Review Problems and Exercises to Accompany

Basic Concepts in Forest Valuation and Investment Analysis

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Review Material for Basic Concepts in Forest Valuation and Investment Analysis

Review Problems
Four Basic Formulas, Pages 2.1-2.14

1. You invest $1,000 in a savings account for 10 years at 10% interest. How much money will be in the account in 10 years?
2. You purchase a C.D. that will pay $2,593.74 in 10 years. If you desire to earn 10%, how much can you pay for the C.D.?
3. How long will it take $1,000 to accumulate to $2,593.74 at 10% interest?
4. If you invested $1,000 today and received $2,593.74 in 10 years, what rate of return did you earn?
5. You invest $1,000 on 1/1/07 at 8% interest. How much will in the account on 12/31/13?
6. Consider Figure 2.2. What is the combined present value of the three single sum cash flows (ignore annual costs)? What is the future value of the cash flows at year 30? Use a 6% interest rate.
7. Which is worth more: $20,000 today or $30,000 in 10 years? Why?
8. Consider two single sums: $1,000 and $2,000. State a set of conditions that will make them equivalent.
9. You deposit $1,000 into an account that pays 5% interest on January 1, 2001 and remove the principal and interest on December 31, 2012. How much money is in the account?!
10. How long will it take a stand of timber worth $2,000 today to double in value at 8% interest?
11. You deposit $100 into a savings account that pays 4% interest. After 8 years what is the balance of the account? How much represents principal and how much interest?
12. You deposited a sum of money into a savings account 8 years ago. The account pays 4% interest. Today there is $1,000 in the account. How much did you deposit?
13. You deposited $100 into a savings account that pays 4% interest. Today the balance of the account is $250. How long has the money been in the account?
14. You deposited $100 into a savings account 10 years ago. Today the account has a balance of $179.09. What interest rate did the account pay?
15. You deposit $1,000 into a savings account for 20 years. The account pays 4% interest for the first 10 years and 8% interest for the second 10 years. What is the balance of the account at year 20?
16. You deposit $100 into a savings account that pays 8% simple interest. How much is in the account after 10 years?
17. You placed $500 into a savings account that pays 5% interest at the beginning of year 3. How much was in the account at the end of year 10?
18. You placed $500 into a savings account that pays 5% interest at year 3. How much was in the account at year 10?
19. You place $500 into a savings account that pays 5% interest at the beginning of year 3. How much is in the account at the beginning of year 10?
20. You place $500 into a savings account that pays 5% interest on January 1, 2005. How much is in the account on January 1, 2012?

Solutions to Review Problems
Four Basic Formulas, Pages 2.1-2.14

1. Future value of a single sum. Formula 2.1, where n = 10, i = 0.10, and V_o = $1,000. Solve for V_n = $2,593.74.
2. Present value of a single sum. Formula 2.2 where n = 10, i = 0.10, and V_n = $1,000. Solve for V_o = $1,000.00.
3. Number of years or compounding periods problem. Formula 2.4, where V_n = $2,593.74, V_o = $1,000.00, and i = 0.10. Solve for n = 10.
4. Solve for interest rate or rate of return. Formula 2.3, where V_n = $2,593.74, V_o = $1,000.00, and n = 10. Solve for i = 0.10.
5. The money will be in the account for seven years. All of 2007, 2008, 2009, 2010, 2011, 2012, and 2013. So the account will contain $1,000(1.08)^7$ = $1,713.82.
6. Figure 2.2 has three single sum cash flows: a cost of $120 at year 0, revenue of $600 at year 17, and revenue of $2,800 at year 30. To get the value at year 30, compound the $120 for 30 years (-$689.22), compound the $600 for 13 years ($1,279.76), and do nothing to the $2,800 as it is already at year 30. They sum to $3,390.54. We also need the present value of three separate single sums. The $120 is at year 0, so does not need to be discounted. It is a cost, so its value is -$120.00. The $600 is at year 17, so must be discounted for 17 years at 6%. This equals $222.82. The $2,800 is at year 30, so must be discounted for 30 years at 6%. This equals $487.51. Combining the two positive and one negative numbers gives a combined present value of $590.33. Note that V_0 and V_{30} are now single sums themselves. So if you compound the $590.33 for 30 years you obtain $3,390.56 (small rounding error).
7. It depends on your time preference for money. What is your interest rate for discounting?
We know that $30,000 in ten years is equivalent to $20,000 at about 4.14%. (Use Formula 2.2 to determine the 4.14 %.) So if my personal interest rate is less than 4.14% I'd prefer the $30,000 in ten years. For example, if my interest rate is 2%, the $30,000 has a present value of $24,610.45 to me. But if my personal interest rate is greater than 4.14%, I'd prefer the $20,000 today. For example, if my personal interest rate is 6%, the $30,000 has a present value of $16,751.84 to me. At 4.14% I should be indifferent as the two sums are equivalent.
8. There are an infinite number of conditions that will make them equivalent, but only one for
any particular n or i. For example, at 10% interest, they will be equivalent if n = 7.273 (solved via Formula 2.4) or it n = 10, they will be equivalent at i = 7.177 (solved via Formula 2.3). Or, at 5% interest, they will be equivalent at if n = 14.21 or if n = 5, they will be equivalent at 14.87%.
9. $1,795.86
10. 9 years
11. $136.86. Principal is $100 and interest is $36.86.
12. $730.69
13. 23.36 years
14. 6%
15. $3,195.74
16. $180.00
17. $738.72
18. $703.55
19. $703.55
20. $703.55

Exercise – Compound Interest

The power of compound interest is a simple concept, but sometimes it is not easy to understand. Several examples will illustrate the concept.

1. Let's plot present value for various interest rates. Consider the future value of $1,000 due in 10 years. What is its value at 0%, 1%, 2%, …, 10%? Draw a simple graph or sketch. Put interest rate on the x-axis and present value on the y-axis. Interest rate ranges from 0 to 10 and present value ranges from $0 to $1,000. Discuss the results. What is the relationship between interest rate and present value?

2. Let's plot present value for various time periods. Consider the value of $1,000 due at the end of the time period. Use 5% interest for all calculations and vary time period from 0 to 10 years. Draw a simple graph or sketch. Put time period on the x-axis and present value on the y-axis. Time period ranges from 0 to 10 years and present value ranges from $0 to $1,000. Discuss the results. What is the relationship between time period and present value?

3. Examples 1 and 2 involved discounting. Let's try one more example, but use compounding. What is the value of $1,000 compounded for various time periods? Use 10% interest for all calculations. Put time period on the x-axis and present value on the y-axis. Time period will be 5-year increments ranging from 0 to 50 years. Discuss the results.

Solutions:

1. 0% = $1,000; 1% = $905.29; 2% = $820.35; 3% = $744.09; 4% = $675.56; 5% = $613.91; 7% = $508.35; 8% = $463.19; 9% = $422.41; and 10% = $385.54.

2. 0 years = $1,000; 1 year = $952.38; 2 years = $907.03; 3 years = $863.84; 4 years = $822.70; 5 years = $783.53; 6 years = $746.21; 7 years = $710.68; 8 years = $676.84; 9 years = $644.61; 10 years = $613.91.

3. 0 years = $1,000; 5 years = $1,611; 10 years = $2,594; 15 years = $4,177; 20 years = $6,728; 25 years = $10,835; 30 years = $17,499; 35 years = $28,102; 40 years = $45,259; 45 years = $72,890; and 50 years = $117,391.

Exercise - Do You Understand Single Sums and Equivalence?

For the following problems assume three **single sum** cash flows (similar to the figure on page 1.5): a cost of $150 at year 0, a revenue of $500 at year 15, and a revenue of $2,900 at year 30. Use a 5% interest rate in all problems and ignore the few cents rounding errors.

1. What is the present value of the three single sums in the figure on page 1.5?

$V_0 = -\$150.00 + 500.00/(1.05)^{15} + \$2,900.00/(1.05)^{30}$

$V_0 = -\$150.00 + \$240.51 + \$671.00 = \$761.51$

2. What is the future value at year 30?

$V_{30} = -\$150.00(1.05)^{30} + \$500(1.05)^{15} + \$2,900.00$

$V_{30} = -\$648.29 + \$1,039.46 + \$2,900.00 = \$3,291.17$

Notice a second solution is:
$V_{30} = \$761.51(1.05)^{30} = \$3,291.20$

3. What is the value at year 15?

$V_{15} = -\$150.00(1.05)^{15} + \$500 + \$2,900.00/(1.05)^{15}$

$V_{15} = -\$311.84 + \$500.00 + \$1,394.95 = \$1,583.11$

Notice a second solution is:
$V_{15} = \$761.51(1.05)^{15} = \$1,583.12$

Notice a third solution is:
$V_{15} = \$3,291.17/(1.05)^{15} = \$1,583.11$

4. What is the value at year 20?

$V_{20} = -\$150.00(1.05)^{20} + \$500(1.05)^{5} + \$2,900.00/(1.05)^{10}$

$V_{20} = -\$397.99 + \$638.14 + \$1,780.35 = \$2,020.50$

Or, $\$761.51(1.05)^{20} = \$2,020.51$
Or, $\$3,291.17/(1.05)^{10} = \$2,020.49$
Or, $\$1,583.12(1.05)^{5} = \$2,020.51$

5. What is the future value at year 50?

$V_{50} = -\$150.00(1.05)^{50} + \$500.00(1.05)^{35} + \$2,900.00(1.05)^{20}$

$V_{50} = -\$1,720.11 + \$2,758.01 + \$7,644.56 = \$8,732.46$

Or, $V_{50} = \$761.51(1.05)^{50} = 8,732.54$
Or. $V_{50} = \$3,291.17(1.05)^{20} = \$8,732.45$
Or, $V_{50} = \$1,583.12(1.05)^{35} = \$8,732.51$
Or, $V_{50} = \$2,020.50(1.05)^{30} = \$8,732.48$

6. Why are rounding errors, while still a few pennies, larger in problem 5 than the earlier problems? (There are longer time periods involved and due to the exponential nature of compounding the small rounding errors become larger.)

Note in the prior examples that at 5% interest $761.51 today is equivalent to $3,291.17 in 30 years and both are equivalent to $1,483.11 in 15 years and all are equivalent to $2,020.50 in 20 years and all are equivalent to $8,732.46 in 50 years. There are an infinite number of cash flows that are equivalent to the original cash flow.

7. What if the $2,900 at year 30 was combined with the cash flow at year 15?

V_{15} = $2,900.00/(1.05)^{15} = $1,394.95

Combined Value = $1,394.95 + $500.00 = $1,894.95

A payment of $150.00 today plus revenue of $1,894.95 in 15 years is equivalent to all the single sums above and is equivalent to the original cash flow.

8. The payment at year 0 can be combined with the revenue at year 15 to produce an equivalent cash flow.

V_{15} = -$150(1.05)^{15} = -$311.84
Combined Value = $188.16

Revenue of $188.16 in 15 years plus revenue of $$2,900.00 in 30 years is equivalent to all of the single sums above and is equivalent to the original cash flow (and is equivalent to the new cash flow in problem 7.)

9. In problems 7 and 8, show that these new cash flows are equivalent to the earlier ones.

V_0 = -$150.00 + $1,894.95/(1.05)^{15} = $761.50

V_0 = $188.16/(1.05)^{15} + $2,900.00/(1.05)^{30} = $761.51

10. There are an infinite number of cash flows equivalent to the original cash flow. Give another one that is different from the ones above. This would involve moving cash flows around to different years. This can be accomplished many, many ways. For example, after much work, it can be shown that revenue of $175.89 per year for the first five year is equivalent to the original cash flow.

V_0 = $175.89/(1.05)^1 + $175.89/(1.05)^2 + $175.89/(1.05)^3 + 175.89/(1.05)^4 + $175.89/(1.05)^5

V_0 = $167.51 + $159.54 + $151.94 + $144.71 + $137.81

V_0 = $761.51

The point to take away from this exercise is that single sums can be moved around the cash flow line. They can be combined. They can be broken up. As long as a single interest rate is used, the cash flows produced will be equivalent. Probably the easier way to determine if a cash flow is equivalent to another cash flow is to solve for present value of each. If they have the same present value at the same interest rate, then they are equivalent. If they have the same value at any point on a time line at the same interest rate, then they are equivalent. This concept is fundamental to forest valuation.

Review Test – Single Sums

1. What is the future value of $100 placed into a savings account today paying 4% interest and withdrawn at the end of 20 years?
2. What is the present value of $5,000 due in twenty years at 4% interest?
3. You deposit $1,000 into an account that pays 5% interest and withdraw the balance of $2,182.87 later. How long was the money in the account?
4. You deposit $1,000 into an account for 20 years and then withdraw the balance of $4,660.96. What was the interest rate?
5. You deposit $5,000 into a savings account for twenty years. The account paid 6% interest for the first ten years and 12% interest for the second ten years. What was the balance of the account after 20 years?
6. You deposit $1,000 into a savings account that pays 8% simple interest. How much will be in the account in 7 years?
7. You placed $1,000 into a savings account that pays 5% interest on 1/1/2000. What will the balance of the account be on 1/1/2011?
8. You place $1,000 into a savings account at year 3 and withdraw the balance at year 8. The interest rate is 7%. How much do you withdraw?
9. You place $1,000 into a savings account at the beginning of year 3 and withdraw the balance at the beginning of year 8. The interest rate is 7%. How much do you withdraw?
10. You place $1,000 into a savings account at the beginning of year 3 and withdraw the balance at the end of year 8. The interest rate is 7%. How much do you withdraw?
11. How long will it take $4,000 to triple in value at 10% interest?
12. Improved genetic stock increases timber revenue in year 25 by $85.84. At 6% interest how much can you afford to pay today for improved stock (and earn 6%)?
13. You are considering a $20/ac. investment in a fertilizer treatment. Pulpwood is worth $10/cord. How much additional revenue is needed 12 years hence to ensure you earn 8% on the investment?
14. In problem 13, how much additional wood yield in cords is necessary for you to earn 8% on the investment?
15. A thinning in year 16 will increase harvest revenue by $280 in year 26. The thinning cost is $129.70. What was the rate of return (i.e., what was the interest rate)?
16. You place $1,000 into a savings account that pays 5% interest. You withdraw $1,628.89 in ten years. How much of the $1,628.89 represents principal and how much interest?
17. How long would it take to triple your money at 5% interest?
18. Timber revenue is expected to be $4,000 per acre in 15 years. If you thin the stand today timber revenue is projected to increase by 10%. If you have a 5% interest rate, what is the most you can pay for the thinning to earn 5%?
19. What interest rate will double your money in ten years?
20. Which cost will have a greater impact on present value: regeneration cost or intermediate thinning?

Solutions.
1. $219.11. **2.** $2,281.93. **3.** 16 years.
4. 8%. **5.** $27,810.51. **6.** $1,560.
7. $1,710.34. **8.** $1,402.55. **9.** $1,402.55.
10. $1,500.73. **11.** 11 ½ years. **12.** $20.00
13. $50.36. **14.** 5.04 cords. **15.** 8%.
16. Principal is $1,000 and interest is $628.89. **17.** 22.52 years. **18.** $192.41.
19. 7.18%. **20.** Regeneration because it occurs closer to year 0 and has less discounting.

Review Test – Single Sums

1. What is the future value of $1,000 placed into a savings account that pays 10% annual interest and the money remains in the account for ten years?

a. $2,593.74
b. $385.54
c. $1,000.00
d. $2,000.00

2. In problem 1 obviously compound interest is implied. What is the answer to problem 1 if simple interest is used?

a. $2,593.74
b. $385.54
c. $1,000.00
d. $2,000.00

3. What is the present value of $1,000,000 payable in 20 years using at 10% interest?

a, $1,000,000
b. $100,000
c. $148,643.63
d. $135,130.57

4. If $100 doubles to $200 over 10 years, what interest rate is earned?

a. 10.0%
b. 7.18%
c. 11.6%
d. 20.0%

5. If $100 is placed into a savings account that pays 10% interest and it triples in value to $300, what length of time was it in the account?

a. 10 years
b. 20 years
c. 11 ½ years
d. 15 ½ years

6. A thinning in year 16 will increase harvest revenue by $280/ac. in year 26. The thinning cost $129.70/ac. What interest rate or rate or return was earned?

a. 3%
b. 5%
c. 7 ¼%
d. 8%

7. Improved genetic stock increases timber revenue in year 25 by $85.84/ac. At 6% interest, how much can you afford to pay today for improved planting stock (and earn 6%)?

a. $85.84/ac.
b. $20.00/ac.
c. $7.92/ac.
d. $47.93 /ac.

8. Assuming today is January 25, 2007 and you place $10,000 into an account paying 10% interest, how much will be in the account on January 25, 2020?

a. $34,522.41
b. $31,38428
c. $37,974.98
d. $41,772.48

9. Assuming today is January 25, 2007 and you place $10,000 into an account paying 10% interest, how much will be in the account on December 31, 2020?

a. $34,522.41
b. $31,38428
c. $37,974.98
d. $41,772.48

10. In 1987 you place $10,000 into an account paying 10% interest and remove the balance in 2007, what is the balance removed (Clue: end of time period assumption holds unless stated otherwise)?

a. $61,159.09
b. $67,275.00
c. $74,002.50
d. $81,402.75

11. At the beginning of the year 1987 you place $10,000 into an account paying 10% interest and remove the balance at the end of the year 2007, what is the balance removed.

a. $61,159.09
b. $67,275.00
c. $74,002.50
d. $81,402.75

12. At the beginning of the year 1987 you place $10,000 into an account paying 10% interest and remove the balance at the beginning of the year 2007, what is the balance removed.

a. $61,159.09
b. $67,275.00
c. $74,002.50
d. $81,402.75

13. What is the present value of a bond that pays $100,000 on January 25, 2025 using a 7% interest rate? (Assume today is January 25, 2007).

a. $29,586.39
b. $31,657.44
c. $27,650.83
d. $25,841.90

14. What is the present value today (January 25, 2020) of $100 placed in to an account earning 5% on 1/25/10?

a. $162.89
b. $171.03
c. $179.59
d. $165.67

15. In problem 14, how much of the present value represents interest earned?
a. $100.00
b. $79.59
c. $62.89
d. $71.03

16. You are considering a $25/ac. fertilizer treatment. Pulpwood is worth $15/cord. How much additional revenue/ ac. is needed 15 years hence to ensure you earn 5% on the investment?

a. $20.78/ac.
b. $83.16/ac.
c. $51.97/ac.
d. $31.18/ac.

17. In problem 16, how much additional yield is necessary to ensure a 5% return?

a. 3 ½ cords/ac.
b. 1 2/3 cords/ac.
c. 2 ½ cord/ac.
d. 1 ½ cords/ac.

18. Which of the following costs associated with a planted pine stand would have the greatest impact on present value of the investment? Not in absolute terms but in terms of a single dollar spent on the practice (i.e., is a dollar spent on planting likely to have a higher or lower impact than a dollar spent on thinning)?

a. planting cost
b. prescribed burning in years 12, 18, and 22
c. precommercial thinning cost in year 9
d. timber sale preparation cost

19. You place $1,000 into a savings account that pays 5% interest for 5 years and then 10% interest for the next 5 years. How much is in the account after 10 years?

a. $1,628.89
b. $2,593.74
c. $2,061.03
d. $2,055.46

20. You place $1,000 into a savings account that pays 10% interest for 5 years and then 5% interest for the next 5 years. How much is in the account after 10 years?

a. $1,628.89
b. $2,593.74
c. $2,061.03
d. $2,055.46

Answers:

1. a. **2.** d. **3.** c. **4.** b. **5.**c. **6.** d. **7.** b. **8.** a.
9. c. **10.** b. **11.** c. **12.** b. **13.** a. **14.** a.
15. c. **16.** c. **17.** a. **18.** a. **19.** d. **20.** d.

Teaching/Learning Example—Single Sums

1. Consider three single sum amounts: $1,000 at year 3, $3,000 at year 7, and $4,000 at year 9. At 5% interest, what is the present value of the three amounts?

Discount $1,000 for 3 years, $3,000 for 7 years, and $4,000 for 9 years, and sum the results to obtain $5,574.32.

2. Consider the same three single sums above, what is V_{10}?

You could compound the $4,000 for 1 year, the $3,000 for 3 years, and the $1,000 for 7 years. However, you have a single sum value for all three. Just compound $5,574.32 for ten years to obtain $9,079.98.

3. Consider the same three single sums above, what is V_7?

Just use the single sum combined value again. Compound $5,574.32 for 7 years to obtain $7,843.63. You could have compounded $1,000 for 4 years and discounted $4,000 for 2 years and added the two to $3,000.

4. Consider the same three single sums above, what is V_{100}?

Just compound $5,574.32 for 100 years to obtain $733,030.09.

CHAPTER THREE
Learning Objectives
Terminating Annuities – Basic Assumptions

1. End of time period assumption. Students should understand the "standard case." Also, what if a payment occurs today? Also, what if the first payment occurs somewhere else than year 1?

2. Students should understand all cash flow series formulas give a "present value" one time period prior to the first payment. So if the first payment is at year one (the standard case), the present value is one time period prior (at year 0). Likewise, if the first payment happened to be at year 77, the "present value" calculated would be at year 76.

3. Students should understand that all cash flow series formulas calculate a present value that is a single sum. All the payment formulas convert to a single sum. Unless the first payment is where the standard case requires (year 1 for annual annuities), this single sum must be moved using one of the two single sum formulas to year 0. In the case above where the first payment was at year 77, the formulas gives a "present value" that is actually at year 76, and it must be discounted as a single sum for 76 years to be a true present value.

4. Students should understand that the future value of terminating annual cash flows automatically line up with the end of year assumption and no adjustments are necessary if the first payment does not occur at year 1.

5. Students should understand the formulas are for uniform cash flow series. However, many non-uniform series can be converted to several uniform series and solved series-by-series and summed to obtain a present or future value.

6. Students should understand the concept of equivalence. Cash flow series can be equivalent to a single sum and vice versa. Single sums can be equivalent at different years on the time line. Equivalence only exists at a specific interest rate.

7. Students should understand any perpetual annual cash flow series contains an infinite number of terminating cash flow series. Therefore the present value of portions of a perpetual annual series can be calculated.

8. Students should be able to utilize a beginning of time period assumption.

9. Students should understand the importance of time lines in performing calculations.

What Are Cash Flow Series or Annuities?

Annuities are shown on television often in late afternoon enticing people to convert their annuities into cash. The announcer might call them settlements, annual payouts, or monthly payouts. An annuity is a series of annual payments. It exists for a set period of time, a lifetime, or can run forever. We will start with annual payments, and later address monthly payments. The math is just the same for both, with a minor adjustment. So, when we are talking about a cash flow series or annuity in chapter 3, we are talking about a uniform series of payments over a specified time period. For example, $12,000 per year for the next 30 years, $12,000 per year for the rest of your life, or $12,000 per year forever. Chapter 3 is mainly about calculating the present and future value (as a single sum) for an annuity. Notice the different types in chapter 3:

Terminating annual cash flow series. Page 3.5. It is $5 per year for 20 years. Notice it is uniform. It ends after 20 years (terminates). It occurs once a year (annual).

Perpetual annual cash flow series. Page 3.16. Example 3.4. It is $45 per year forever. Notice it is uniform. It never ends (perpetual). It occurs once a year (annual).

Perpetual periodic cash flow series. Page 3.16. Example 3.5. It is $125,000 every ten years. Notice is it uniform. It never ends (perpetual). It is non-annual or periodic (every 10 years).

Terminating periodic cash flow series also exist. They are not that common in forestry and we will not stress them. They are discussed on page 3.13.

Notice uniform payments need not be perfectly uniform if they follow a pattern and can be converted into different uniform cash flow series to be calculated separately.

Some cash flow series alternate in a pattern. Consider problem 17 on page 17. It is $1,000 in old years and $2,000 in even years. This can be solved using our standard formulas by breaking the problem into odd year and even year problems. Then you have two periodic uniform cash flow series and the standard formulas work.

Some cash flow series alternate by increasing or decreasing after a period of time. Example: a cash flow series of $1,000 annually for 40 years and then $2,000 annually for another 40 years. The simple solution is to solve this as two uniform cash flow series.

An assumption of the formulas is that the cash flows begin at the end of the first year. Two standard exceptions are easy to handle.

First, the series might start later. Just discount the result to year 0 to compensate. Example: $1,000 per year starting at year 21. Use the standard formula to get a "present value" at year 20, and then just discount that single sum for twenty years.

Second, the series might start today. Just add the payment amount to the calculated present value (since the extra payment occurs today, no discounting is necessary).

Cash Flow Series – Details to Problem 3.1

The results of this problem showed that a cash flow series of $1,600 per year over 12 years at 6% interest was equivalent to a single sum of $13,414.15 today. That is, you could put $13,414.15 into a savings account today that paid 6% interest and withdraw $1,600 each year for the next 12 years. At that time the account balance would be zero. This is illustrated below:

Year	Amount in Account
0	$13,414.15
1	$13,414.15(1.06) - $1,600 = $12,619.00
2	$12,619.00(1.06) - $1,600 = $11,776.14
3	$11,776.14(1.06) - $1,600 = $10,882.71
4	$10,882.71(1.06) - $1,600 = $9,935.67
5	$9,935.67(1.06) - $1,600 = $8,931.81
6	$8,931.81(1.06) - $1,600 = $7,867.72
7	$7,867.72(1.06) - $1,600 = $6,739.78
8	$6,739.78(1.06) - $1,600 = $5,544.17
9	$5,544.17(1.06) - $1,600 = $4,276.82
10	$4,276.82(1.06) - $1,600 = $2,933.43
11	$2,933.43(1.06) - $1,600 = $1,509.44
12	$1,509.44(1.06) - $1,600 = 0

Notice that the $13,414.15 is a single sum. Using the future value of a single sum formula, the future value of $1,600 per year at year 12 at 6% interest would be $13,414.15(1.06)^{12} = $26,991.91. Or, the future value of a terminating series gives the same result. In table form, as above:

Year	Amount in Account
0	0
1	$1,600
2	$1,600 + $1,600(1.06) = $3,296.00
3	$1,600 + $3.296.00(1.06) = $5.093.76
4	$1,600 + $5,093.76(1.06) = $6,999.39
5	$1,600 + $6,999.39(1.06) = $9,019.35
6	$1,600 + $9,019.35(1.06) = $11,160.51
7	$1,600 + $11,160.51(1.06) = $13,430.14
8	$1,600 + $13,430.14(1.06) = $15,835.95
9	$1,600 + $15,835.95(1.06) = $18,386.11
10	$1,600 + $18,386.11(1.06) = $21,089.28
11	$1,600 + $21,089.28(1.06) = $23,954.64
12	$1,600 + $23,954.64(1.06) = $26,991.92

Cash Flow Series – Details to Example 3.3

The results of this problem showed two alternatives: one pay $500 today and the other was to pay $25 today and for each of the next 40 years. The first alternative had a present value of $500. The second alternative had a present value of $293.93. The second option was preferable with the lower net present value (since both alternatives were costs, the lower present value was preferred). Here is a way to think of the problem. The hunter could spend $500 today or he or she could spend $293.93. The $293.93 would go into a savings account that paid 9%. Then payments would be made out of that account. Later a formula to convert a single sum into an equivalent series of payments will be introduced. For now, accept an equivalent annual payment based on $500 today would be $42.52. That also shows the $25 option is better. Always compare apples and apples (single sums or payments). Later when the payment formula is introduced you may want to return to this page and compare the tables.

Year Paid	Account Balance	Account Balance
0	$293.93 -$25 = $268.93	$500.00 -$42.52 = $457.48
1	$268.93(1.09) - $25 = $268.13	$457.48(1.09) - $42.52 = $456.12
2	$268.13(1.09) - $25 = $267.26	$456.12(1.09) - $42.52 = $454.65
3	$267.26(1.09) - $25 = $266.32	$454.65(1.09) - $42.52 = $453.04
4	$266.32(1.09) - $25 = $265.29	$453.04(1.09) - $42.52 = $451.28
5	$265.29(1.09) - $25 = $264.16	$451.28(1.09) - $42.52 = $449.37
6	$264.16(1.09) - $25 = $262.94	$449.37(1.09) - $42.52 = $447.29
7	$262.94(1.09) - $25 = $261.90	$447.29(1.09) - $42.52 = $445.02
8	$261.90(1.09) - $25 = $260.15	$445.02(1.09) - $42.52 = $442.54
9	$260.15(1.09) - $25 = $258.56	$442.54(1.09) - $42.52 = $439.85
10	$258.56(1.09) - $25 = $256.83	$439.85(1.09) - $42.52 = $436.90
11	$256.83(1.09) - $25 = $254.95	$436.90(1.09) - $42.52 = $433.70
12	$254.95(1.09) - $25 = $252.89	$433.70(1.09) - $42.52 = $430.21
13	$252.89(1.09) - $25 = $250.66	$430.21(1.09) - $42.52 = $426.40
14	$250.66(1.09) - $25 = $248.22	$426.40(1.09) - $42.52 = $422.25
15	$248.22(1.09) - $25 = $245.56	$422.25(1.09) - $42.52 = $417.72
16	$245.56(1.09) - $25 = $242.66	$417.72(1.09) - $42.52 = $412.79
17	$242.66(1.09) - $25 = $239.50	$412.79(1.09) - $42.52 = $407.41
18	$239.50(1.09) - $25 = $236.05	$407.41(1.09) - $42.52 = $401.55
19	$236.05(1.09) - $25 = $232.30	$401.55(1.09) - $42.52 = $395.17
20	$232.30(1.09) - $25 = $228.21	$395.17(1.09) - $42.52 = $388.21
21	$228.21(1.09) - $25 = $223.74	$388.21(1.09) - $42.52 = $380.62
22	$223.74(1.09) - $25 = $218.88	$380.62(1.09) - $42.52 = $372.35
23	$218.88(1.09) - $25 = $213.58	$372.35(1.09) - $42.52 = $363.33
24	$213.58(1.09) - $25 = $207.81	$363.33(1.09) - $42.52 = $353.50
25	$207.81(1.09) - $25 = $201.51	$353.50(1.09) - $42.52 = $342.79
26	$201.51(1.09) - $25 = $194.65	$342.79(1.09) - $42.52 = $331.12
27	$194.65(1.09) - $25 = $187.16	$331.12(1.09) - $42.52 = $318.39
28	$187.16(1.09) - $25 = $179.01	$318.39(1.09) - $42.52 = $304.52
29	$179.01(1.09) - $25 = $170.12	$304.52(1.09) - $42.52 = $289.40
30	$170.12(1.09) - $25 = $160.43	$289.40(1.09) - $42.52 = $272.92
31	$160.43(1.09) - $25 = $149.87	$272.92(1.09) - $42.52 = $254.96
32	$149.87(1.09) - $25 = $138.36	$254.96(1.09) - $42.52 = $235.38
33	$138.36(1.09) - $25 = $125.82	$235.38(1.09) - $42.52 = $214.03
34	$125.82(1.09) - $25 = $112.14	$214.03(1.09) - $42.52 = $190.77
35	$112.14(1.09) - $25 = $97.23	$190.77(1.09) - $42.52 = $165.41
36	$97.23(1.09) - $25 = $80.99	$165.41(1.09) - $42.52 = $137.77
37	$80.99(1.09) - $25 = $63.28	$137.77(1.09) - $42.52 = $107.64
38	$63.28(1.09) - $25 = $43.97	$107.64(1.09) - $42.52 = $74.81
39	$43.97(1.09) - $25 = $22.93	$74.81(1.09) - $42.52 = $39.01
40	$22.93(1.09) - $25 = 0	$39.01(1.09) - $42.52 = 0

Cash Flow Series – Assumptions Exercise

Consider an annuity or cash flow series. The payment is $5,000 and it occurs annually for five years starting in year 16 and ending in year 20. That is, the series is $5,000 in years 16, 17, 18, 19, and 20. The interest rate for this problem is 8%. First, draw a time line for this cash flow series.

1. Use the present value (at year 15) and future value of a terminating annual series to obtain the values of this cash flow series. Use the single sum formula to show the two results are equivalent.

2. On the time line, where do the present value and future value occur?

3. Convert the "present value" from the formula to a present value that is at year 0.

4. Using just the single sum formulas, show that the $19, 963.55 present value is correct.

5. Using just the single sum discounting formula, show that the $6,293.34 present value is correct. **Notice that any cash flow series problem can be solved using the single sum formulas; it just might require a tremendous amount of work for a large problem.**

6. What is the value of the cash flow series at year 10? At year 100?

7. Using the single sum formulas show the results of answer 6 are equivalent.

8. What is the value of the cash flow series at year 18? Use the single sum formulas to show your answer is correct. Now use the present and future values of a cash flow series formulas to show the same thing.

Answers: 1. Present value is $19, 963.55 and future value is $29,333.00. To show equivalence, discount the $29,333.00 for five years at 8% or compound the $19,963.55 for five years at 8%. **Notice that the $19,963.55 and $29,333.00 are single sums and that they can be moved on the time line by just using the single sum formula.**

2. The present value occurs one year prior to the first payment, or year 15. The future value occurs at year 20.

3. Discount $19, 963.55 for 15 years at 8% or discount $29,333.00 for 20 years at 8%. Either way the present value at year 0 is $6,293.34.

4. $\$5,000/(1.08)^1 = \$4,629.63$
$\$5,000/(1.08)^2 = \$4,286.69$
$\$5,000/(1.08)^3 = \$3,969.16$
$\$5,000/(1.08)^4 = \$3,,675.15$
$\$5,000/(1.08)^5 = \underline{\$3,402.92}$
$19,963.55

5. $\$5,000/(1.08)^{16} = \$1,459.45$
$\$5,000/(1.08)^{17} = \$1,351.34$
$\$5,000/(1.08)^{18} = \$1,251.25$
$\$5,000/(1.08)^{19} = \$1,158.56$
$\$5,000/(1.08)^{20} = \underline{\$1,072.74}$
$6,293.34

6. Value at year 10 equals 13,586.86 and the value at year 100 equals $13,843,851.15.

7. Compound the value at year 10 for 90 years and expect a small rounding error. Or, discount the value at year 100 for 90 years.

8. $25,148.32. Compound the year 16 and 17 values and discount the year 19 and 20 values, and add the result to the year 18 value. That is, $5,832.00 + $5,400.00 + $4,286.69 + $4,629.63 + $5,000.00 = 25,148.32. Second question: PV of two payments = $8,916.32 and FV of three payments = $16,232.00. $8,916.32 + $16,232.00 = $25,148.32

Review Problems
Terminating Annual Series, Pages 3.5-3.12

1. Consider a cash flow series of $250/year starting in one year and ending in 50 years. What is the present value of the cash flow series at 5% interest?
2. What is the future value of the cash flow series in Problem 1?
3. Use the single sum formula to show the answers in Problems 1 and 2 are consistent.
4. Consider the same cash flow series in Problem 1. What is the value of the series at year 20 using 5% interest?
5. Consider a cash flow series of $250/year starting today and ending in 50 years. That is, 51payments starting today. What is the present value of the series at 5% interest?
6. Consider the same cash flow as in Problem 5. What is the future value at year 50?
7. Consider a cash flow series of $500/year for years 1 to 50 and $1,000/year for years 51 to100. What is the present value at 5% interest?
8. Consider the same cash flow as in problem 7. What is the future value at year 100?
9. What is the present value of a series of 50 payments of $1,000, the first payment due today, at 5% interest? That is, one payment today and 49 payments over the next 49 years.
10. What is the future value of a series of 50 payments of $1,000, the first payment due today, at 5% interest?
11. What is the present value of the last 25 payments of a series of 50 annual payments of $1,000? The first payment is one year from today and the interest rate is 5%
12. What is the future value of the last 25 payments of a series of 50 annual payments of $1,000? The interest rate is 5%.

Solutions

1. $4,563.98
2. $52,337.00
3. In both cases the cash flow series has been converted into a single sum. So if $4,563.98 is compounded at 5% for 50 years we ought to get the answer in Problem 2. Or, the answer to Problem 3 can be discounted for 50 years to produce the same answer as in Problem 1.
4. One way to solve the problem is to recognize the answer to Problems 1 and 2 are single sums. So $4,563.98 can be compounded for twenty years or $52,337.00 can be discounted for 30 years to equal $12,109.60. A second way is to calculate the value of the first twenty payments using Formula 3.3 ($8,266.49) and the value of the last thirty payments using Formula 3.4 ($3,843.11). Combined these equal $12,109.60.
5. Solution 1. The extra payment occurs today and has a present value of $250.00. The remaining 50 payments are the same ones in Problem 1 (the standard end-of-year problem). So the solution is $4,563.98 + $250 = $4,813.98. **Solution 2.** You can just solve the problem at a terminating annual series with 51 payments using Formula 3.4 to get a value of $4,584.74. Recognize that value is at -1 on the time line and needs to be compounded for one year to become a year 0 value of $4,813.98.
6. This is an important problem. Note that the payments line up perfectly on the time line. Use Formula 3.3 and 51 payments to get the answer. No adjustment is needed. $55,203.84. Note you could have taken the result from Problem 2 and recognized there was one extra payment at year 0. If you compound that payment for 50 years and add it to the Problem 2 result you also obtain $55,203.84.
7. Solution 1a. Think of this as a $500 series for all 100 years and then a $500

series for just the last 50 years. Use Formula 3.4 to get the value of the 100 $500 payments ($9,923.96). Then use Formula 3.4 to get the value of the last 50 $500 payments. (9,127.96). Keep in mind this value is one year prior to the first payment, or at year 50. So it must be discounted for 50 years to equal $795.99. Then the combined cash flows are $9,923.96 + $795.99 = $10,719.95. **Solution 1b.** What if for the second step you calculated the future value of the residual $500 payments? That would be $104,674. It would be at year 100 and if that single sum was discounted for 100 years would also equal $795.99. **Solution 2a.** First, calculate the value of just the first 50 payments using Formula 3.4 ($9,127.96). Next, calculate the value of the last 50 payments using Formula 3.4 ($18,255.93). This value is actually at year 50 and needs to be discounted for 50 years ($1,591.99). Then the combined cash flows are $10,719.96. **Solution 2b.** What if for the second 50 payments you calculated the future value? The future value is $209,348. It would be at year 100 and if this single sum is discounted for 100 years it also equals $1,591.99. **Solution 3.** Like Solution 1 think of this cash flow series as one series of $500 per year for 100 years and a second series of $500 for just the last 50 years. The future value of the 100 payments series is $1,305,012.58. The future value of the 50 payments is $104,674.00. The combined future value is $1,409,686.58. This is a single sum and if discounted for 100 years equal $10,719.96. **Solution 4.** Solve for the future value again, but for the first and second 50 years separately. The $1,000 series has a value at year 100 of $209,348.00. The $500 series has a value at year 50 of $104,674.00. Compound that value as a single sum for 50 years to obtain $1,200,338.61. Then V_{100} = $1,409,686.61. Discount that number for 100 years to obtain V_0 = $10,719.95. **Solution 5.** Obtain the future value for the series as if all payments were $1,000. This value is $2,610,025.16. We know from above the first 50 payments of $500 have a future value of $104,674.00 and that value compounded to year 100 is $1,200,338.61. Subtract that value from $2,610,025.16 to obtain V_{100} = $1,409,686.55 and that number discounts to V_0 = $10,719.95. **Solution 6**. Obtain the present value of $1,000 for 100 years. This is $19,847.91. Then obtain the present value of 50 payments of $500. This is $9,127.96. Subtract $9,127.96 from $19,847.91 to obtain the present value of the cash flow series of $10,719.95.

8. Solution 1. Of course, we can compound the answer to problem 7 for 100 years to get the answer of $1,409,688.20**. Solution 2.** Calculate the future value of the first 50 payments using Formula 3.3 ($104,674.00) and then compound it for 50 years ($1,200,338.60). Then calculate the value of the last 50 payments using Formula 3.3 ($209,347.99). The combined value of the series is $1,409,686.60. **Solution 3.** Calculate the future value of the $500 cash flow for the full 100 years using Formula 3.3 ($1,305,012.60). Then calculate the value of the remaining $500 cash flow series over the last 50 years using Formula 3.3 ($104,674.00). The combined values are $1,409,686.60. Due to long time period involved a rounding error $1.60 develops.

9. $19,168.72

10. $209,347.99 (Realize this value is at the end of the series or year 49. What if I wanted the value at year 50 where no payment occurs? Simply compound the single sum for one year to obtain $$219,815.40.)

11. $4,161.98

12. $47,727.10

Teaching/Learning Example
Annual Terminating Series

Consider the same cash flow series from page 3.5. This is, twenty annual payments of $5 per year at 8% interest. However, the first payment is at year 5 and the last payment is at year 24. Calculate V_0, V_{24}, V_{10}, and V_{100}.

Calculate V_0

In the original problem with the first payment at the end of the first year (standard assumption), we calculated V_0 as $49.09. That value now falls at V_4. So, discount $49.09 for four years to get V_0 for the new situation. ($36.08)

Calculate V_{24}

Notice the cash flow series perfectly lines up with the end value. We previously calculated a future value of $228.81 and that is also the value at V_{24}. As a check if you compound $36.08 for 24 years you obtain$$228.81 or if you discount $228.81 for 24 years you obtain $36.08.

Calculate V_{10}.

You could compound $36.06 for 10 years or discount $228.81 for 14 years and obtain $77.90. Or, you could use both of the terminating cash flow series formulas. First, consider the first six payments. Use the future value formula to obtain the future value of 6 payments at V_{10}. This value is $36.68. Then use the present value formula to obtain the value of the last 14 payments at V_{10}. This value is $41.42. Add the two values to obtain $77.90.

Calculate V_{100}.

Compound $36.08 for 100 years, or $228.81 or 76 years, or $79.90 for 90 years.

An extra problem. Consider an annual cash flow series of $1,000 for the first 10 years, $2,000 for the second 10 years, and $3,000 for the third 10 years. Using just four steps (formula calculations), obtain the present value. Use a 5% interest rate.

One way to solve this problem, ignoring steps, is to first discount the first 10 payments. That value is $7,721.73. Then address the second ten years. The value at year 10 for those ten payments is $15,443.47. That is a single sum. Discount it for ten years to obtain a present value of $9,480.95. Then address the last ten year series. Discount the $3,000 payments to year 20 to obtain $23,165.20. Discount that single sum for twenty years to obtain $8,730.72. Combine the three present values to obtain $25,933.40. But, this took five steps.

Another way would be to consider a $1,000 cash flow series for all 30 years. That discounts to $15, 372.45. It needs no further discounting as the value is at year 0. Then there is a second $1,000 series from year11 to year 30. That discounts to 1$2, 462.21 at year 10. That single sum is discounted to year 0 and is $$7,650.72. Last, there is a final $1,000 cash flow series from year 21 to 30 that discounts to $7,721.73. That single sum discounts 20 years to $2,910.24. The three single sums add up to $25,933.41. The problem is this method also took five steps.

What is a method to accomplish this task in four steps? First, look at the cash flow in segments and notice that there is a $1,000 cash flow from year 1 to year 30. Compound that cash flow to year 30 and obtain $66,438.85. Second, notice there is a second cash flow series on top of the original one that runs from year 11 to 30. Compound that cash flow and obtain $33,065.95. Third, notice there is one last cash flow series left of $1.000 at years 21 to 30. Compound that series to year 30 and obtain $12,577.89. The future value then is the sum of the three separate future values, or $$112.082.69. Discount that single sum for 30 years and obtain $$25,933.41. This was accomplished in four steps.

Review Test
Terminating Cash Flow Series

1. Consider a cash flow series of eight $1,000 payments, the first payment at year 4 and the last one at year 11. At 5% interest what are the PV and FV of the cash flow series?

2. Consider the cash flow series in problem 1. Same eight payments, but the first cash flow is today. That is, payments occur from year 0 to year 7. What are the PV and FV at 5% interest?

3. Consider a cash flow series of $1,000 payments from year 1 to year 100 and $2,000 payments from year 101 to year 250. What are the PV and FV at 5% interest?

4. Consider a cash flow of $1,000 at year 26. What is that cash flow worth at year 36 at 5% interest?

5. You place $1,000 in a savings account at year 1. At year 5 you take $500 out of the account. At year 10 you close the account and take out all the money. The account paid 5%. How much did you take out at year 10?

6. You place $1,000 in a savings account each year from year 1 to year 10. At year 5 you take $500 out of the account. At year 10 you close the account and take out all the money. The account paid 5%. How much did you take out at year 10?

7. What is the present value of the cash flow series in problem 6?

8. You need to buy a piece of logging equipment in ten years. It will cost $350,000. You plan to put $25,000 into a savings account today and at the end of each of the ten years. How much extra money will you need to come up with at year 10?

9. Same problem as in question 8. Except you put $50,000 in bank today and $10,000 annually for ten years. How much extra money will you need to come up with?

10. In problem 9 you can't come up with $142,776.34 at the end of ten years. So you'll put 15,000 per year in the savings account for ten years. You want to have exactly $350,000 in bank in ten years. How much do you need to put into the bank today (year 0)?

11. What is the amount $X.XX that is equivalent to a cash flow of $1,000 today and $1.000 in year 5? The amount $X.XX is at year 10. Use a 5% interest rate.

12. What is the present value of $X.XX in problem 11?

Answers: **1.** PV = $5,583.17 and FV = $9,547.55. **2.** PV = $6,786.37 and FV = $$9,549.11 (at year 7). **3.** PV = $20,151.89 and FV = $3,996,138,305. **4.** $1,628.89. **5.** $990.76. **6.** $11,939.75. **7.** $7,329.97. **8.** $355,169.68. **9.** $207,223.66. **10.** $99,043.62. **11.** $2,905.17. **12.** $1,783.53

Review Problems
Perpetual Series, Pages 3.13-3.18

1. What is the present value of $1,000 per year forever at 8% interest?
2. What is the future value of $1,000 per year forever at 8% interest?
3. What is the present value of $1,000 every three years, forever at 8% interest?
4. What is the answer to problem 1 if the first payment is at year 6?
5. What is the answer to problem 3 if the first payment is at year 6?
6. Show that the answer to problem 4 is correct.
7. Show that the answer to problem 5 is correct.
8. What is the present value of a perpetual annual series of $1,000 per year at 5% interest?
9. What is the present value of a perpetual periodic series of $1,000 every ten years, beginning in 10 years at 5% interest?
10. What is the present value of a perpetual annual series of $1,000 per year at 5% interest, the first payment being paid today?
11. What is the present value of a perpetual periodic series of $1,000 every ten years, beginning today at 5% interest?
12. What is the present value of a perpetual annual series of $1,000 per year at 5% interest if the first payment is due at the end of the fifth year (year 5)?
13. What is the present value of a perpetual periodic series of $1,000 every ten years at 5% interest if the first payment is due at the end of the fifth year?
14. Consider a perpetual annual series of $1,000 every year at 5% interest. What is the present value of all the payments past the first 200 years (payments at years 201, 202, 203, 204...)?
15. At 7% interest, calculate the present value of thinning revenues of $500 starting in 10 years and continuing every 20 years after that in perpetuity.
16. Principal of $20,000 generated $2,000 of interest per year on a perpetual basis. What must the interest rate be?
17. Consider a perpetual cash flow series. All odd years have a payment of $1,000 and all even years have a payment of $2,000. At 5% interest what is the present value of the cash flow series?
18. Consider a perpetual cash flow series. Every year a payment of $1,000 occurs, with the exception of every third year when a payment of $4,000 occurs (that is, a payment of $4,000 occurs in years 3, 6, 9 ...). What is the present value of the cash flow series at 5% interest?
19. Consider the same problem on the top of page 3.17, but the first payment in the series won't be until year 14. What is present value or the value of the forest then?

Solutions to Review Problems
Perpetual Series, Pages 3.13-3.18

1. This is a perpetual annual series. Formula 3.5. $1,000 / 0.08 = $12,500.
2. There is no future value for a perpetual series as it goes to infinity.
3. This is a perpetual periodic series. Formula 3.6. $\$1,000 / [(1.08)^3 – 1] =$ $3,850.42. Note that you can confirm the answer if you study the timeline. This is the same as the timeline in problem 1, with two three-year periodic perpetual series removed. The first one occurs in years 1, 4, 7, … and the second one occurs in years 2, 5, 8 …. Both are three-year periodic series, so both have a "present value" of $ $3,850.42. Both occur too "soon" and the calculated present values have to be compounded for 1 or 2 years to move to year 0. So $\$12,500 – (\$3,850.42)(1.08)^1 – (\$3,850.42)(1.05)^2 = \$3,850.42$.

4. Problem 1 is an annual series and the "present value" is calculated one time period (one year in this case) prior to the first payment. So the present value of the perpetual series will be calculated for year 5. That means the answer to question 1 must be discounted for five years. $12,500 / $(1.08)^5$ = $8,507.29.

5. Problem 3 is a periodic series and the "present value" is calculated one time period (three years in this case) prior to the first payment. So the present value of the perpetual series will be calculated for year 3. That means the answer to question 3 must be discounted for three years. $3,850.42 / $(1.08)^3$ = $3,056.59.

6. We know that the standard perpetual annual series starting at year 1 is worth $12,500 from problem 1. If the first payment is now at year 6, we know that five payments have been removed. These five payments are a terminating annual series and the present value can be calculated using Formula 3.4. The present value of the five payments is $3,992.71. $12,500 - $3,992.71 = $8,507.29.

7. We know that the standard perpetual periodic series starting at year 3 is worth $3,850.42 from problem 3. If the first payment is now at year 6, we know that one payment (the one at year 3) has been eliminated. The present value of that payment is $1,000 / $(1.08)^3$ = $793.83. $3,850.42 - $793.83 = $3,056.59.

8. Formula 3.5. $20,000.

9. Formula 3.6. $1,590.09.

10. Draw this out on a time line. It is the standard payment series for a perpetual period series (Formula 3.5), but there is one additional payment at year 0. One way to solve this problem is to solve the standard problem as in Problem 8, then add in the value of the additional payment at year 0. The value of the additional payment at year 0 is $1,000, so the present value of the whole series must be $20,000 + $1,000 = $21,000. A second approach is to simply solve for the "present value" of the series as it is on the time line. This gives a "present value" of $20,000, but at year -1 on the time line. To get to year 0 one must compound the value for one year. $20,000$(1.05)^1$ = $21,000.

11. Draw this out on a time line. It is the standard payment series for a perpetual periodic series (Formula 3.6), but there is one additional payment at year 0. One way to solve this problem is to solve the standard problem at in Problem 9, then add in the value of the additional payment at year 0. The value of the additional payment at year 0 is $1,000, so the present value of the whole series must be $1,590.09 + $1,000 = $2,590.09. A second approach is to simply solve for the "present value" of the series as it is on the time line. This gives a "present value" of $1,590.09 at year -10 on the time line. To get to year 0 one must compound the value for ten years. $1,590.09$(1.05)^{10}$ = $2,590.09.

12. Formula 3.5. The "present value" is given at year 4. Draw a time line. So the present value at year 0 must be calculated by taking this value and discounting for 4 years to get to year 0. $20,000 / $(1.05)^4$ = $16,454.05.

13. Formula 3.6. The "present value" is given at year -5. So the present value at year 0 must be calculated by taking this value and compounding it for 5 years to get to year 0. $1,590.09$(1.05)^{5=}$ $2029.40.

14. From Problem 8 we know the present value of all the payments is $20,000. The first 200 payments represent a terminating annual series. Formula 3.4 can be used to get the present value of a terminating annual series of $1,000 payments at 5% interest. This present value is $19,998.84. So the present value of all the payments past year 200 is $20,000 - $19,998.88 = $1.16. Also notice that $20,000 - $20,000/$(1.05)^{200}$ = $1.16.

15. Draw a time line for any problem like this. First, let's get the present value of the payment at year 10. Discount $500 at 7% for ten years. This is $254.17. Then consider the 20-year periodic series. The "present value" will be 20 years prior to the first payment or year 10. So that present value must also be discounted at 7% for ten years. The "present value" of the 20-year series is $\$500/[(1.07)^{20} - 1] = \$174.24$. Discounting the $174.24 yields $88.57. So the present value is $254.17 + $88.57 = $342.74. Of course, since the 20-year series gave a "present value" at year 10 and the "extra" payment of $500 was also at year 10, the $174.24 could be added to the $500 to sum to 674.24 and that could be discounted also to $342.74.
16. Formula 3.5. Solve for $i = \$2,000 / \$20,000$. $i = 0.10$ or 10%.
17. Draw a time line. **Solution 1.** Every year has a cash flow of $1,000 and every even year has an extra cash flow of $1,000. Since every year has a cash flow of $1,000, Formula 3.5 gives a present value of $20,000. The extra $1,000 occurs every even year, or is a standard periodic perpetual cash flow series occurring every 2 years. It has a present value of $9,756.10 according to Formula 3.6. The combined present value is $29,756.10. **Solution 2.** First, consider just the perpetual $1,000 cash flow. It is a periodic perpetual cash flow that occurs every two years. We know its value is $9,756.10. But this value occurs two years prior to the first payment, or at year -1. So this value must be compounded for one year to equal $\$9,756.10(1.05)^1 = \$10,243.91$. Second, the $2,000 represents a perpetual periodic cash flow series that is properly aligned on the time line so that its value occurs at year 0. The present value of the $2,000 series is $19,512.20. The combined present value is $29,756.11.

18. Draw a time line. **Solution 1.** There is an annual perpetual cash flow of $1,000 per year its present value is $20,000. There is a $3,000 periodic perpetual cash flow every three years and its present value is $19,032.51. The combined present value is $39,032.51. **Solution 2.** The first $1,000 payment can be looked at as occurring every three years. That is, at years 1, 4, 7, ... That makes it a perpetual periodic series of three years that calculates the present value at year -2. Formula 6 gives a present value of $6,344.17. Compounded for two years that is $6,994.45. The second $1,000 payment can be looked at as occurring every three years. That is, at years 2, 5, 8, ... That makes it a perpetual periodic series of three years that calculates the present value at year -1. Formula 1 gives a present value of $6,344.17. Compounded for one year that is $6,661.38. The $4,000 payment is a perpetual periodic series of three years that gives a present value at year 0. Formula 6 gives a present value of $25,376.69. The combined present value is $6,994.45 + $6661.38 + $25,376.69 = $39,032.52.
19. Draw another cash flow diagram and note the "present value" will occur at year 4. So that same value of $166,876.67 discounted for 4 years becomes $133,436.34. Look at both cash flow diagrams. The difference is that payment at year 4. Discount $125,000 for 4 years to obtain a present value of $99,951.31. Then note $233,387.66 - $99,951.31 = $133,436.35.

Review Test
Cash Flow Series

1. Consider a cash flow series of $1,000 per year starting in one year and ending in 40 years. What is the present value of the cash flow series at 6% interest?
2. What is the future value (at year 40) of the cash flow series in question 1?
3. What is the value of the cash flow series in question 1 at year 20?
4. What if the cash flow series in question 1 had 41 payments? These payments include the original 40 payments, plus one additional payment today. What is the present value then?
5. Consider the cash flow series in question 1. What is the present value of the last 20 payments in the series?
6. Consider the cash flow series in question 1. What is the future value (at year 40) of the last 20 payments in the series?
7. Consider a cash flow series of 40 payments. The first 20 payments are $1,000 per year and the last 20 payments are $2,000 per year. At 6% interest what is the present value of the cash flow series?
8. Consider a cash flow series of 40 annual payments. The first payment occurs at year 21. The interest rate is 6% and the payment is $1,000. What is the present value of this series?
9. Consider the cash flow series in question 8. What is the future value (at year 60) of the cash flow series?
10. What is the present value of $500 per year forever at 6% interest?
11. What is the future value of $500 per year forever at 6% interest?
12. What is the present value of $500 every 5 years, forever at 6% interest?
13. What is the future value of $500 every 5 years, forever at 6% interest?
14. In question 10, what is the answer if the first payment is at year 40?
15. In question 12, what is the answer if the first payment is at year 40?
16. In question 12, what is the answer if the first payment is at year 2? That is, the payments occur at years 2, 7, 12, 17, 22, ….
17. Consider the cash flow in question 10. What is the present value of all the payment past year 100? That is, subtract off the value of the first 100 payments?
18. You need the present value of a simple forestry investment. There are three cash flows. First, you regenerate today (year 0) at a cost of $200/ac. Second, you harvest timber at year 30 for a revenue of $2,000/ac. Third, you have to pay property taxes over the thirty years of $5/ac. At 6% interest, what is the present value of the forestry investment?
19. Consider a perpetual cash flow series. Every year you receive $1,000, but every fifth year you receive $6,000 (total). What is the present value of the cash flow series at 6% interest?
20. What is $1,000,000 every 200 years forever worth at 10% interest? That is, the payments occur at years 200, 400, 600, ….

Answers: **1.** $15,046.30 **2.** $154,761.97 **3.** $48,255.52 **4.** $16,046.30 **5.** $3,576.38 **6.** $36,785.59 **7.** $18,622.68 **8.** $4,691.51 **9.** $154,761.97 **10.** $8,333.33 **11.** infinity **12.** $1,478.30 **13.** infinity **14.** $858.80 **15.** $192.33 **16.** $1,760.68 **17.** $24.56 **18.** $79.98 **19.** $31,449.70 **20.** ½ cents.

Review Problems
Annual Payments, Pages 3.23-3.37

1. What is the payment on a $100,000 loan, 10 annual payments at 10% interest?
2. What is the annual payment that will accumulate $100,000 over 10 years at 10% interest?
3. What is the annual payment to accumulate $100,000 over 10 years at 10% interest, the first payment today plus 10 additional payments?
4. What is the annual payment to accumulate $100,000 over 10 years at 10% interest, if you have $20,000 today to put into the account?
5. What if we had a problem like Problem 4, but with a repayment formula. That is, what is the annual payment to repay $100,000 over 10 years at 10% interest, if you have $20,000 today to contribute to the repayment?
6. Consider the loan in Problem 1. What is the present value of the first five payments?
7. Consider the loan in Problem 1. What is the present value of the last five payments?
8. Consider the loan in Problem 1. You have just made the last payment. How much principal have you paid back and how much interest?
9. Consider the loan in Problem 1, at the end of year 5 you decide to pay off the loan, what is the amount due?
10. Calculate an amortization schedule for Problem 1.
11. You agree to pay back a $100,000 loan at 10% interest in ten annual payments of $16,274.54. However, the lender agrees to forego the first payment. How much did you really pay back in present value terms?

Solutions to Review Problems
Annual Payments, Pages 3.23-3.37

1. This is a standard annual payment to repay a loan problem. It is also called installment payments and capital recovery. Formula 3.9 with n = 10, i = 0.10, and V_o =$100,000. Solve for $P_{ann.}$ = $16,274.54.
2. This is a standard annual payment to a future sum problem. It is also called a sinking fund problem. Formula 3.7 with n = 10, i = 0.10, and V_n = $100,000. Solve for $P_{ann.}$ = $6,274.54.
3. Draw the time line for this problem. Note that this is exactly the same as a standard sinking fund problem with 11 payments. Use Formula 3.7 to solve with n = 11, i = 0.10, and V_n = $100,000. Solve for $P_{ann.}$ = $5,396.31. Let's see if this is correct. The first payment today of $5,396.31 has a value of $13,996.64, calculated as $5,396.31$(1.10)^{10}$. That means only $86,003.36 now needs to be accumulated, calculated as $100,000 - $13,996.64. So now solving the standard sinking fund problem, n = 10, i = 0.10, and V_n =$86,003.36. Solve for P_{ann}. = $5,396.31.
4. This is similar to the second part of answer 3. The $20,000 contributes to the $100,000 needed in ten years. Compound it to calculate that contribution. $20,000$(1.10)^{10}$ = $51,874.85. That means the annual payments only need to accumulate $48,125.15. Using Formula 3.7, n =10, i = 0.10, and V_n = $48,125.15. Solve for $P_{ann.}$ = $3,019.63.
5. The $20,000 is equivalent to a down payment. The present value of $20,000 today is $20,000 and if you "contribute" it to the loan, in effect, you only owe $80,000. Use Formula 3.9, with n = 10, i = 0.10, and V_o = $80,000. Solve for $P_{ann.}$ = $13,019.63.
6. The first five payments represent a terminating annual series. Formula 3.4 can be used to obtain the present value of series, with n = 5, i = 0.10, and a = $16,274.54. Solve for V_o = $61,693.31.
7. We can calculate the value by subtraction. Since the present value of all payments must be $100,000 and the present value of the first five payments is

$61,693.31, then the present value of the last five payments must be $100,000 - $61,693.31 = $38,306.69. Or, you might have calculated the present value of those five payments using Formula 3.4 as in Problem 6 and obtained an answer of $61,693.31. However, you would look at the time line and see that this "present value" is at one year prior to the first payment or at Year 5. So you'd discount it for five years to obtain $61,693.31/(1.10)^5 = $38,306.69. Or, you might have calculated the future value of those five payments using Formula 3.3, with n = 5, i = 0,10, and a = 16,274.54. Solve for V_n = $99,357.69. However, you again look at the time line and notice this value is at year 10, so you discount it for ten years to get the present value of the last five payments. $99,357.69/(1.10)^10 = $38,306.69.

8. Total amount paid on the loan was 10 X $16,274.54 =$162,745.40. So you have paid back $100,000 in principal and have paid $62,745.40 in interest.

9. We know from Problem 7 that the "present value" of the last five payments at year 5 is $61,693.31. But you still owe the fifth payment at the end of year 5, so the amount due is $61,693.31 + $16,274.54 = $77,967.85.

As a check the present value of $77,967.85 at year 5 is $48,411.90 at year 0 and the present value of the first four payments is $51,588.10 and that adds up to $100,000. Note that the value of the six last payments (at year 4) via Formula 3.4 is $70,879.86 and if that amount is compounded for one year to year 5 it also equals $77,967.85.

10. An amortization schedule shows the interest paid and remaining balance on a loan. It is an interesting exercise as it illustrates that most of the interest on a loan is paid early in the loan because the outstanding principal is larger then. First, the equal annual payment is calculated using Formula 3.9. We know from Problem 1 the equal annual payment for a 10-year 10% loan of $100,000 is $16,274.54. The amortization schedule is below. Note that the loan balance in the amortization schedule agrees with the result in Problem 6. The one cent difference is simply a rounding error. Also note the loan balance can be directly calculated by:

Loan Balance = $V_0 (1 + i)^n - a[((1 + i)^n – 1)/ i]$

where V_0 = original loan balance and n = payment number.

Loan Balance for payment 5 = $100,000(1.1)^5 - $16,274.54[(1.1)^5 – 1]/i.

Loan Balance for payment 5 = $161,051 - $99,357.69 = $61,693.31.

11. The present value of the first payment is $14,795.04. So you paid back $85,204.96.

Amortization Schedule.

Payment Number	Interest	Principal	Loan Balance
0	-	-	$100,000.00
1	$10,000.00	$6,274.54	93,725.46
2	9,372.55	6,901.99	86,823.47
3	8,682.35	7,592.19	79,231.28
4	7,923.13	8,351.41	70,879.87
5	7,087.99	9,186.55	61,693.32*
6	6,169.33	10,105.21	51,588.11
7	5,158.81	11,115.73	40,472.38
8	4,047.24	12,227.30	28,245.08
9	2,824.50	13,450.04	14,795.04
10	1,479.50	14,795.04	-0-

Problem 3.13 from Textbook
Accumulate $150,000 over 8 years with annual payments of $15,155.39
An "Amortization Schedule" for a Sinking Fund Problem

An "amortization schedule" for a sinking fund problem will illustrate exactly how each payment is treated. Notice the first payment was at the end of the first year. Therefore there was no interest for the first year. Notice the last payment was at the end of eight year. Therefore, that last payment never earned any interest. That schedule is shown below.

Year	Payment	Interest	Balance
0	0	0	0
1	$15,155.39	0	$15,155.39
2	15,155.39	$909.33	31,220.11
3	15,155.39	1,873.21	48,248.71
4	15,155.39	2,894.93	66,299.03
5	15,155.39	3,977.94	85,432.36
6	15,155.39	5,125.94	105,713.69
7	15,155.39	6,342.82	127,211.90
8	15,155.39	7,632.71	150,000.00

Uniform/Non-Uniform Series
Non-Uniform Cash Flows Problems

1. Consider the following cash flow series:
   - $5,000 every odd year forever, years 1, 3, 5, …
   - $10,000 every even year forever, years 2, 4, 6, …

At 5% interest, what is the cash flow series worth today n present value terms?

V_{0odd} = $\{\$5,000/[(1.05)^2 – 1]\}(1.05)^1$ = $51,219.51

V_{0even} =$\$10,000/[(1.05)^2 – 1]$ = $97,560.98

V_0 = $51,219.51 + $97,560.98 = $148,780.49

2. Consider the same cash flow series in problem 2. Instead of the irregular cash flows, you desire a single uniform annual cash flow of equivalent value at 5% interest. Convert the irregular cash flow series into a uniform perpetual annual cash flow series. What is the uniform payment?

Use the present value of an annual perpetual cash flow series to solve this problem. Recall $V_0 = a/i$. Or, solving for a, then $a = V_0(i)$.

a = $148,780.49(0.05) = $7,439.02

Review Problems
Cash Flow Series-Review of Assumptions

There are six basic cash flow series formulas: Present and Future Value of a Terminating series, Perpetual Annual and Periodic series, Installment Loan, and Sinking Fund. All use the end-of -time period or end-of-year assumption. That is, all are based on the first payment occurring one compounding period from today. When this assumption is violated, the technique is to solve using the standard formula for a "present value" that is misplaced on the time line, that is, not located at year 0, and then to move this single sum to year zero using simple compounding and discounting.

Beginning of year problems.

1. What is the present value of a series of 40 annual payments of $1,000, the first payment occurring today? The interest rate is 6%.
2. What is the future value of a series of 40 annual payments of $1,000, the first payment occurring today? The interest rate is 6%. Note that the future value will be at year 39.
3. What is the future value of a series of 40 annual payments of $1,000, the payment occurring today, but the future value at the end of year 40 (year 40)? The interest rate is 6%. Draw a time line and note that no payment occurs at year 40.
4. What is the present value of perpetual annual series of $1,000 payments, the first payment occurring today? The interest rate is 6%.
5. What is the present value of a perpetual periodic series of $1,000 payments every three years, the first payment occurring today? The interest rate is 6%.
6. You borrow $40,000 at 6% interest to be repaid over ten years, with the first payment today. That is, there are 11 payments. What is the payment?
7. You wish to accumulate $40,000 using an account that pays 6% interest with 11 annual payments, the first payment today. The future value at the end of year 10 will be $40,000.
8. In problem 7 calculate the present value and future value of the 11 payments and show the answer is correct.

Delayed payments problems

9. What is the present value of a series of 40 annual payments of $1,000, the first payment occurring at year 11? The interest rate is 6%.
10. What is the future value of a series of 40 annual payments of $1,000, the first payment occurring at year 11? The interest rate is 6%. The future value would be at year 50.
11. What is the future value of a series of 40 annual payments of $1,000 at 6% interest, the first payment occurring in one year and the future value at year 100? That is, after the 40^{th} payment the amount sits in the account untouched for 60 additional years earning interest.
12. What is the present value of a perpetual annual series of $1,000 payments at 6% interest, the first payment occurring at year 11?
13. What is the present value of a perpetual periodic series of $1,000 payments every three years at 6% interest, the first payment occurring at year 11?
14. You borrow $40,000 to be repaid over ten years in annual payments at 6% interest. However, the lender gives you a one-year grace period with no interest. The first payment is not due until the end of year 2. What is the payment?

15. What is the payment in Problem 14 if interest is charged over the grace period?
16. You need to accumulate $1,000,000 over 40 years using a savings account that pays 6% interest. You decide to make 20 uniform payments over the first 20 years and no payments over the last 20 years. What is the uniform annual payment?
17. What is the answer to Problem 16 if you make payments only during the last 20 years?

Non-uniform Series

18. Consider a perpetual cash flow series that pays $1,000 every even year and an additional $2,000 every five years. At 6% interest what is the present value of the cash flows?
19. Consider a perpetual cash flow series that pays $1,000 every three years and an extra $1,000 every sixth year. At 6% interest what is the present value of the cash flows?
20. Consider a perpetual cash flow series that pays $1,000 annually for the first 100 years and then $5,000 annually forever. At 6% interest what is the present value of the cash flow series?
21. You need to accumulate $1,000,000 over 40 years. You have a 6% savings account and place $20,000 into it today. What is your annual payment into the account necessary to accumulate $1,000,000?

Solutions to Review Problems
Cash Flow Series-Review of Assumptions

1. Consider the problem to be a series of 39 annual payments and use Formula 3.4 to obtain a present value of $14,949.08. Then add in the additional first payment that has a present value of $1,000. Answer: $15,949.08. Alternative Solution is to consider the series to be 40 payments and recognize that that "present value" will be calculated for year -1. So use Formula 3.4 and 40 payments to get a "present value" of $15,046.30 and compound that result for one year to get a present value of $15,949.08.
2. Recognize that all 40 payments align in the regular manner in this problem. The answer will be at year 39 and not year 40. Use Formula 3.3 with 40 payments to obtain a future value of $154,761.97. Note that $\$15{,}949.08(1.06)^{40} = \$154{,}761.97$.
3. The value at year 39 was calculated in Problem 2 as $154,761.91. This is a single sum. Simply compound it for one year: $\$154{,}761.97(1.06)^{1} = \$164{,}047.69$. A second solution would be to solve for the future value of 41 annual payments. This would be $165,047.68. Note the last payment occurs at year 40, so it has a value of $1,000 then. Subtract that last payment from $165, 047.68 and you have the identical result.
4. One way to solve the problem is to recognize that this is the standard problem with one additional payment at year 0. Ignoring the first payment, the value of the series using Formula 3.6 is $16,666.67. The first payment occurs at year 0 and is worth $1,000. So the present value of the whole series is $16,666.67 + $1,000.00 = $17,666.67. A second way to look at the series is as a standard series that places the "present value" at year -1. So compounding the calculated "present value" for one year will give you $\$16{,}666.67(1.06)^{1} = \$17{,}666.67$.
5. Just like problem 4, use Formula 3.5 to calculate a "present value" of $5,235.16 and add $1,000 to it. The present value is $6,235.16. Or, recognize the computed "present value" is at year -3, so just compound $5,235.16 for three years at 6% to get $6,235.16.
6. This is a little silly. If the first payment is today, then it is a down payment and reduces the amount borrowed. But what would the payment be? Draw a time line. You have 11 payments running from year 0 to year 10. If this is treated as a terminating annual series the present value is one year prior to the first payment, or year -1. So we

must discount the loan amount for one year to calculate the amount actually borrowed. This is $40,000 / (1.06)^1 = $37,735.85. Using Formula 3.9 with n = 11, i = 0.06, and V_o = $37,735.85, we calculate a payment of $4,784.64. Let's see if that is correct. Using Formula 3.4 the present value of ten payments is $35,215.38. The present value of the additional payment at year 0 is $4,784.64. The combined present value is $40,000.02. The 2 cents is a rounding error. The payment is $4,784.64.

7. This lines up fine on the time line with 11 payments. No adjustment is necessary. Use Formula 3.8 and 11 payments to obtain a payment of $2,671.72.

8. Since we know the future value at year 10 is $40,000, the present value must be $40,000 /(1.06)^{10} = $22,335.79. Using Formula 3.3, the future value of the 11 payments of $2,671.72 is $40,000.04. Using Formula 3.4, the present value of 10 payments of 2,671.72 is $19,664.09. But there is an additional payment at year 0 that must be added to that value, so the present value at year 0 is $19,664.09 + $2,671.72 = $22,335.81. Or, using Formula 3.4 the "present value" of the 11 payments can be calculated at year -1 as $21,071.52. That value must be compounded for one year to obtain a present value at year 0. $21,071.52(1.06)^1 = $22,335.81. The 4 cents and 3 cents differences are due to rounding.

9. The "present value" of the 40 payments using Formula 3.4 is $15,046.30. But this value occurs one year prior to the first payment or year 10. So it must be discounted for ten years to give a present value at year 0 of $8,401.78.

10. The 40 payments line up correctly on the time line to give a future value at year 50. Use Formula 3.3 to obtain a future value of $154,761.97. As a check if the future value is discounted for 50 years as a single sum it ought to equal the present value obtained in Problem 9 and it does.

11. First, obtain the future value of the 40 payments as $154,761.97. Realize this is a single sum at year 40 and compound it for 60 years to get the value at year 100. $154,761.97(1.06)^{60} = $5,105,240.10.

12. Use Formula 3.5 to obtain a "present value" of $16,666.67. Realize this value is on the time line one year earlier than the first payment, or year 10. So discount the result for ten years. $16,66.67 / (1.06)^{10} = $9,306.58.

13. Use Formula 3.6 to obtain a "present value" of $5,235.16. Realize this value is on the time line 3 years earlier than the first payment, or year 8. So discount the result for eight years. $5,235.16 / (1.06)$^{8 =}$ $3,284.60.

14. Since there is no interest for the first "grace" year, you still owe $40,000 and the payment will be the standard one calculated using Formula 3.9, or $5,434.71.

15. The lender charges interest for the first year. So the loan amount increases to $40,000(1.06)^1 = $42,400.00. Use Formula 3.9 and the new principal amount to recalculate the payment. The new payment is $5,760.80.

16. Since there are no payments during the last 20 years and you need $1,000,000, you need to discount the $1,000,000 for 20 years as a single sum to see how much you need at year 20. At year 20 you need $1,000,000 / (1.06)^{20} = $311,804.73. Use Formula 3.7 to calculate the payment of $8,476.27.

17. The last 20 payment line up perfectly with the standard assumptions. Use Formula 3.7 to calculate a payment of $27,184.56.

18. Draw a time line. For the first twelve years cash flows occur at year 2 ($1,000), year 4 ($1,000), year 5 ($2,000), year 6 ($1,000), year 8 ($1,000), year 10 $3,000), and year 12 ($1,000). Of course, these cash flows continue forever. On the time line the cash flow series looks very non-uniform. There are $1,000, $2,000, and $3,000 cash flows scattered about. Closer observation shows it is really just two uniform perpetual cash flow series. The first one is a perpetual periodic cash flow series of $1,000 every two years and it lines up in the standard manner on the time line. Using Formula 3.6 the present value is $8,090.62. The second one is a perpetual periodic cash flow series of $2,000 every five years and it lines up in the standard manner on the time line. Using Formula 3.6 the present value is $5,913.21. The combined present value is $14,003.83.

19. Draw a time line. For the first twelve years cash flows occur at year 3 ($1,000), year 6 ($2,000), year 9 ($1,000), and year 12 ($2,000). Of course, the cash flows continue forever.

Solution 1. Close observation shows two uniform periodic perpetual cash flow series. The first one is $1,000 every three years forever. Using Formula 3.6 the present value is $5,235.16. The second one is $1,000 every six years forever. Using Formula 3.6 the present value is $2,389.38. The combined present value is $7,624.54.

Solution 2. Look closely at the time line you've drawn. The $1,000 payment occurs every 6 years, but the first payment is ay year 3 (not the standard case of year 6). First use Formula 3.6 to obtain the "present value" of the 6-year cash flow. This is $2,389.38. But this value is at year -3 on the time line. So it must be compounded as a single sum for three years to obtain a present value of $2,845.79. The $2,000 payment occurs every six years and the first payment is at year 6. This is the standard case. Formula 3.6 gives a present value of $4,778.75. The combined present value is $7,624.54.

20. Draw a time line. **Solution 1**. Consider the first 100 payments to be a terminating annual series and use Formula 3.4 to obtain a present value of $16,617.55. Consider the payments after the 100^{th} payment to be a perpetual annual series of $5,000 payments. Use Formula 3.5 to obtain a "present value" of $83,333.33. However, this present value is at year 100 on the time line. It needs to be discounted as a single sum for 100 years. The year 0 present value is $245.60. The combined present value is $16,863.15.

Solution 2. Consider the $1,000 to occur every year and $4,000 to occur starting in year 101. Then the present value of the $1,000 perpetual annual series is $16,666.67. The $4,000 cash flow series begins in year 101 and continues forever. Formula 3.5 gives a year 100 "present value" of $66,666.67 for this cash flow. However, it must be discounted for 100 years to produce a year 0 present value of $196.48. The combined present value is $16,863.15

21. Compound the $20,000 single sum to year 40 to see the contribution it makes towards the $1,000,000. The contribution is $205,714.36. This means the payments must accumulate to $794,285.64. Use Formula 3.7 to obtain the payment of $5,132.31.

Extra Problems

1. Consider a perpetual annual cash flow series consisting of $1,000 payments beginning at year 21. Use a 5% interest rate. What is the present value of this cash flow series? Which of the following is not a method to obtain that result? Given: the present value of a perpetual annual cash flow series of $1,000 beginning at year 1 (the standard case) is $20,000.

a. Obtain the future value of the first 20 (missing payments) at year 20 and subtract that from the "present value" of a perpetual annual series starting at year 21 (year 21, 22, 23, …).

b**.** Obtain the present value of the first 20 payments and subtract that from $20,000.

c. Obtain the "present value" at year 20 of the perpetual annual series ($20,000). Discount that single sum for 20 years.

d. Obtain the future value at year 20 of the full perpetual cash flow series with no missing payments . Obtain the future value at year 20 of the missing 20 payments. Subtract the future value of the missing payments from the future value of the full series. Discount that value for 20 years.

2. Consider the same cash flow series in question 1. What is the value of all payments past year 100 (that is, payments 101,102, 103, …)? Which of the following is not a method to obtain that value?

a. Discount payments of $1,000 per year from year 1 to year 100, and then take that present value and subtract it from $20,000.

b. Assume a perpetual annual cash flow series starting at payment 101 and get the "present value" of that cash flow series at year 100. Discount that amount for 100 years.

c. Compound the 80 payments from payment 21 to payment 100 to get a future value at year 100. Discount that value to year 0 and subtract it from $20,000.

d. Compound 100 payments (from year 1 to year 100) to get a future value at year 100 and discount that value to year 0 and subtract it from $20,000.

3. Consider the same cash flow series in question 1. What is the value of the first 80 payments in the series (payments 21, 22, 23, …, 99, 100)? Which of the following is not a method to obtain that value?

a. Obtain the present value of a terminating annual series that runs from payment1 to payment 20 and subtract that from the present value of a terminating annual series that runs from payment 1 to payment 100.

b. Obtain the present value of a terminating annual series that runs from payment 1 to payment 20 and subtract that from the present value of $20,000.

c. Compound the 80 payments to year 100 and discount that value for 100 years.

d. Discount the present value of 80 payments to year 20 and discount that value for 20 years.

Answers: 1. a, 2. c, 3. b. Show yourself that the other three methods in each case produce the same result.

Review Test
Uniform Cash Flow Series

Use a 4% interest rate for all problems.

1. Consider a perpetual periodic cash flow series of $5,000 every ten years. However, the first payment is in year 5. What is the present value of the cash flow series? (The payments are in years 5, 15, 25). Answer: $12,667.02

2. Consider the same cash flow series as in problem1, except the first payment is today. What is the present value of the cash flow series? (The payments are in years 0, 10, 20,…). Answer: $15,411.37

3. Consider a terminating annual cash flow series of $5,000 per year that ends at year 80. What is the present value of the last 40 payments? (Payments in years 41, 42, 43, …, 78, 79, 80). Answer: $20,613.09

4. What is the future value at year 80 of the same last 40 payments in problem 3? Answer: $475,127.58

5. Consider a perpetual periodic cash flow series of $5,000 every ten years, beginning in ten years. What is the present value of all cash flows occurring after year 120? (Payments at years 130, 140, 150, …). Answer:$94.08

6. Consider a perpetual periodic cash flow series of $5,000 every ten years, beginning in ten years. Convert this cash flow series to an equivalent perpetual annual cash flow series. What is the equivalent annual payment? Answer: $416.45

7. You purchase a skidder for $150,000. You make a $30,000 down payment. You agree to make six annual payments. Five of them will be equal payments, one for each of the first five years. However, the sixth payment is a balloon payment of $50,000. What is the amount of the five equal annual payments? Answer: $18,078.95

8. You place $50,000 into a savings account today. Your goal is to accumulate $250,000 at the end of ten years. You will make ten equal annual payments to accomplish this goal. What amount will the equal annual payment be? Answer: $14,658.19

9. Consider an annual perpetual cash flow series that pays $2,000 every year except every fifth year when it pays $4,000. Plus, it pays $4,000 today. So the first ten years cash flow is: year 0 = $4,000, year 1 = $2,000, year 3 = $2,000, year 4 = $2,000, year 5 = $4,000, year 6 = $2,000, year 7 = $2,000, year 8= $2,000, year 9 = $2,000, year 10 = $4,000, …). What is the present value of the cash flow series? Answer: $63,231.36

10. You take out a $400,000 loan. Repayment will be with ten equal annual payments. On the day the fifth payment is due you choose to repay the entire outstanding balance. How much will be due to pay off the loan on that date? Answer: $268,864.14

Review Problems
Non-Annual Interest

Non-annual interest must account for inter-period compounding within the year. The APR is the annual percentage rate. The custom is to state all interest rates in terms of a year. However, if the compounding period is less than a year (daily, monthly, quarterly, or semi-annual), then the interest rate for that period is the APR divided by the number of compounding periods in a year. That is for daily it is divided by 365, for monthly it is divided by 12, for quarterly it is divided by 4, and for semi-annual it is divided by 2.

For example, if the APR is 12%, the monthly interest rate is 1%. Effectively then, since compounding takes place 12 times over the year, the interest rate would work out to more than 12%. Take $1,000 and place it into a savings account paying 12% annually. Your account balance at the end of the year would be $1,120. But if you earned 1% a month for twelve months your balance would be $\$1{,}000(1.01)^{12}$ = $1,126.83. Where did the extra $6.83 come from? This is due to compounding from month to month; the effect of compound interest. Page 9.19 illustrates that the more compounding periods the year is split up into, the higher the effective interest rate.

For a non-annual compounding problem use the following two rules. The number of compounding periods, n, equals the number of compounding periods in a year times the number of years involved. The interest rate equals the APR divided by the number of compounding periods in a year.

Example 1. You buy a new truck for $35,000 at 6% interest for four years. Repayment will be monthly. That means the number of compounding periods is 48 and the interest rate is 0.5%.

Example 2. You borrow $100,000 for ten years at 18% interest and agree to make quarterly payments. The number of payments is 40 and the interest rate is 4.5%.

Consider problem 28 on page 9.18. It asks for the account balance of a savings account if $1,000 is held in it for six years earning 8% interest, compounded quarterly. The problem notes that this involves 24 compounding periods and a 2% quarterly interest rate. So the account balance will be $V_n = \$1{,}000(1.02)^{24} = \$1{,}608.44$.

Notice that using the same method the account balance can be calculated for different non-annual interest rates. For example:

Annual: $\$1{,}000(1.08)^{6} = \$1{,}586.37$
Semi-Annual: $\$1{,}000(1.04)^{12} = \$1{,}601.03$
Quarterly: $\$1{,}000(1.02)^{24} = \$1{,}608.44$
Monthly: $\$1{,}000(1.00667)^{72} = \$1{,}613.50$
Daily: $\$1{,}000(1.000219178)^{2{,}190} = \$1{,}615.99$

Notice the effective annual interest rates are calculated on page 9.19. They can be used to calculate the same account balances (subject to rounding errors):

Annual: $\$1{,}000(1.08)^{6} = 1{,}586.37$
Semi-Annual: $\$1{,}000(1.0816)^{6} = \$1{,}601.03$
Quarterly: $\$1{,}000(1.0824)^{6} = \$1{,}605.15$
Monthly: $\$1{,}000(1.0830)^{6} = \$1{,}613.51$
Daily: $\$1{,}000(1.0833)^{6} = \$1{,}616.19$

Review Problems – Non-Annual Interest

1. You agree to repay a loan of $30,000 in 48 monthly payments at 12% interest. What is the payment?

2. What if problem 1 is changed to 4 annual payments? What is the payment? Note that in problem 1 you paid $790.02 X 12 = $9,480.24. The effective interest rate is $(1.01)^{12} - 1 = 0.1268$ or 12.68%. Shouldn't the payment be higher due to higher interest?

3. Using a savings account that pays 6% compounded semi-annually, if you place $1,000 in the account today, how much will be in the account in 10 years?

4. What is the effective interest rate in problem 3?

5. Using a savings account that pays 6.09% annually, if you place $1,000 in the account today, how much will be in the account in 10 years?

6. A radio station offers a grand prize of one day's interest on $1,000,000. At 9% interest, how much is that?

7. You need to purchase a new truck for $40,000. You make a $10,000 down payment and finance the rest at 6% over 48 months. What is your monthly payment?

8. You need $30,000 to purchase a new truck in 4 years. You decide to establish a sinking fund and make monthly payments. The interest rate is 6%. What is the payment?

9. You purchase a new truck for $30,000 and pay for it over 60 months at 5% interest. The dealer has a promotion and pays the first payment for you. What did the truck really cost?

Solutions
Non-Annual Interest

1. The monthly interest rate is 1%. Use Formula 3.10 with n = 48 and i = 0.01 to calculate a payment of $790.02.

2. The annual payment is $9,877.03. Yes, higher interest means higher payments. However, repayment starts much earlier and this more than makes up for the higher interest, so the payments accumulate to a smaller sum.

3. There are 20 compounding periods and the interest rate is 3% per compounding period. $(1.03)^{20}$ X $1,000 = 1,806.11.

4. $(1.03)^2 - 1 = 0.0609$ or 6.09%.

5. $1,806.11.

6. 0.09/365 X $1,000,000 = $246.58.

7. The amount borrowed is $30,000. n = 48 and i = 0.005. Payment is $704.55.

8. Payment is $554.55.

9. $29,436.21.

Review Test
Uniform Series/Payments

1. What is the present value of $1,000 per year for 40 years, the first payment in one year, using a 5% discount rate?

2. What is the present value of $1,000 per year, for a total of 40 payments, the first payment due today, using a 5% interest rate?

3. What is the present value of $1,000 per year for 40 years, the first payment at year 10, using a 5% interest rate?

4. What is the future value of $1,000 per year for 40 years, the first payment in one year, using a 5% interest rate?

5. What is the future value of forty $1,000 payments, the first payment due today, using a 5% discount rate? (Future value at year 39.)

6. What is the future value of $1,000 per year for 40 years, the fist payment at year 11, using a 5% interest rate? (Future value at year 50.)

7. Show by example that the answers to problems 1 and 4 are equivalent at a 5% interest rate.

8. What is the present value of a perpetual annual series of $1,000 payments, the first payment due in one year, using a 5% interest rate?

9. What is the present value of a perpetual annual series of $1,000 payments, the first payment due today, using a 5% interest rate?

10. What is the present value of a perpetual annual series of $1,000 payments, the first payment at year 11, using a 5% interest rate?

11. What is the present value of a perpetual periodic series of $1,000 payments, the first payment in 10 years and every 10 years thereafter, using a 5% interest rate?

12. What is the present value of a periodic perpetual series of $1,000 payments, the first payment today, and payments every 10 years thereafter, using a 5% interest rate?

13. What is the present value of a perpetual periodic series of $1,000 payments, the first payment at year 100, and every 10 years thereafter, using a 5% interest rate?

14. Consider a perpetual annual series of $1,000 payments and a 5% interest rate. What is the present value of all the payments past year 100 (that is, payments 101, 102, 103, ...)?

15. Draw a time line. Every third year on the time line a payment of $1,000 occurs and every fourth year a payment of $1,000 occurs. (Every twelfth year they combine for a payment of $2,000). Treat this as a perpetual series. What is the present value at 5% interest?

16. Draw another time line. Every second year a payment of $1,000 occurs and every fourth year a payment of $2,000 occurs. That means they combine for a payment of $3,000 every fourth year. For the first twelve years the payments are: Year 2 = $1,000 Year 4 = $3,000 Year 6 = $1,000 Year 8 = $3,000 Year 10 = $1,000 and Year 12 = $3,000. Treat this as a perpetual series. What is the present value at 5% interest?

17. You purchase a used Ford truck for $25,000. You pay $5,000 down. You agree to make 48 monthly payments at a 6% APR. What is the monthly payment?

18. You need to purchase a truck in four years. You decide to establish a sinking fund using a bank account that pays 6% interest, compounded monthly. You need $20,000 in four years. What monthly deposit will accumulate $20,000 in four years?

19. At 7% interest, calculate the present value of thinning revenues of $500 starting in 12 years from now and continued every 30 years in perpetuity.

20. Consider a perpetual annual series of $1,000 payments at 5% interest. What is the present value of the 100 payments that occur in the second hundred years? That is payments 201, 202, 203, ..., 300.

21. Principal of $100,000 generates perpetual annual payments of $8,000. What interest rate must the money be earning?

22. Consider a 50-year terminating annual series of payments at 5% interest. The payment is $500 for the first 25 years and $1,500 for the second 25 years. What is the present value of the series?

23. What is the future value of the cash flow series in problem 22?

24. You need $10,000,000 at the end of 10 years to replace logging equipment. You plan to accumulate the capital by making monthly payments into an account that pays 12% interest, compounded monthly. What is the payment?

25. In problem 24, what is the present value of the first 24 payments?

26. What would the payment be in problem 24 if you had $2,000,000 today to put into the account, in addition to making the 120 monthly payments?

27. A radio contest offers a grand prize of the interest on $1,000,000 for one day. Using a 7% interest rate, what is one day's interest?

28. What if they doubled the prize? What would two days interest be?

29. In problem 24 you are expecting good profits for the first five years and poor profits for the second five years. So you plan to accumulate the $10,000,000 with just 60 annual payments and then letting the funds accumulate interest for the second five years. What is the new monthly payment?

30. What is the monthly payment in problem 29 if the last 60 months are the good ones?

Solutions

1. $17,159.09**.**
2. $18,017.04.
3. $11,060.90**.**
4. $120,799.77.
5. $120,799.77.
6. $120,799.77.
7. Either compound $17,159.09 for 40 years or discount $120,799.77 for 40 years.
8. $20,000.
9. $21,000.
10. $12,278.27.
11. $1,590.09.
12. $2,590.09.
13. 19.70.
14. $152.09
15. $10,984.41.
16. $19,036.57.
17. $469.70.
!8. $369.70.
19. $255.59.
20. $1.15.
21. 8%.
22. $13,289.94.
23. $152,401.10
24. $43,470.95.
25. $923,470.23
26. $14,776.76.
27. $191.78.
28. $383.60
29. $67,399.51.
30. $122,444.48

Review Test

1. You deposit $1,000 a year into a 4% savings account for 30 years. You leave the balance in the account for another 30 years. How much is in the account at year 60?

2. You deposit 1,000 a year into a 4% savings account and make 30 deposits. The first payment is today (beginning of the year). You leave the balance in the account until year 60. How much is in the account at year 60?

3. An annuity pays you $25,000 a year for 15 years and then $50,000 a year for the next 15 years. At 6% interest what is the present value of the annuity?

4. Consider a perpetual cash flow series that pays $6,000 every year except the fifth years when it pays $3,000. That is, it pays $3,000 in years 5, 10, 15, … and $6,000 in all other years. What is the present value of the series at 6% interest?

5. You borrow $100,000 and agree to pay it back in 20 equal annual payments of $8,718.46 at 6% interest. After you make 10 payments, the lender agrees to let you pay off the remainder of the loan over another 20 years. The interest rate will not change. What is your new payment?

6. If interest is compounded daily, what is the interest on a loan of $1,000,000 for one year at 12% interest?

7. You borrow $250,000 and agree to pay it back over 20 years with semi-annual payments at 6% interest. What is the payment?

8. You buy your first house. You have a $25,000 down payment. You use a 30 year mortgage at 6% interest. You purchase an $185,000 house. What is your monthly payment?

9. The present value of $100,000 in ten years at 10% interest is $38,554.32.

a. Now you find out that the time period increased to 20 years and the interest rate changed, and that the present value is still $38,554.32. Is this possible? Y or N?

b. Now you find out that the time period increased to 20 years and the interest rate changed and the present value is now $138,554.32. Is this possible? Y or N?

c. Now you find out that the time period is still ten years and the interest rate has decreased. The present value is now $48,554.32. Is this possible? Y or N?

d. Now you find out that both the time period and interest rate increased. Present value is now $32,554.32. Is this possible? Y or N?

Solutions

1. $181,905.75
2. $189,181.98
3. $445,435.33
4. $91,130.18
5. $5,594.51
6. $127,474.62
7. $10,815.59
8. $959.28
9. Y, N, Y, Y

CHAPTER FOUR
Net Present Value

1. You are considering the purchase of bare land to grow timber. Planting costs are $100/ac. and annual management and tax cost is $5/ac./yr. Thinning revenue at year 17 is $220/ac. and final harvest revenue at year 28 is $2,600/ac. You expect to sell the land at the end of rotation for $650/ac. after replanting that will cost $150/ ac. If you desire to earn 8% on this investment, how much can you pay for the bare land?

2. You are considering the purchase of land that is fully forested. You plan to develop a selection forest managed on an uneven-aged basis. It will take 25 years for the forest to develop and then you expect perpetual periodic harvests yielding $75,000 every 7 years. The first harvest is at year 25. Annual management fees and property taxes are $6,000 and annual hunting lease revenue is $5,000. If your discount rate is 8%, and you expect to earn at least that much, what is the most you can pay for the land? Assume natural regeneration after each harvest.

3. If you calculate the present value of $50,000 in 25 years at a 12% interest and then recalculate the present value using the situations below, how will the present value change:

(a) $50, 000 in 50 years at a higher interest rate,

(b) $50,000 in 50 years at a lower interest rate,

(c) $50,000 in 15 years at a higher interest rate, or

(d) $50,000 in 15 years at a lower interest rate?

4. A forest has three revenues that will occur on a perpetual basis: Year 15 has thinning revenue of $350/ac., year 28 has harvest revenue of $3,500/ac., and net annual revenues are $6.00/ac. year. At 3% interest, what is the present value of the three revenues? It is year 0 relative to the three cash flows.

5. The USDA Forest Service constructs a recreational area for a cost of $500,000. The area is expected to have a perpetual life and annual maintenance costs of $40,000/year. The expected use is 5,000 visits per year. Using a 4% discount rate, what is the minimum value per visit to justify the recreational area?

6. Suppose that the USDA Forest Service constructs a recreational area with a 100-year life. Construction costs $500,000 today. What uniform annual recreation benefits must be generated to earn a 3.5% rate of return?

7. You desire to earn 8% interest. You purchase bare land that requires $200/ac. in regeneration costs today and at the end of each harvest. Harvest is every 25 years in perpetuity and will yield $2,700/ac. before regeneration. Annual hunting lease revenue pays for management fees and property taxes. How much can you afford to pay for the land?

8. The current management regime on a forested tract produces a present value of $500,000. An alternative management regime is has an objective of increased recreation and wildlife benefits. The present value of that management regime is $350,000. Using a 4% discount rate, what additional annual value must the increased recreation/wildlife produce to account for the decrease in present value?

Solutions

1. $263.54. **2.** $13,793.13. **3.** (a) lower, (b) can't tell, (c) can't tell, (d) higher. **4.** $3,316.63. **5.** $12.00 per visit. **6.** $18,079.64. **7.** $227.46/ac. **8.** $6,000.

Chapter 4
EAI and B/C Ratio Problems

Notice that Formula 3.9 for annual installment payments (capital recovery factor) is the same one that is used to convert a NPV into an EAI.. Notice that NPV, EAI, and the B/C ratio will produce consistent results as all three are mathematical versions of the same thing.

1. Consider the following forest management regime:

Year	Item	Amount/Acre
0	Buy land	-$400.00
1	Site preparation	- 100.00
2	Plant	- 60.00
15	Thinning revenue	+320.00
26	Harvest revenue	+4,300.00
26	Sell land	+750.00
1-26	Annual taxes	- 2.00

What is EAI and B/C ratio at 6% interest?

2. The USDA Forest Service constructs a recreation site for $250,000. It has an expected life of 25 years. It is expected to have 1,000 visitors per year. If the investment must return 4%, what must the visitor fee be?

3. What if in problem 2 we add that $9,000 in maintenance is incurred by the site each year?

4. The USDA Forest Service constructs a recreation site for $250,000. It is expected to have a perpetual life. Annual maintenance costs are $9,000 per year and every 25 years major repairs of $50,000 are expected. The site is expected to have 5,000 visitors per year. If the recreation site is expected to return 5%, how much must the visitor fee be?

5. NPV = 0 at 5% interest. If NPV is recalculated using 6%, what do we know about the EAI and B/C ratio?

6. If the B/C ratio equals 1, what is the NPV and EAI?

7. If the interest rate used to calculate the B/C ratio in problem 6 is lowered, what happens to NPV and EAI?

8. The NPV of a 25-year timber rotation is calculated as $24,899 at a 7% interest rate. What is the EAI and B/C ratio?

9. Assume NPV is $1. As interest rate increases, what happens to NPV, EAI, and B/C ratio?

Solutions

1. Since B/C is being calculated, one would save time by keeping track of costs and revenues separately. The present value of the revenues is $1,243.57 and the present value of the costs is $573.75. Net present value equals $669.82. EAI = $51.51. B/C ratio = 2.17.

2. EAI for the $250,000 NPV equals $16,002.99. Visitors must produce $16,002.99 in revenue each year or about $16.00 per visit.

3. The $9,000 is already on an annual basis, so the fee would go up to about $9.00 per visit.

4. The NPV of the three costs is $250,000 + $180,000 + $20,952.46 = $450,952.46. If we multiply the NPV by 0.05 we find that we need annual revenues of $22,547.62 or about $4.51 per visit.

5. We know NPV will be negative if interest rate is increased, so EAI will be negative and the B/C ratio will be less than 1.

6. Zero.

7. Both become positive.

8. EAI = $2,136.60 and all we can say about the B/C ratio is that it is greater than 1.

9. All three decrease. Eventually NPV and EAI = 0 and B/C ratio = 1. Then if interest rate continues to decrease NPV and EAI become negative and B/C ratio becomes less than 1.

Chapter 4
IRR Problems

Notice that in the case of a single cost and single revenue IRR can be directly calculated using Formula 2.3. Otherwise iterations are required. If NPV is positive, IRR must be higher. If NPV is negative, IRR must be lower. If NPV = 0, then discount rate equals IRR.

1. What is the IRR of the first problem under EAI and B/C ratio?

2. What if a silvicultural treatment in year 13 that cost $25/ac. increased timber sale revenue in year 25 by $100/ac., what would the rate of return be?

3. A timberland investment has an IRR of 10%. For the same investment the B/C ratio is calculated using an 8% interest rate. What do we know about the B/C ratio?

4. The B/C ratio of a timberland investment calculated with an 8% discount rate is 1. What is the IRR?

5. Actually, there is an exception to the statement that with more than 1 cost or revenue you must use iterations to solve the problem. If the cost or revenue is a uniform series you can directly solve for the interest that equates costs and revenues. You invest in a timberland investment that will return $10,000 every year for the next 30 years. You pay $150,000 for the investment. What is your rate of return?

6. You pay $100,000 for a timberland investment that returns $30,000 every three years forever. What is your rate of return?

7. You pay $100,000 for a timberland investment that returns $10,000 annually forever. What is your rate of return?

8. Investment A has an IRR of 10%. Investment B has an IRR of 10%. Which one is better? Or, are they equal investments and the investor should be indifferent.

9. Investment A has a negative NPV of -$1,000 at a discount rate of 5%. What do we know about the IRR? Is this a good investment?

10. A timberland investment pays $10,000 annually for 20 years and $250,000 at year 21. You desire a 10% rate of return. How much should you pay for the investment?

Solutions. 1. 9.49%. **2.** Use Formula 2.3 to calculate 12.25%. **3.** We know at 10% the B/C ratio must equal 1 and NPV = 0. Since a lower interest rate is being used, we expect NPV to increase or become positive. That means the present value of revenues exceeds the present value of costs and we expect the B/C ratio to be greater than 1. **4.** If B/C = 1, the present value of revenues must equal the present value of costs and this means NPV = 0 and IRR = the discount rate of 8%.

5. Many financial calculators will solve this directly as 5. 2166%. N = 30, PMT = $10,000, PV = $150,000, CPT %i. You are equating $100,000 to the present value of a terminating annual cash flow series of $10,000 for 30 years and just solving for i.

6. Note that $100,000 = $30,000 / [(1 + i)^3 – 1] and you can solve for i. i = 9.13929%.

7. i = $10,000 / $100,000 = 0.10 = 10%.

8. Assuming equal risk, you'd think the investor would not care which investment his or her money ended up in. But scale is not apparent with any of these criteria. Maybe Investment A is a $100 investment and Investment B is a $100,000 investment. There is not enough information to tell which investment would be better for a particular investor.

9. We know the IRR is less than 5%. We don't know the investor's alternative rate of return, so we have no way of knowing if this is a "good" investment.

10. Simply discount the two cash flows at 10% and pay that much for NPV = 0. The terminating annual series is worth $85,135.64 and the $250,000 is worth $33,782.64, so the investor should pay $118,918.28 to earn 10%.

Chapter 4
Land Expectation Value Problems

1. Consider the management regime for Problem 1 under EAI and B/C ratio. What is the LEV for that regime at 6% interest?
2. In problem 1, what if the annual costs double to \$4.00/year? How is LEV affected?
3. Consider the simple management regime in problem 4 under Net Present Value. Calculate LEV using "net future value" and explain why you got the same answer as in the original problem.
4. For an even-aged forest I expect annual harvest revenue of \$4,200 per acre at age 35 and regeneration costs of \$200 per acre today and every 35 years thereafter. Annual management costs are \$5.00/ac./yr. What is LEV at 4% interest?
5. Obtain the same result as in Problem 4 using only Formulas 3.5 and 3.6.
6. Consider a fully-regulated forest that produces \$40,000 of net timber revenue on an annual basis and has net annual revenues from other sources of \$10,000 per year. What is LEV at 3 ½% interest?
7. Consider a fully-regulated forest that produces \$40,000 of net timber revenue every three years and has net annual revenue from other sources of \$10,000 per year. What is LEV at 3 ½% interest?
8. The LEV of a tract of timber is \$750/ac. at 4% interest. A farmer notes he can grow annual crops on the tract and earn \$50/ac./yr. At 4% interest which is the better option?
9. A tract has an LEV of \$750/ac. at 4% interest. A farmer asks what that is equivalent to on an annual basis. What is EAI at 4%?
10. Consider Example 4.7 on page 4.20. Calculate NPV for that problem. Note that since net future value = $NPV(1.09)^{35}$ we can use NPV to directly calculate net future value. Actually another way to picture this problem is NPV occurring at years 0, 35, 70, 105, ... Show that this value is equal to LEV. Calculate EAI using NPV and LEV. Why are they the same?

Solutions

1. First, notice buying land and selling land is in that regime. That is not part of an LEV calculation. The "net future value" of the other items is \$4,117.01 and LEV is calculated as \$1,159.92.
2. Many LEV formulas calculate a net future value without the annual cost or revenue and add it into the value after dividing by the interest rate, since it is a perpetual annual series. In this case it is an extra \$2.00 per year in cost and \$2 / 0.06 = \$33.33
So LEV is reduced by \$33.33.
3. Net future value of the three cash flows is \$4,271.58 and using the LEV formula that is an LEV of \$3,316.63. The original problem had three perpetual cash flows in it and that is equivalent to an LEV calculation.
4. Net future value = \$3,042.52 and LEV = \$1,032.73.
5. The present value of the regeneration today is \$200. Then every 35 years forever the net harvest revenue (after regeneration) will be \$4,000. Using Formula 3.6 this present value is \$1,357.73. The annual cost has a present value of \$125. Then net present value can be calculated as \$1,357.73 -\$200 -\$125 = 1,032.73.
6. Technically this does not meet the classic definition of LEV. But an equivalent LEV value can be calculated. This is the situation on page 7.7. There is a net perpetual annual revenue of \$50,000. This has a present value of \$50,000/ 0.035 = \$1,428,571.40.
7. This is similar to the last problem.
The \$40,000 cash flow series has a present value of \$367,924.78 and the \$10,000 cash flow series has a present value of \$285,714.29. The combined net present value is \$653,639.07.

8. Note the relationship between EAI and LEV on page 4.22. If we convert EAI to its perpetual present value equivalent, we see annual crops has a value of $1,250. He'd be better off with the agricultural crops.
9. EAI = $750(0.04) = $30/ac/.yr.
10. NPV = $351.78. Note that $\$351.78(1.09)^{35} = \$7,181.23$ (same answer, but rounding error). Using NPV directly, $LEV = \$351.78 + \$351.78 / [(1.09)^{35} - 1] = \$351.78 + \$18.12 = \$369.90$. EAI for the 35-year rotation, using the equation on page 4.4, equals $33.29. EAI for infinite rotations, using the equation on page 4.22, equals $369.90(0.09) = $33.29.

What Does NPV Really Mean?

Consider Example 4.1 on page 4.3. The net present value was $1,388.07 at 6% interest. What does this really mean? It means that this investment will earn 6% plus an additional $1,338.07 in present value terms. OK, what does that mean?

What is the payment that equates $4,500 today to ten equal annual payments at 5% interest? Using the installment payment formula we find that it is $611.41. What is the net present value in Example 4.1 if you use a $611.41 payment? It is zero. That means you'd earn exactly 6% (and that is IRR also). The extra payment in the original problem is $800.00 - $611.41 = $188.59. What is the present value of ten annual payments of $188.59 at 6% interest? It is $1,388.04 or the same as the net present value in the original problem (ignoring rounding error).

Another way to look at it is if you borrowed $4,500 and paid it back over ten years at 6%. Your annual payment would be $611.41. But in this case you pay an extra 188.59 per year.

At the end the future value of ten annual payments of $188.59 at 6% interest is $2,485.77. You have nearly $2,500 extra dollars in the bank. What is the present value of this $2,485.77? It is $1,388.04.

If you put $4,500 in the bank at 6% annual interest, how much will be in the account in ten years. Answer is $8,058.81. If you put ten annual payments of $611.41 in the bank at 6% interest, how much will be in the account in ten years? Answer is 8,058.87. Difference is due to rounding.

What is the IRR in this problem? IRR = 12.1%. At 12.1% interest the NPV = 0 (again, approximately due to rounding).

Land Expectation Value Problem

One of the FORVAL manuals contains a problem that calculates financial criteria values for a 27-year rotation. There are two costs: $160 for establishment at year 0 and an annual management cost of $2.50 per year. There are three revenues: thinning at age 16 of $97.50, thinning at age 22 of $156, and final harvest revenue of $1,287 at age 27. The problem uses a 4% discount rate. Financial criteria values are calculated as:

NPV = $363.41
ROR = 8.71%
EAI = $22.25
B/C Ratio = 2.8
LEV = $556.37

While LEV has quite specific assumptions, note that it is nothing more than the net present value of bare land in perpetual timber production. So if each cost and revenue is capitalized as a perpetual series and they are summed, LEV will be calculated. Many quasi-LEV calculations appear in the literatures that recognize this fact. Note that the five cash flows in the rotation above can be summed to equal LEV.

The establishment cost occurs every 27 years forever. The present value of $160 as a perpetual periodic cash flow series every 27 years at 4% interest is $84.95. However, one additional cash flow of $160 occurs at year 0 and is not included in the calculation. Since this cash flow occurs at year 0 it can be added to the $84.95 to equal a $244.95 cost contribution to LEV. The other cost is the $2.50 annual cost. This is a perpetual annual cash flow series and represents a cost contribution of $62.50 to LEV.

The thinning at year 16 occurs every 27 years forever. That is a perpetual periodic cash flow series worth $51.77. However, the first payment occurs at year 16 rather than year 27, or 11 years "too early." So the $51.77 must be compounded for 11 years at 4% to equal a revenue contribution of $79.70 to LEV. The second thinning at year 22 occurs every 27 years forever. The first payment occurs 5 years "too early." So $82.83 must be compounded for 5 years at 4% to equal a revenue contribution of $100.78 to LEV.

The final harvest at year 27 occurs every 27 years forever and needs no "adjustment." It represents a revenue contribution of $683.35 to LEV. The three revenues add up to $863.83 and the two costs add up to $307.45. LEV then equals $863.83 - $307.45 = $556.38.

Also note that EAI = $22.25. This is annual income that occurs on a perpetual basis. So if EAI is divided by the discount rate, the results will be LEV. So, $22.25/0.04 = $556.25. There is a small rounding error since EAI is only carried to two places.

FORVAL Exercise

FORVAL (FORest VALuation) is a timber and timberland valuation software package developed at Mississippi State University. It is available on-line at www.cfr.msstate.edu/forval or can be downloaded at fwrc.msstate.edu/software.asp. There is a manual at the download site. However, the program is set up for simple use and a manual should not be necessary.

1. Note that four options are available: (1) Financial Criteria, (2) Monthly or Annual Payments, (3) Precommercial Timber Value, and (4) Projected Stumpage Price. We will cover the first two options today and leave the other two for later.

2. First, click on Financial Criteria. Notice a menu. First you choose a type of calculation. Notice all the ones we've covered so far are there, plus a couple of others. We'll use **net present value** first, as it does all the compounding and discounting that involved all the formulas we've learned so far.

3. Second, notice there are four kinds of Cost/Revenue Types: (1) Single Sum, (2) Terminating Annual, (3) Perpetual Annual, and (4) Perpetual Periodic. These are the exact same type of cash flow series as we've discussed in going over the formulas.

4. Third, notice you have to enter two or three numbers directly. You enter the Cost/Revenue Amount. Enter it without a sign and to two places. Then enter where the cost or revenue occurs or begins. This is a year. Since a single sum, perpetual annual, and perpetual periodic don't have an end, **only for the terminating annual series,** enter the year it ends.

5. Then click "Add Cost" or "Add Revenue" and repeat the process if necessary. When you've entered all cash flows, click on "Calculate." Note the program will ask you for the interest rate. Enter it as a whole number.

FORVAL Sample Problems

I. What is the present value of $1,000 received in ten years at 6% interest? Use net present value and single sum, amount of revenue is 1000 and year revenue occurs is 10. Answer is $558.39.

II. What is future value of $1,000 received today in ten years at 6% interest? Use future value and single sum, amount of revenue is 1000 and year revenue occurs is 0. Note that after the program asks for interest rate, it will ask for ending year. Answer is $1,790.84.

III. What is the present value of a series of 40 annual payments of $1,000, the first payment occurring in one year? Use net present value and terminating annual, enter 1000 for amount, beginning at 1 and ending at 40. Answer is $15,046.29.

IV. Problem 1 on page 19. What is the present value of 40 annual payments of $1,000, the first payment occurring today? The interest rate is 6%. Use net present value and terminating annual, amount is 1000, beginning is 0, and ending is 39. Answer is $15,949.07.

V. Problem 2 on page 19. What is the future value of 40 annual payments of $1,000, the first payment occurring today? The interest rate is 6%. Use future value and terminating annual and enter rest as in problem IV. Answer is $154,761.96.

VI. Problem 9 on page 19. What is the present value of a series of 40 annual payments of $1,000, the first payment occurring at year 11? The interest rate is 6%. Use net present value and terminating annual and enter 11 for beginning and 50 for end. Answer is $8,401.77.

VII. Problem 10 on page 19. Solve the last problem for future value. Use future value and terminating annual series and enter 11 for beginning and 50 for end. Answer is $154,761.96.

VIII. Problem 11 on page 19. What is the future value at year 100 of a series of 40 annual payments of $1,000 at 6% interest, the first payment occurring one year from now and the last payment occurring at year 40? That is, after the 40^{th} payment the amount sits in the account untouched for 60 additional years earning interest. Answer is $5,105,239.87. Use future value and terminating annual, enter 1000 for amount, 1 for beginning, and 40 for end. It will prompt you for interest rate, enter 6, then prompt you for rotation length, enter 100.

IX. What is the present value of a perpetual annual series of $1,000 payments at 6 percent interest? The first payment occurs at year 1. Answer is $16,666.66.

X. What is the present value of a perpetual periodic series of $1,000 payments every 3 years at 6% interest, the first payment at year 3? Answer is $5,235.16

Note that the two periodic formulas are **set at year 1.** That is, the model won't allow you to start elsewhere. So it only works for the standard assumptions. That means I can't ask you to solve problems IX and X starting at year 0 or some other year besides year 1.

6. Now let's try the Monthly or Annual Payments option. Note that there are two options and they are the ones we discussed in class: repay a loan (installment payments) and accumulate a future sum (sinking fund). There are two options for period of payment: monthly and annual.

XI. What is the monthly payment to repay a loan of $35,000 over 4 years at 3.25% interest? Answer is $778.57.

XII. What is the annual payment to repay a loan of $35,000 over 4 years at 3.25% interest? Answer is $$9,472.30.

XIII. What is the monthly payment to accumulate $35,000 over 4 years at 3.25% interest? Answer is $683.78.

XIV. What is the annual payment to accumulate $35,000 over 4 years at 3.25% interest? Answer is $$8,334.80.

XV. Recall problem 10 on page 17. It asked for an amortization schedule for a $100,000 loan to be repaid over 10 years at 10% interest. You have the answer for that problem in your notes. Notice you can put that problem into FORVAL and check the amortization schedule box to get an amortization schedule for any problem.

Let's go back to financial criteria. Note that you can solve for all the standard financial criteria. Recall the first simple problem we discussed in class on Page 4.3 of your text, Example 4.1.

XVI. For Example 4.1 what is NPV, IRR, EAE, and B/C ratio? Recall this problem had a cost of $4,500 at year 0 for putting in food plots and the hunting club would pay $800 per year more on its lease for 10 years. The interest rate was 6%. So there are one cost and one revenue. Enter the cost as a single sum and the revenue at a terminating annual series. The NPV is $1,388.06 and the IRR is 12.1%. Use "all of the above" to get all four criteria.

XVII. Let's look at one last problem. Example 4.2 on page 4.5. It was a 30-year timber rotation with regeneration costs of $150 at year 0, thinning revenue of $500 at year 16, and final harvest revenue of 2,900 at year 30. Enter all three as single sums. Answers are IRR = 12.83%, NPV = $400.33, EAI = $32.26, and B/C ratio = 3.66.

XVIII. Solve Example 4.2, but add one more cost of annual property taxes of $5. Now what is NPV? NPV decreases to $338.28.

Second FORVAL Exercise

1. FORVAL On-line is at www.cfr.msstate.edu/forval
Download at www.cfr.msstate.edu/fwrc/products/software/forval.htm
Use FORVAL to solve Example 4.4. (1) Select Financial Criteria, (2) Select Rate of Return, (3) Enter Cost # 1, 100000 at year 0 as a single sum, (4) Enter Cost # 2, 5000 at year 5 as a single sum, (5) Enter Revenue # 1, 50000 at year 10 as a single sum, (6) Enter Revenue # 2, 140000 at year 12 as a single sum. Press Calculate.

2. Use FORVAL to solve Example 4.1. (1) Select Financial Criteria (2) Select All of the Above (3) Enter Cost # 1 as a single sum of 4500 at year 0 (4) Enter Revenue as a Terminating Annual Series starting at year 1 and ending at year 10 (5) Press Calculate (6) Enter 6 for interest rate.

3. Use FORVAL to solve Example 4.7. (1) Select Financial Criteria, (2) Select Land Expectation Value (3) Enter Cost # 1 of 95 at year 0 (4) Enter Cost #2 of 4 as a

Terminating Annual Series starting at year 1 and ending year 35 (5) Enter Revenue # 1 as a single sum of 550 at year 15 (6) Enter Revenue #2 as a single sum of 1500 at year 25 (7) Enter Revenue # 3 as a single sum of 3350 at year 35 (8) Press Calculate (9) Enter interest rate as 9 (10) Enter rotation length as 35.

4. Note that the NPV of the rotation described in Example 4.7 is $351.78. Perform the following calculation: (1) Select Financial Criteria (2) Select All of the Above (3) Enter 351.78 as a single sum Revenue at year 0 (4) Enter 351.78 as a perpetual periodic Revenue occurring every 35 years. (5) Press Calculate (6) Enter 9 as the interest rate. Note that the result is the same as for Problem 3. How could that be? (Ignore any small rounding errors.) Also the rate of return came out 399.99%, what happened there?

5. Use FORVAL to solve Problem 3.15. Select Monthly or Annual Payments. Select Accumulate a Future Sum and Monthly. Enter 75000, 12 and 7.5

6. Use FORVAL to solve Example 3.8. Select Monthly or Annual Payments. Select Repay a Loan and Annual. Enter 19000, 10, and 9

7. Use FORVAL to solve Example 3.9. Select Monthly or Annual Payments. Select Repay a Loan and Monthly. Enter 19000, 10, and 9

8. Assume the price of timber is $300/MBF and the price is increasing 1.33% per year. Use FORVAL to determine the price in 20 years. Select Projected Stumpage Price. Enter 300, 1.33 and 20. Projected price is $390.73.

9. Use FORVAL to solve the Valuation of Immature Timber Problem on pages 7.16 to 7.21 of your text. Select Precommercial Timber Value. Enter 150 as a single sum cost at year 0 and *2550* as single sum revenue at year 25. Note that the program will take any sort of cost or revenue. Press Calculate. Enter current age of stand (12). Enter stands rotation age (25). Enter the beginning value of land (400). Enter the ending value of land (400). Value comes up as $1231.69 for land and timber and with land valued at $400, timber is worth $831.69. *This is a unique program that will come in handy on the job.*

10. Use FORVAL to solve Problem 4.7

Inflation Example Using Example 6.1 from Text.

Initial cost = \$450/ac.
Thinning revenue = \$950/ac. in year 20.
Final harvest and land sale revenue = \$3,800.00/ac. in year 30.
Discount rate = 4% and inflation rate = 3%.
Then i = (1.04)(1.03) – 1 = 0.0712 = 7.12%.

In addition to the original data, now assume land and timber experience a 2% annual price appreciation. Now what is NPV?

Current dollar solution:
Prices appreciate by (1.03)(1.02) – 1 = 0.0506 = 5.06%.
Discount rate = 7.12%.
Thinning revenue = \$950$(1.0506)^{20}$ = \$2,549.60.
Final revenue = \$3,800$(1.0506)^{30}$ = \$16,707.30.
NPV = \$2,549.60/$(1.0712)^{20}$ + \$16,707.30/$(1.0712)^{30}$ - \$450 = \$2,316.47.

Constant dollar solution:
Thinning revenue = \$950$(1.02)^{20}$ = \$1,411.65.
Final revenue = \$3,800.00$(1.02)^{30}$ = \$6,883.30.
NPV + \$1,411.65/$(1.04)^{20}$ + \$6,883.17/$(1.04)^{30}$ - \$450 = \$2,316.47.

Simple Example

Pulpwood is worth \$30.00 per cord today.
Inflation = 3%.
Real interest rate = 4%.
i = 0.04 + .03 + (0.03(0.04) = 0.0712 = 7.12%.

In current dollars what will pulpwood be worth in 10 years? \$30.00$(1.03)^{10}$ = \$40.32.

So in current dollars,
PV = \$40.32/$(1.0712)^{10}$ = \$20.27.
And in constant dollars, PV = \$30.00/$(1.04)^{10}$ = \$20.27.

Same problem, but assume price appreciation of 1% annually.
In current dollars the price becomes:
\$30.00$[(1.03)(1.01]^{10}$ = \$44.53.
In constant dollars the price becomes \$30.00$(1.01)^{10}$ = \$33.14.
So, in current dollars PV = \$44.53/$(1.0712)^{10}$ = \$22.38.

And in constant dollars PV + \$33.14/$(1.04)^{10}$ = \$22.38.

Timber Tax Accounting
Comprehensive Problems

Problem 1. The following transactions take place on a cumulative consecutive basis. What are the basis, depletion rate, and depletion allowance at each point you sell timber?

1. Purchase land, timber, building, and forestry equipment for $480,000. Acquisition costs were $20,000. Land was 100 acres and appraised at $500/acre. The building and equipment had a fair market value (FMV) of $49,000 each. There are 8 MBF/acre on 90 acres of the land. Pine sawtimber sells for $666.66/MBF. There is also a 10-acre precommercial plantation worth $200/acre.

2. Sell equipment for $50,000. Qualifies for capital gains treatment. Capital gains rate equals 20%. How much tax is due on the sale?

3. Building has a remaining life of 37.5 years. Take first year straight-line depreciation of $38,900 / 37.5 = $1,037.33.

4. Sell the building for $40,000 (to be moved). How much tax is due?

5. You clearcut 45 acres that contain 480 MBF of 800 total MBF and sell for $500/MBF.

6. You plant the 45 acres at $200/acre.

7. You prescribe burn the remaining 55 acres at $4/acre.

8. It is four years after you plant the 45 acres and you apply herbicide at a cost of $60/acre.

9. You sell all the remaining commercial timber (350 MBF) at $600/MBF and use some of the proceeds to replant 45 acres at $200/acre.

10. The young growth becomes merchantable.

11. The first 45 acre plantation becomes merchantable.

12. The timber is thinned and 6 cords of 30 cords/acre are removed and sold for $20/cord.

13. The land and timber are sold for $100,000.

Problem 2. You purchase a 1,000 acre forested tract for $900,000 (including costs for legal fees, surveying, appraisal, and a timber cruise). The tract has a house on it that appraised for $100,000, outbuildings that appraised for $20,000, equipment that appraised for $10,000, and a road system with bridges that appraised for $70,000. The timber cruise established that there is 3,500 MBF of timber on the tract and a 5-year old plantation. The plantation cost $80 an acre to establish and the timber is worth $250/MBF. The forester appraised the plantation at $120/ac. for the entire 50 acres. Bare land appraised at $100/ac.

1. Determine the basis of each capital account.

2. One year later you sell 1,800 MBF. The total inventory was 3,600 MBF. You receive $260/MBF for the timber. Your capital gains rate is 20%. How much tax is due?

3. Now you plant 100 acres of pine at a cost of $100/ac. How are the accounts affected?

4. The next year you sell the house for $50,000 (to be removed from the tract). You depreciated it over 20 years using straight-lone depreciation. How much tax is due on the sale?

5. The plantation and young growth becomes merchantable. How does this affect the capital accounts?

6. You now have a second timber sale of 1,000 MBF. Current inventory is 2,000 MBF. Timber is selling for $300/MBF. What is the depletion allowance, depletion rate, taxable capital gain, and tax due? The capital gains rate is 20%.

7. You now sell the equipment. It has a basis of $1,000. How did the basis become $1,000? The selling price was $500. What are the tax consequences? You buy new equipment worth $50,000. How is the equipment account affected?
8. You finally sell all the remaining timber for $300,000. What is the depletion allowance and tax due if the capital gains rate is 20%?

Solutions

Problem 1

1.

Acccount	FMV	Proportion	Original Cost Basis
Land	$50,000	0.0793	$39,650
Timber	480,000	0.7619	380,950
Young Growth	2,000	0.0032	1,600
Building	49,000	0.0778	38,900
Equipment	49,000	0.0778	38,900
	$630,000	1.0000	$500,000

2. $50,000 – $38,900 = $11,100 taxable capital gain. Tax due is $2,220. The balance of the equipment account becomes $0.
3. This is depreciation expense. The building account balance becomes $37,862.67.
4. $40,000 - $37,862.67 = $2,137.33 taxable capital gains. Tax due is $427.47.
5. Depletion rate = $380,950 / 800 MBF = $476.1875/MBF. Depletion allowance = 480 MBF X $476.1875/MBF = $228,570. $240,000 - $228,570 = $11,430.
Tax due is $2,286. The balance of the timber account is now $380,950 - $228,570 = $152,380.
6. The plantation account now has a balance of $9,000.
7. This can be expensed in the current year.
8.This can be expensed in the current year.
9. Depletion rate = $152,380 / 350 MBF = $435.37/MBF. Depletion allowance = 350 MBF X $435.37 = $152,380. Tax due equals $210,000 - $152,380 = $57,620. Tax due is $11,524. Timber account has a balance of $0. The plantation account has an additional balance of $9,000 for the new plantation. So the plantation account now has a total balance of $18,000.
10. Young growth now has a balance of $0. Timber has a balance of $1,600 as the balance of the young growth account is transferred there.
11. The plantation account now has a balance of $9,000 as half its balance is transferred to the timber account. The timber account now has a balance of $10,600.
12. There is $10,600 in the timber account. The total timber volume is 1,650 cords. Depletion rate = $10,600 / 1,650 cords = $6.4242/cord. Depletion allowance = 330 cords X $6.4242/cord = $ 2,120. $6,600 - $2,120 = $4,480. Tax due is $896. The balance of the timber account is now $8,480.
13. Three accounts have balances: Timber = $8,480, Plantation = 9,000 and Land = $39,650. The basis of the assets sold is $57,130. $100,000 - $57,130 = $42,870 taxable capital gain. Tax due is $8,574. All accounts now have a basis of $0.

Problem 2

1.

Account	FMV	Proportion FMV	Original Cost Basis
House	$100,000	0.084674	$76,207
Outbuildings	20,000	0.0169348	15,241
Equipment	10,000	0.0084674	7,612
Road System	70,000	0.0592718	53,345
Land	100,000	0.084674	76,207
Timber	875,000	0.7408975	666,807
Young Growth	6,000	0.0050804	4,572
	1,181,000	1.0020659	$900,000

2. Depletion rate = $666,807/3,600/MBF = $185.22/MBF. Depletion Allowance = 1,800 MBF X $185.20/MBF = $333,403.50. Timber revenue = $468,000. Taxable capital gain = $468,000 - $333,403.50 = $134,596.50. Tax due = $134,596.50 X 0.20 = $26,919.30.
3. Plantation account now has a balance of $10,000.
4. You have depreciated the house for two years. The basis of the house is now $68,586.30. You have a capital loss of $18,586.30. No tax is due on the sale and the capital loss may be used to offset other capital gains.

5. The plantation and young growth amounts moves into the Timber Account. Timber has a balance of $347,975.50. The plantation and young growth accounts now have balances of zero.
6. Depletion rate = $347,975.50/2,000 MBF = $173.7018/MBF. Depletion allowance = 1,000 MBF X $173.98775/MBF = $173,987.75 Timber revenue = $300,000. Taxable capital gain = $300,000 - $173,987.75 = $126,012.25. Tax due = $1226,012.25 X 0.20 = $25,202.45.
7. You have a capital loss of $500. The equipment account now has a balance of $50,000.
8. Depletion allowance = $173,987.75. Tax due = 25,202.45.

Timber Tax Questions

1. You purchased a forested property. The fair market values (FMV) of the assets were precommercial timber $100,000, merchantable timber $800,000, and land $100,000. You paid $1,200,000 for the property. What is the original cost basis of the land?

a. $100,000
b. $120,000
c. $80,000
d. $0

2. The year after you purchase the tract in question 1 you plant 40 acres of it for $50,000. How does this affect the merchantable timber account?

a. It increases by 800,000/1,000,000 of $50,000 or $40,000.
b. It increases by $50,000.
c. It increases by 800,000/1,200,000 of $50,000 or $33,333.33.
d. It has no effect on merchantable timber.

3. Two years after you purchase the tract in question 1 the original precommercial timber (not the newly planted 40 acres) becomes merchantable. How does this affect the merchantable account?

a. It increases by $100,000.
b. It increases by 120,000
c. It increases by 80,000.
d. It has no effect.

4. Three years after you purchase the tract in question 1 you sell half of the timber for $1,500,000. Your capital gains rate was 28%. How much tax is due on the sale?

a. $420,000
b. $302,400
c. $268,800
d. $151,200

5. How much tax is due on the following timber sale?
Timber Revenue = 2,000 MBF X $175/MBF = $350,000.
Total merchantable timber volume, after sale = 6,000 MBF.
Adjusted basis, before sale = $400,000.
Capital gains tax rate = 28%.

a. $98,000
b. $112,000
c. $70,000
d. $42,000

6. You are a consulting forester and attend a professional short course to maintain your skills. The cost of course was $500 and it is a business expense. Your marginal tax rate is 28%. What was the after-tax cost of attending the short course?

a. $360
b. $500
c. $140
d. $280

7. The adjusted basis of the merchantable timber on a tract is $20,000. You harvest 40% of the timber and receive $20,000. Your tax rate is 28%. How much tax is due?

a. $3,360
b. $5,600
c. $2,240
d. $4,480

8. A landowner sells a stand of timber lump sum for $100,000. Sales expenses were 10%. His adjusted basis for the timber was $150,000 and he cut 20,000 of 100,000 cords of pulpwood. His tax rate was 28%. What was his profit on the sale, after taxes?

a. $100,000
b. $90,000
c. $60,000
d. $73,200

9. You purchase land and timber for $1,000,000. The fair market values of the assets are land = $200,000, merchantable timber =$500,000, and young growth = $100,000. Soon after the purchase one quarter of the young growth becomes merchantable. After that you have a timber inventory completed and find a timber inventory of 2,000 MBF. You harvest 1,000 MBF and sell it for $400,000. The ordinary income tax rate is 35% and the capital gains tax rate is 15%. (i) What is the depletion rate for the timber sale? (ii) What is the depletion allowance? (iii) What is the adjusted basis of the timber before the timber sale? (iv) What is the adjusted basis of the timber after the timber sale? (v) What is the after-tax cash flow resulting from the timber sale?

10. Continue the transactions of question 9. After the year 3 timber sale, you sell all the remaining assets for $676,875. The cash flows (before-tax) are given below.

Year	Item	Before-Tax Amount
0	Purchase	$1,000,000.00
1	Plant	5,000.00
3	Timber sale	400,000.00
4	Land and timber sale	676,875.00
1-4	Annual revenue	48,451.10

(i) What is the after-tax NPV of the investment at 5% interest? (ii) What is the after-tax IRR? (iii) What is the after-tax LEV or BLV at 5% interest? (iv) Is this a good investment? (v) Assume that land represented $250,000 of the purchase price and $250,000 of the sales price, what then was the before-tax land opportunity cost in this problem using a 5% interest rate? (vi) What is the after-tax land opportunity cost?

11. Assume you are in the 28% federal income tax bracket and in the 7% state income tax bracket. You are a forestry commission forester and earn $35,000 from that job. You take on a consulting job and earn $5,000 additional income. FICA and Medicare taxes are 15.3%. How much of the $5,000 do you get to keep after taxes?

12. You purchase timber for $50,000 in 2001 and sell it 2005 for $73,205. What is the after-tax rate of return over the four years if the capital gains rate is 15%?

13. What is the land expectation value (bare land value per acre for the following investment before taxes?
Reforest at the beginning of each rotation = $100.
Harvest every 25 years in perpetuity = $2,000.
Annual cost = $2/ac./yr.
Interest rate = 5%.

14. In question 13, with a 15% capital gains tax rate and a 25% ordinary income tax rate, what is the after-tax LEV?

Answers
1. b. **2.** d. **3.** b. **4.** c. **5.** c. **6. a**. **7.** a. **8.** d. **9.** (i) $328.125/MBF (ii) $328,125 (iii) $656,250 (iv) $328,125 (v) $389,218.75. **10.** (i) $0 (ii) 5% (iii) $0 (iv) It depends. It is a good investment if earning 5% is your goal. (v) $44,324.38 (vi) $44,324.38 **11.** $2,485. **12.** 8.67%. **13.** $656.19. **14.** $546.76.

Amortization of Reforestation Expenses

Prior to 2004 qualified taxpayers could elect to amortize up to $10,000 of qualified reforestation expenses over 84 months. In addition, a 10 percent tax credit could be claimed on the same reforestation expenses. This changed in 2004. The current law allows up to $10,000 of qualified reforestation costs to be expensed (deducted) against current year's income. In addition, qualified reforestation costs over $10,000 can be amortized over 84 months. There is a half-year convention that requires only one-half of regular amortization can be claimed the first year (this effectively makes 7-year amortization become 8-year amortization.

Example. You spend $25,000 in qualified reforestation expenses in November and December of this year. Your marginal ordinary income tax is 33% and your capital gains rate is 15%. What is the present value at 6% interest of the tax savings due to this provision? Without the provision you would use the capitalized $25,000 to reduce timber revenue and capital gains income at a year 25 harvest.

Without the provision taxable capital gains would be reduced by $25,000 in 25 years. This is a $3,750 tax reduction (calculated as 0.15 X $25,000). At 6% interest the present value of the tax reduction is $873.74.

With the tax provision the tax savings are calculated as follows. Note that because the tax savings occur very late in the year and the owner receives the benefits very early in the next year, we assume the first benefits are at year 0. If the owner had to wait a year for the benefits, we would use years 1 to 8 for the analysis.

Year	Deduction	Tax Savings	Present Value @ 6%
0	$10,000 + 1/14(15,000)	$11,071.43(0.33)	$3,653.57
1	(1/7)$15,000	$2,142.86(0.33)	667.12
2	(1/7)$15,000	$2,142.86(0.33)	629.35
3	(1/7)$15,000	$2,142.86(0.33)	593.73
4	(1/7)$15,000	$2,142.86(0.33)	560.12
5	(1/7)$15,000	$2,142.86(0,33)	528.42
6	(1/7)$15,000	$2,142.86(0.33)	498.51
7	(1/14)$15,000	$1,071.43(0.33)	235.15
			$7,365.97

Questions

1. A landowner incurs $12,000 of reforestation expenses this year. What is his/her allowable amount of amortization (deduction) for this year?

a. $ 857.14
b. $1,714.28
c. $10,000.00
d. $10,142.86

2. A landowner incurs $12,000 of reforestation expense this year. What is his/her allowable amount of amortization deduction next year?

a. $857.14
b. $142.85
c $285.71
d. $1,714.29

3. A landowner incurs $12,000 of reforestation expense this year. His/her marginal ordinary income tax rate is 33% and the capital gains rate is 15%. How much is his/her taxes due reduced this year?

a. $3,347.14
b. $3,394.29
c. $3,960.00
d. $1,800.00

4. A landowner has $20,000 of reforestation expense that he/she expects to incur about at year's end. He/she has asked the contractor to speed things up and complete the project before Christmas. Considering the potential to amortize these expenses, what advice might you give the landowner to maximize the tax savings?

a. Amortization allows you to deduct these expenses, so the quicker you incur them, the quicker you deduct them.
b. The time value of money says that any tax savings are best incurred sooner than later, so it is wise to speed up the reforestation.
c. The entire amount is deductible, so timing is not important.
d. Perhaps you might want to do have the reforestation this year and half early next year.

5. The prior tax law on amortization of reforestation expenses included a tax credit. If you are a taxpayer with a 33% marginal ordinary income tax rate and a 15% capital gains rate, which of the following is most valuable to you in terms of reducing your taxes?

a. $1,000 tax credit
b. $2,500 tax deduction
c. $5,000 reduction in capital gain
d. $2,000 tax deduction and a $2,000 reduction in capital gain

Answers

1. d
2. c
3. a
4. d
5. a (a is worth $1,000, b is worth $825, c is worth $750, and d is worth $960.

After-tax Investment Analysis Example

You purchase a forested tract for $350,000. Your marginal ordinary income tax rate is 33% and your capital gains rate is 15%. The accounts of the transaction are:

Account	Original Cost Basis
Land	$60,000
Timber	250,000
Young growth	40,000

A year after the transaction you sell 8,000 of 12,000 cords for $25/cd.
Six years after that you sell 2,000 of 5,000 cords for $30/cd.
The next year the young growth becomes merchantable.
The next year you sell all the assets for $180,000.

What is the before-tax net present value of the transactions above at 6% interest?

Year	Item	Amount	Present Value
0	Purchase	-$350,000	-$350,000
1	Timber Sale	+ 200,000	188,679
7	Timber Sale	+60,000	39,903
9	Total Sale	+180,000	106,542
			-$14,876

What is the after-tax net present value of the transaction above at 6% interest?

First, let's convert each cash flow from a before-tax basis to an after-tax basis. The purchase price was $350,000 before-tax. How much was it after-tax? How did it affect taxes at year zero? The owner capitalized the amount and obtained no tax reduction at the beginning. So, after-tax, the cash flow is unchanged. The initial timber sale does have tax consequences. Since 2/3 of the timber was cut, the depletion allowance is $166,667. You owe capital gains taxes on 33,333. Taxes due are $5,000. So the after-tax cash flow is $195,000. For the second timber sale the depletion allowance is $33,333, so you owe taxes on 26,667. Taxes due are $4,000. The after-tax cash flow is $56,000. For the final sale of land and timber the depletion allowance is $150,000, so you owe taxes on $30,000. Taxes due are $4.500. The after-tax cash flow is 175,500.

Year	Item	B/T Amount	A/T Amount	Present Value
0	Purchase	-$350,000	-$350,000	-$350,000
1	Timber Sale	+200,000	+195,000	+183,962
7	Timber Sale	+60,000	+56,000	+37,243
9	Total Sale	+180,000	75,500	+103,878
				-$24,917

After-Tax Analysis with Inflation

Consider the before tax inflation example. Note that the constant dollar approach is not appropriate for after-tax forestry investment analysis. This is because some of the costs are held in capital accounts and do not change with inflation. Taxes are paid in current dollars and the analysis must be in current dollars.

The simple example had one cost and two timber revenues. Assume that the original establishment cost is capitalized, the thinning volume is 20% of total volume, and the capital gain tax rate is 25%. What is the net present value after taxes, considering inflation?

Year	Item	Amount	Amount with 3% increase	After-tax cash flow	PV @8.15%
0	Establishment	-$160	-$160.00	-$160.00	-$160.00
15	Thinning	+350	545.29	419.97	129.66
23	Final Harvest	2,200	4,341.89	3,288.42	542.47
				After-tax NPV =	$512.13

Calculation of Year 15 After-tax Cash Flow.

Timber revenue	$549.29
Depletion Allowance	32.00
Taxable Capital Gain	$517.29
Tax Rate = 25%	X 0.25
Tax Due	$129.32

After-tax cash flow = $549.29 - $129.32 = $419.97

Calculation of Year 23 After-tax Cash Flow.

Timber Revenue	$4,341.89
Depletion Allowance	128.00
Taxable Capital Gain	4,213.89
Tax Rate = 25%	X 0.25
Tax Due	$1,053.47

After-tax cash flow = $4,341.89 - $1,053.47 = $3,288.42

What about annual costs? Those are a little tricky. Say annual cost is $5.00 per acre per year and increases at the inflation rate. Then it is $5.00 at year 1, $5.15 at Year 2, $5.30 at Year 3, … and $9.87 at Year 23. If the ordinary income tax rate is 33%, then each of those are converted into an after tax amount by the factor (1 – 0.33) = 0.67. Then the 8.15% interest rate is used to obtain a present value. The geometric cash flow series formula can be used to discount a series like this.

Problem:

A forest owner establishes a small pine plantation on a portion of his farm. Site preparation and planting cost $10,000 (both occurred this year). Assume establishment cost was in Year 1. He expects a thinning at year 14 to produce $9,000 of timber revenue and a final harvest at year 23 that yields $45,000. Capital gains tax rate is 10% and ordinary income is taxed at 28%. What is the net present value of the investment at 4% interest? All costs and revenues are in today's dollars and all are expected to increase at the inflation rate of 3%.

Year	Item	B/T Amount	A/T Amount	PV@7.12%
0	Establishment	-$10,000	-$7,200	-$7,200
14	Thinning	13,613	12,252	4,678
23	Harvest	88,811	79,930	16,432
				$13,910

Since all the capitalized costs were expensed at the beginning, there was no depletion allowance in this problem. Does that mean the constant dollar approach would work in this case?