract Mathematics*. 2nd Edition. New York: Ardsley House, Publishers, Inc. 1985, 1989, 1990. Print.

The Mathematics. *The History of the Concept of Function and Some Educational Implications*. Trans. João Pedro Ponte. Vol 3.

Viirman, Olov. *The Function Concept and University Mathematics Teaching*. Diss. Karlstad University Studies, 2014. Universitetstryckeriet: Karlstad, 2014. Print.