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Simulation Study on Confidence Interval Estimation for Standard Deviation with Non-Normal Distributions

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SIMULATION STUDY ON CONFIDENCE INTERVAL ESTIMATION FOR STANDARD DEVIATION WITH NON-NORMAL DISTRIBUTIONS

by

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ABSTRACT

This study explores innovative approaches to constructing confidence intervals for the population standard deviation, σ, in non-normal data scenarios. While the sample standard deviation, **s**, is widely used, its reliability is compromised when dealing with skewed or heavy-tailed distributions and exhibits sensitivity to outliers. Our research addresses these limitations by investigating alternative estimation methods that offer greater robustness and accuracy.

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1 Introduction

Standard deviation is a measure of dispersion. In measuring the average deviation of each data point from the sample mean, the sample standard deviation provides valuable insights into the spread of a dataset, indicating how tightly or loosely the data points cluster around the average value. The larger the standard deviation, the more spread out the data points are from the mean, which is an indication that there is greater variability and less consistency. However, a smaller standard deviation suggests that the data points are more concentrated around the mean, which is an indication that there is greater consistency and less variability. The sample standard deviation is calculated as the square root of the sample variance. All data points are used in the sample variance calculation. Many other measures of dispersion do not use all the sample data. For example, the range only uses the maximum and minimum data points.

Standard deviation plays a crucial role in statistical inference and data analysis by providing a measure of the spread of data. In statistical inference, the standard deviation is used when conducting a hypothesis test or building confidence intervals as it provides a measure of uncertainty related to these inference methods. In an attempt to estimate a population parameter, the standard deviation provides a means to determine a margin of error for the estimate. Beyond statistical inference, standard deviation finds extensive application in data analysis and descriptive statistics. Standard deviations are used to assess the normality of distribution, compare the variability of different datasets, and detect outliers. Also, the standard deviation is used to help calculate probabilities, assess risk, and make informed decisions.

Understanding standard deviation is necessary for many fields, including statistics, finance, and engineering. Its role in multiple fields makes it relevant for making meaningful decisions and conclusions from data. For instance, in engineering the standard deviation is used to establish valid control limits for manufacturing processes. In finance, standard deviation provides a measure of an investment risk. Understanding standard deviation is crucial for anyone who works with data.

If a family of probability distributions is such that there is a parameter η (and other parameters $\boldsymbol{\theta}$) for which the cumulative distribution function satisfies:

$$F(x; \eta, \boldsymbol{\theta}) = F(x/\eta; 1, \boldsymbol{\theta}), \tag{1.1}$$

then η is called a scale parameter since its value determines the "scale" or statistical dispersion of the probability distribution, and an estimator of a scale parameter is often simply called an estimator of scale. Scale estimators are important in many statistical applications and the most common scale estimator is the sample standard deviation, $\mathbf{s}$, which for a random sample $x_1, ..., x_n$ is defined as $\mathbf{s} = \sqrt{\sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{(n-1)}}$, where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. The sample standard deviation provides a point estimate for the population standard deviation, σ. The sample standard deviation, $\mathbf{s}$, is not a resistant estimator as it is very sensitive to the presence of outliers. Also, $\mathbf{s}$ is not necessarily the most efficient estimator of scale in skewed and leptokurtic distributions, and, notably, it is not robust to slight deviations from normality [12]. Although $\mathbf{s}$ is very sensitive to outliers, it is considered an efficient estimator for estimating population standard deviation for a normal distribution. In addition, $\mathbf{s}$ is often used to construct a confidence interval for a population standard deviation, σ.

Point estimation is finding an approximate value for a population parameter. The sample standard deviation, $\mathbf{s}$, is an estimator for the population standard deviation. The single approximation is unlikely to be exactly equal to the population standard deviation, σ. Consequently, it is reasonable to build a range of possible values for the parameter, σ, known as an interval estimate. This gives us a better chance of capturing the actual value of σ. The most common forms of interval estimation are confidence intervals (a frequentist method) and credible intervals (a Bayesian method). Credible intervals are analogous to confidence intervals, however, they differ in philosophical basis.

A confidence interval (CI) is a range of values that has a positive probability of including the unknown parameter. This means that if random samples of the same sample size are taken repeatedly from the same distribution or population, and a confidence interval for a given parameter is produced for each random sample, then a certain proportion of these intervals are expected to contain unknown parameter.

An exact (100 - α)% confidence interval for σ is based on the assumption that the underlying distribution of the sample observations is normal. Suppose $Y_1, Y_2, ..., Y_n$ form a random sample from a normal distribution with mean, μ and variance, σ^2, that is, $Y_i \sim N(\mu, \sigma^2)$ for all i, then $\sum_{i=1}^{n} \frac{(Y_i - \bar{Y})^2}{\sigma^2} \sim \chi^2_{n-1}$, where $\bar{Y} = \sum_{i=1}^{n} \frac{Y_i}{n}$, follows a chi-square distribution with $n-1$ degrees of freedom; then the exact (100-α)% confidence interval for σ^2 is given as

$$(n-1)s^2/U < \sigma^2 < (n-1)s^2/L \tag{1.2}$$

where $s^2 = \sum_{i=1}^{n} \frac{(Y_i - \bar{Y})^2}{(n-1)}$, $L = \chi^2_{\alpha/2, n-1}$ and $U = \chi^2_{1-\alpha/2, n-1}$, and $\chi^2_{p,df}$ is the p^{th} percentile of a chi-square distribution with *df* degrees of freedom. To get the confidence interval for σ, take the square root of the endpoints of (1.2).

The exact CI (1.2) is hypersensitive to minor violations of normality assumption [2]. As stated earlier, the exact confidence interval is based on the assumption that the underlying distribution is normal; however, we do not always get to see situations where the data are normally distributed. So, the question becomes what can be done when we have cases where the observed samples are not from a normal distribution? When we have data from heavy tail distributions or skewed distributions, the exact (100 - α)% confidence interval for σ^2 does not perform well; hence, the need for alternatives to build (100 - α)% confidence intervals for σ^2 for such situations.

Robust methods are not overly sensitive to changes in distributions and are designed to deal with problems associated with skewed distributions and outliers. Some statistical literature shows that robust methods might give more meaningful measures of scale and are indeed more resistant to departures from normality and the presence of outliers than **s**. Such methods can provide alternatives to the exact (100 - α)% confidence intervals for σ^2 (1.2).

In this work, a simulation study evaluates several such alternative confidence interval estimates of scale parameter σ. The simulations assess these estimators when observations are obtained from a variety of heavy-tailed and skewed distributions, as well as the normal distribution. In addition to attempting to verify results in previous work, Bonett [2], and Ahmed Abu Shawiesh et al. [1]; some exploration of potential modifications to existing approaches will also be considered.

2 Methods

2.1 Introduction to Confidence Interval

An interval estimate of a parameter, θ, is any pair of functions $L(x_1, x_2, ..., x_n)$ and $U(x_1, x_2, ..., x_n)$ that satisfies $L(\mathbf{x}) < U(\mathbf{x})$ for all $\mathbf{x} = [x_1, ..., x_n]$. A confidence interval is an interval estimate of θ.

A confidence interval is a range of values with a positive probability (equal to a specified confidence coefficient) of including the unknown parameter to be estimated. The confidence coefficient is the overall capture rate if a specific confidence interval method is used repeatedly, or the method's success rate. Confidence intervals measure the degree of uncertainty in estimating a parameter based on a sample. Although the confidence interval provides an estimate of the parameter, the interval computed might not necessarily include the true value of the parameter. This is why confidence intervals are built with a confidence coefficient usually selected by the researcher.

For example, suppose a researcher chooses a confidence coefficient of 95%. In that case, it does not mean that for a given realized interval there is a 95% probability that the population parameter lies within the interval. It also does not mean that 95% of the sample data lies within the confidence interval. However, it implies that if the estimation process is repeated over and over with the same sample size from the same population, then approximately 95% of the calculated intervals contain the true value of the parameter. For a given confidence interval, the parameter it is attempting to bound, or capture, is either in the interval or not.

A two-sided confidence interval has two bounds called the lower and the upper bound, usually written as $L(\mathbf{x}), U(\mathbf{x})$. Confidence intervals can also be one-sided. A one-sided interval only has an upper or lower bound. For instance if the lower bound,

$L(\mathbf{x}) = -\infty$, then we have the one-sided interval $(-\infty, U(\mathbf{x}))$. A two-sided confidence interval provides a range of plausible values for the parameter.

When estimating a location parameter of a symmetric distribution, such as a population mean, μ, using the best point estimate along with a suitable margin of error provides a confidence interval based on a sample. The point estimate is the best guess for the true parameter based on the sample, and the margin of error defines a range around the point estimate within which the true parameter is expected to be with a specified level of confidence, that is, the confidence coefficient. When such a confidence interval is created, its width is twice the margin of error, a function of the point estimate's standard error. In confidence interval estimation, narrower widths are preferred because there is less uncertainty with narrower intervals.

When estimating the confidence interval for a scale parameter such as variance or standard deviation, a different approach is used. This is because the sampling distribution of the point estimate used in deriving the confidence interval for σ^2 or σ generally is not symmetric or bell-shaped like the sampling distribution of a point estimate used in deriving the confidence interval for a location parameter such as μ.

2.2 Introduction to the Exact Confidence Interval (CI) for σ^2

Suppose we have $Y_1, Y_2, ..., Y_n$ random observations from a normal distribution with mean, μ and variance, σ^2, that is, $Y_i \sim NID(\mu, \sigma^2)$ for all i, then, $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$, where, χ^2_{n-1} is a chi-square distribution with $n-1$ degrees of freedom. Since the parameter of interest is σ^2, the exact (100-α)% confidence interval for σ^2 is given as described in (1.2).

2.2.1 Derivation of the Exact Confidence Interval for σ^2

In conducting a statistical hypothesis test for σ^2 when the samples are normally distributed, the method used is called the chi-square test for variance, and the test statistic of this method is

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}. \tag{2.1}$$

In building the confidence interval, it is necessary to find $L(\mathbf{x})$ and $U(\mathbf{x})$ such that $P[L(\mathbf{x}) \leq \sigma^2 \leq U(\mathbf{x})] = 1-\alpha$, where $1-\alpha$ is the desired confidence coefficient, for $0 < \alpha < 1$. Using the test statistic (2.1) to build a 95% confidence interval for σ^2, we have under the assumption of normally distributed data with common mean, μ, and common variance, σ:

$$P\left(\chi^2_{.025,n-1} \leq \frac{(n-1)s^2}{\sigma^2} \leq \chi^2_{.975,n-1}\right) = .95, \tag{2.2}$$

where $\chi^2_{.025,n-1}$ and $\chi^2_{.975,n-1}$ are 0.025 and 0.975 quantiles from chi-squared distribution with $n-1$ degrees of freedom. The parameter of interest is σ^2. To be able to isolate σ^2 in the middle, inversion of the test statistic results in

$$P\left(\frac{1}{\chi^2_{.975,n-1}} \leq \frac{\sigma^2}{(n-1)s^2} \leq \frac{1}{\chi^2_{.025,n-1}}\right) = .95. \tag{2.3}$$

Isolating σ^2 to be in the middle of the inequality, we get

$$P\left(\frac{(n-1)s^2}{\chi^2_{.975,n-1}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{.025,n-1}}\right) = .95. \tag{2.4}$$

Taking the square root of the endpoints of (2.4) gives a $100(1-\alpha)\%$ CI for σ.

The chi-square distribution with n degrees of freedom (df) can be described as the sum of the squares of n independent standard normal random variables. It is a right-skewed distribution, that is, it has a longer tail towards the right side and the majority of the data points fall to the left side. Chi-square distributions are a family of distributions indexed by their degrees of freedom, df. When the degrees of freedom increase towards infinity, the chi-square distribution approaches a standard normal distribution (bell curve). The probability density function for a random variable X having a chi-squared distribution with k degrees of freedom can be expressed mathematically as:

$$f_X(x) = \frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}, \tag{2.5}$$

where $x > 0$, and $\Gamma(t) = \int_0^\infty v^{t-1}e^{-v}dv$. The key assumption that must be met before this exact confidence interval (2.2) can be used is that the sample observations are generated from a normal distribution.

Although the exact confidence interval for σ^2 is easy to construct, there are limitations. One of the limitations is that the method is sensitive to departure from the normality assumption. This means that when the samples are not normally distributed, it can lead to inaccurate coverage probabilities (that is, the interval will not perform as desired).

Example 2.1. You are a bakery owner and want to estimate the variation in the weight of your loaves of bread. You randomly bake and weigh 14 loaves. The results you obtained in pounds are 1,1.5,2,1,0.8,0.9,1,0.85,0.95,1,1.3,1.2,1.1,1.3. Assume that the weights of the loaves are normally distributed. Construct a 95% confidence interval for the population variance , σ^2, of the weights of your loaves of bread.

From (2.4) we know that we need to know s^2, $U = \chi^2_{1-\alpha/2,n-1}$ and $L = \chi^2_{\alpha/2,n-1}$. s^2 are computed from the sample. The values U and L are computed using computer software, a calculator, or a chi-square table. So, $s^2 = \sum_{i=1}^{n} \frac{(Y_i-\bar{Y})^2}{(n-1)} = \frac{1.297}{13} \approx .0998$. Adopting a confidence level of 95%, an $\alpha = .05$, computer software returns $L = \chi^2_{.025,13} = 5.0088$ and $U = \chi^2_{.975,13} = 24.7356$. Therefore, the 95% confidence interval is:

$$\frac{13(.0998)}{24.7356} \leq \sigma^2 \leq \frac{13(.0998)}{5.0088} \text{ or } 0.0524 \leq \sigma^2 \leq 0.2590. \tag{2.6}$$

Taking the square root of the endpoints of this interval above gives a confidence interval for σ.

$$\sqrt{\frac{13(.0998)}{24.7356}} \leq \sigma \leq \sqrt{\frac{13(.0998)}{5.0088}} \text{ or } 0.229 \leq \sigma \leq 0.509. \tag{2.7}$$

The confidence interval for σ is often more helpful than that for σ^2 because standard deviation has the scale or units of the data while the variance is on a squared units scale. The interpretation is that we are 95% confident the population standard deviation of the weights of loaves of bread lies between 0.229 and 0.509 pounds. If we repeat this process many times and calculate the confidence interval on each sample, we expect 95% of them to capture the true population standard deviation.

We can often get samples that are not normally distributed in the real world. As stated earlier, the exact CI can suffer when the data departs from the normality assumption, thus, in such cases, the interval produced with the exact CI can become inaccurate. For this reason, researchers and statisticians have proposed other methods that do not require the samples to be normally distributed. Some of these methods are discussed below.

2.3 Bonett's Approximate Confidence Interval For Standard Deviation of Nonnormal Distributions

Douglas G. Bonett's 2006 paper "Approximate Confidence Intervals for the Standard Deviation of Nonnormal Distributions" [2] proposes a method to estimate confidence intervals for the standard deviation when data deviates from a normal distribution. When applied to nonnormal data, the exact confidence interval of section 2.2 for standard deviation can be unreliable and inaccurate. Bonett's method addresses this issue by constructing an approximate confidence interval that is more robust to deviations from normality. The proposed interval is nearly exact under normality, has a coverage probability close to $1-\alpha$ under moderate nonnormality, has a coverage probability that approaches $1-\alpha$ as the sample size increases for nonnormal distributions with finite fourth moments, and finally, is not computationally intensive.

Bonett proposed that instead of assuming that the samples are normally distributed, let Y_i (i= 1,2,...,n) be continuous, independent, and identically distributed random variables with $0 < var(Y_i) = \sigma^2$, $E(Y_i) = \mu$ and a finite fourth moment, γ_4.

Given the desired properties of $\ln(\hat{\sigma}^2)$, such as improving the small-sample performance of Shoemaker's (2003) equal variance test, and Bartlett and Kendall (1946) showing that the sampling distribution of $\ln(\hat{\sigma}^2)$ converges to normality much faster than the sampling distribution of $\hat{\sigma}^2$ when $Y_i \sim N(\mu, \sigma^2)$; Bonett proposed a large-sample confidence interval for σ^2 from a reverse-transformed confidence interval for σ^2. The following $100(1-\alpha)\%$ confidence interval was proposed

$$\exp(\ln(c\hat{\sigma}^2) \pm z_{\alpha/2} se) \tag{2.8}$$

where z_p is the p^{th} percentile of the standard normal distribution, $se = c[\frac{\hat{\gamma_4}^* - \frac{n-3}{n}}{n-1}]^{1/2}$, $c = \frac{n}{n - z_{\alpha/2}}$ is an empirically determined, small-sample adjustment that helps equalize the tail probabilities, and $\hat{\gamma_4}^*$ is a pooled estimate of γ_4, which is defined as

$$\hat{\gamma_4}^* = (n_0\tilde{\gamma_4} + n\bar{\gamma_4})/(n_0 + n). \tag{2.9}$$

The value $\tilde{\gamma_4}$ could be a prior point estimate of γ_4 obtained from a previously obtained sample of size n_0, and $\bar{\gamma_4}$ is a proposed estimator of γ_4, which is asymptotically equivalent to Pearson's estimator and is defined as

$$\bar{\gamma_4} = n\sum(Y_i - m)^4/(\sum(Y_i - \hat{\mu})^2)^2, \tag{2.10}$$

where $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} y_i$, m is a trimmed mean with trim-proportion equal to $\frac{1}{2\sqrt{(n-4)}}$

so that m converges to μ as n increases without bound, and in such case, this proposed estimate becomes Pearson's estimator $\hat{\gamma}_4 = n \sum(Y_i - \mu)^4 / (\sum(Y_i - \hat{\mu})^2)^2$.

Bonett explained that Pearson's estimator, $\hat{\gamma}_4$, tends to have a large negative bias in leptokurtic (heavy-tailed) distributions unless the sample size is very large. Taking the square root of the endpoints of (2.8) gives a confidence interval for σ. Bonett also stated that simulations suggest that when $n_0 > n$, replacing $(n-3)/n$ with 1 and replacing $n-1$ with n in se improves the small-sample performance of (2.8); however, when no prior information is available $n_0 = 0$.

Constructing the Confidence Interval in Bonett's Method

Estimates of coverage probabilities and average interval widths of (1.2) and (2.8) were obtained using 50,000 Monte Carlo random samples of a given sample size from various distributions. Prior kurtosis information is not utilized in (2.8) for the simulation, that is, $n_0 = 0$.

The results suggest that (2.8) has a coverage probability close to $1 - \alpha$ when the observations are from a normal distribution with $n > 10$. Bonett's results suggest that (2.8) is slightly conservative in platykurtic distributions and slightly liberal in moderately leptokurtic distributions, and (2.8) improves as n increases. With highly nonnormal distributions the coverage probability of (2.8) was considerably less than $1-\alpha$ unless n is large. However, (1.2) is very conservative in platykurtic distributions, very liberal in leptokurtic distributions, and its coverage probability does not converge to $1 - \alpha$ as n increases.

Bonett explained that the performance of (2.8) depends on the degree of nonnormality of $\ln(\hat{\sigma}^2)$ and the bias of se. The bias of se can be reduced by prior kurtosis information. Also, increasing the sample size tends to improve the normality of $\ln(\hat{\sigma}^2)$. This highlights the importance of taking sufficiently large samples from a highly nonnormal distribution.

2.4 A Simulation Study on Some Confidence Intervals for Population Standard Deviation

In their paper titled "A Simulation Study on Some Confidence Intervals for Population Standard Deviation", Moustafa Omar Abu-Shawiesh et al., 2011 [1], used a robust estimator against outliers and proposed a robust method for estimating the population standard deviation specifically when the data are from skewed distributions and in the presence of outliers.

The sample standard deviation, **s**, is the most common scale estimator and provides a logical point estimate of the population standard deviation, σ. However, **s** is sensitive to the presence of outliers in the data. Furthermore, **s** is not the most efficient or meaningful estimator of scale in skewed and leptokurtic distributions, and **s** is not robust to departures from the normality assumption. This motivated them to look for a robust scale estimator, that has a closed form and is easy to compute as an alternative to **s**.

Rousseeuw and Croux, 1993 [10], proposed two robust estimators for scale, the S_n and Q_n estimators that can be used as initial or ancillary scale estimators in the same way as the median absolute deviation (MAD), but they are more efficient and not slanted towards symmetric distributions. Moustafa Omar Abu-Shawiesh et al., explained that the Rousseeuw-Croux estimator, Q_n, might be a more meaningful measure of variation and may be preferred to **s** because it has high efficiency (82%) at normal distributions, shares desirable robustness properties with the mean absolute deviation (MAD), and does not depend on symmetry.

2.4.1 Definition of MAD, S_n, and Q_n

Suppose $x_1, x_2, ..., x_n$ are random samples. Let $\tilde{x}$ denote the sample median, which is the middle-order statistic when we have odd sample sizes. When the sample size is even, the median is the average of the two middle-order statistics.

Median Absolute Deviation (MAD) Estimator

The median absolute deviation about the median (MAD) is a robust measure of the variability of a sample. It is the median of the absolute deviations from the data's median:

$$MAD = \text{median}\{|x_i - \tilde{x}|\} \tag{2.11}$$

Because MAD is a more robust estimator of scale than the sample variance or standard deviation, MAD works better with skewed or heavy-tailed distributions. The formula

$$\hat{\sigma} = b \cdot MAD \tag{2.12}$$

is used to make MAD a consistent estimator for the estimation of the standard deviation σ. The constant scale correction factor, b, depends on the distribution. In the case of Gaussian distributions, it has been shown that we need to set $b = 1.4826$.

S_n Estimator

The estimator S_n is defined as

$$S_n = c \cdot \text{median}_\text{i}\{\text{median}_\text{j}|x_i - x_j|\}, i \neq j \tag{2.13}$$

where c is again a correction factor for consistency. Rousseeuw and Croux [10] explained that $c's$ default value is 1.1926.

Q_n Estimator

The estimator Q_n is defined as

$$Q_n = d\{|x_i - x_j|; i < j\}_{(k)} \tag{2.14}$$

where d is again a correction factor for consistency and $k = \binom{h}{2} \approx \frac{\binom{n}{2}}{4}$, where $h = [\frac{n}{2}] + 1$ is roughly half the number of observations. In the case of Gaussian distributions, Rousseeuw and Croux [10] explained that $d = 2.2219$.

2.4.2 Proposed Confidence Interval

Moustafa Omar Abu-Shawiesh et al. [1], proposed a new robust confidence interval for estimating the population standard deviation σ. Suppose $x_1, x_2, ..., x_n$ are random observations from continuous, independent, and identically distributed random variable. The random variable T is defined as

$$T = \frac{d_n Q_n}{\sigma} \tag{2.15}$$

where $d_n Q_n$ is an unbiased estimator for σ, so that $E(T) = 1$ for normal distribution. Based on the work by Rousseeuw and Croux in 1993 [10], for larger values of n, the following asymptotic result can be used:

$$T = \frac{d_n Q_n}{\sigma} \sim N\left(1, \frac{1}{1.65n}\right). \tag{2.16}$$

The following approximation result can be obtained

$$\sigma T = d_n Q_n \sim N\left(\sigma, \frac{1}{1.65n}\sigma^2\right). \tag{2.17}$$

Therefore from (2.17), the authors obtained the following pivotal quantity:

$$\frac{d_n Q_n - \sigma}{\frac{1}{1.28\sqrt{n}}\sigma} \sim N(0, 1). \tag{2.18}$$

Now, using the above pivotal quantity, they derived a $100(1 - \alpha)\%$ robust confidence interval for σ as follows:

$$P\left(z_{\frac{\alpha}{2}} < \frac{d_nQ_n - \sigma}{\frac{1}{1.28\sqrt{n}}\sigma} < z_{1-\frac{\alpha}{2}}\right) = 1 - \alpha. \tag{2.19}$$

where $z_{\frac{\alpha}{2}}$ and $z_{1-\frac{\alpha}{2}}$ are the $(\frac{\alpha}{2})^{th}$ and $(1-\frac{\alpha}{2})^{th}$ percentiles of the standard normal distribution.

Note that (2.19) is equivalent to

$$P\left(\frac{z_{\frac{\alpha}{2}}}{1.28\sqrt{n}} + 1 < \frac{d_nQ_n}{\sigma} < \frac{z_{1-\frac{\alpha}{2}}}{1.28\sqrt{n}} + 1\right) = 1 - \alpha.$$

Isolating σ gives

$$P\left(\frac{1.28\sqrt{n}\cdot d_nQ_n}{z_{1-\frac{\alpha}{2}} + 1.28\sqrt{n}} < \sigma < \frac{1.28\sqrt{n}\cdot d_nQ_n}{z_{\frac{\alpha}{2}} + 1.28\sqrt{n}}\right) = 1 - \alpha.$$

Therefore, their $100(1-\alpha)\%$ robust confidence interval for σ is as follows:

$$\left(\frac{1.28\sqrt{n}\cdot d_nQ_n}{z_{1-\frac{\alpha}{2}} + 1.28\sqrt{n}}, \frac{1.28\sqrt{n}\cdot d_nQ_n}{z_{\frac{\alpha}{2}} + 1.28\sqrt{n}}\right). \tag{2.20}$$

Rousseeuw and Croux, 1993 [10], derived the unbiasing factor d_n so that d_nQ_n becomes an unbiased estimator of σ for the case of normal distribution. Values of d_n for $n < 10$ are provided in Table 2.1.

n	2	3	4	5	6	7	8	9
d_n	0.399	0.994	0.512	0.844	0.611	0.857	0.6969	0.872

Table 2.1: Unbiasing Factor (d_n) Values

For $n \geq 10$, d_n can be defined as

$$d_n = \begin{cases} \frac{n}{n+3.8}, & n \text{ even} \\ \frac{n}{n+1.4}, & n \text{ odd} \end{cases}$$

3 Simulation Studies

3.1 Simulation Framework

A simulation study was conducted to evaluate and compare the performance of various intervals due to the impracticality of theoretically comparing them. In such studies, artificial datasets are generated based on specified probability distributions, simulating real-world scenarios. These datasets mimic the characteristics and variability observed in actual data, allowing for a comprehensive evaluation of statistical methods. The simulation follows a structured flowchart designed to evaluate interval estimation methods systematically. Below is the flowchart of our simulation:

1. Choose distributions with features usually seen in real-world data.

2. Draw or simulate random samples from the selected distributions.

3. Construct a confidence interval with the simulated samples.

4. Evaluate the performance of the constructed interval per distribution by computing the proportion of times the parameter is within the interval.

Various probability distributions are chosen to represent both symmetric and skewed data scenarios. These distributions capture some of the real-world datasets' different characteristics and complexities. The distributions considered by Abu-Shawiesh et al. [1] are:

1. Normal distribution with mean 3 and standard deviation 1.

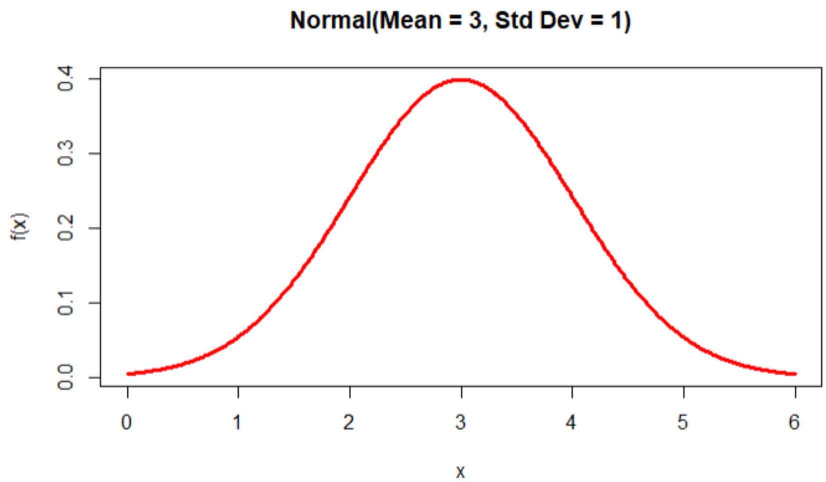


Figure 3.1: Normal Distribution

2. Chi-square distribution with one degree of freedom ($df = 1$).

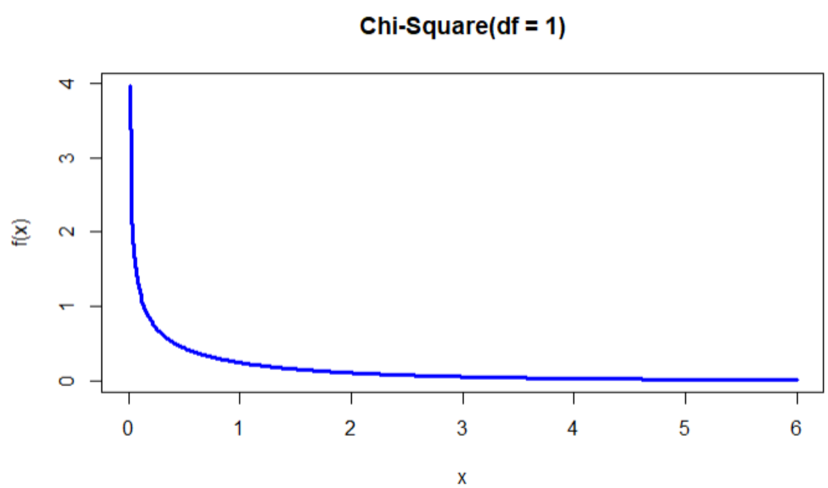


Figure 3.2: Chi-square Distribution

3. Lognormal distribution with mean 1 and standard deviation 0.80

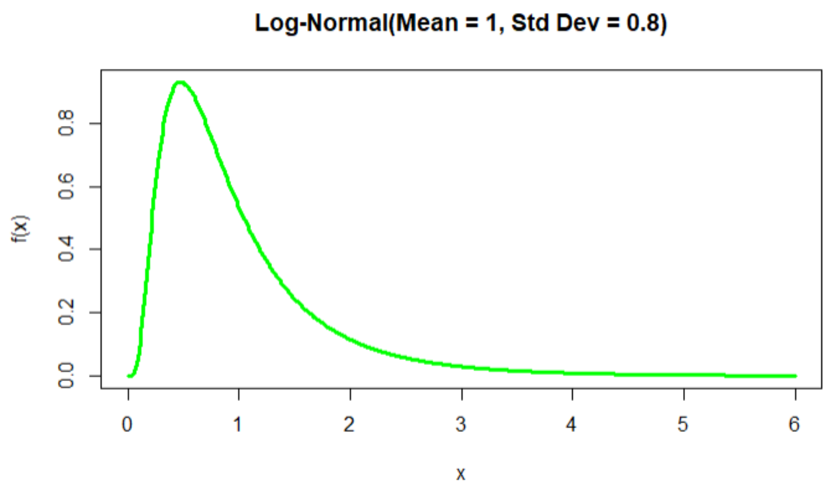


Figure 3.3: Lognormal Distribution

Random samples are generated from the selected distributions to simulate data reflecting the underlying population. These samples are drawn with specific sample sizes, enabling the emulation of diverse data collection scenarios and complexities. Sample sizes ranging from 5 to 100 were employed by Abu-Shawiesh et al. [1], covering a spectrum of data collection scales. This range encompasses smaller sample sizes (e.g., 5 and 10) typical in certain fields or studies with resource constraints, as well as larger sample sizes (e.g., 50, 70, and 100) often seen in well-funded research or large-scale surveys.

Interval estimation methods were applied to each generated sample. The evaluation of each method's performance relied on predefined criteria, including coverage probability, the average width of intervals, and the standard deviation width of intervals.

- **Coverage Probability**: This metric assesses the proportion of intervals that successfully contain the true parameter within their bounds. It is desirable for the coverage probability to be at least as large as the targeted coverage rate.

- **Average Width of Intervals**: The average width reflects the estimation precision achieved by the interval method. Smaller widths are more desirable.

- **Median Width of Intervals**: This measure provides insight into the central tendency of interval widths, offering a robust precision assessment. Again, smaller widths are preferred.

- **Standard Deviation Width of Intervals**: Capturing the variability in interval widths across multiple estimates, the standard deviation width portrays the spread of individual interval widths around the average width. This metric elucidates the consistency and reliability of the estimation method under consideration, and smaller values are desirable.

The median width of the intervals is an additional criterion not considered by Abu-Shawiesh et al. [1].

Our simulation study obtained estimates of the above quantities, with coverage probability as the most important of the metrics. These estimates were derived from 1000 simulation replications for each sample size, and for intervals involving bootstrap methods (Section 3.2.2), 1000 bootstrap samples for each sample were considered. The commonly used 95% confidence interval ($\alpha = 0.05$) was employed, where the confidence coefficient reflects the level of confidence in the estimation and is the targeted coverage rate.

It is widely acknowledged that in cases where data originate from a symmetric distribution or the sample size is large, the coverage probability for most methods closely approximates $1 - \alpha$. Moreover, a shorter interval width is indicative of a more precise confidence interval. When comparing methods with the same coverage probability, a smaller width suggests that the technique is better suited for the particular sample under consideration. The simulation programs were written and executed in R.

3.2 Methods

Given the sensitivity of the exact confidence interval method (1.2) to departures from normality, and the presence of outliers, this study evaluates some robust alternative approaches discussed in the literature for estimating confidence intervals for a population standard deviation σ when dealing with non-normal data distributions, including some bootstrap procedures.

3.2.1 Robust Alternative Approaches

Bonett's [2] interval as given in (2.8) and the method proposed by Abu-Shawiesh et al. [1] as given in (2.19) are considered robust alternatives to (1.2) for constructing an approximate confidence interval for σ.

3.2.2 Some Bootstrap Approaches

In addition to the exact confidence interval and the robust methods proposed by Bonett [2] and Abu-Shawiesh et al. [1], we also investigate some bootstrap-based approaches for constructing confidence intervals for the population standard deviation σ.

Let $X^*_{(i)} = X^*_{(1)}, X^*_{(2)}, ..., X^*_{(B)}$, be the i^{th} bootstrap sample, for $i = 1, 2, 3, ..., B$, and B is the number of bootstrap samples. Abu-Shawiesh et al. [1], investigated the following bootstrap confidence intervals for σ.

Nonparametric Bootstrap Confidence Interval

Calculate the sample standard deviation, $S^*_{(i)}$, $i = 1, 2, 3, ..., B$, for each bootstrap sample and then order the bootstrap standard deviations so that

$$S^*_{(1)} \leq S^*_{(2)} \leq S^*_{(3)} \leq ... \leq S^*_{(B)}$$

The $(1-\alpha)100\%$ non-parametric bootstrap (CI) for the population σ is given by

$$LCL = S^*_{(\alpha/2)} \text{ and } UCL = S^*_{(1-\alpha/2)} \tag{3.1}$$

where LCL and UCL are the lower and upper confidence bound respectively This method constructs the CI by taking the empirical $\alpha/2$ and $1-\alpha/2$ quantiles of the bootstrap standard deviations as the lower and upper limits, respectively. This is the common percentile bootstrap interval for σ.

Parametric Bootstrap Confidence Interval

This approach assumes normality but uses the bootstrap to estimate the quantiles of the chi-square distribution. The $(1-\alpha)100\%$ parametric bootstrap CI for the population σ is given by

$$LCL = S\sqrt{(n-1)/\chi^{*2}_{1-\alpha/2,(n-1)}} \text{ and } UCL = S\sqrt{(n-1)/\chi^{*2}_{\alpha/2,(n-1)}} \tag{3.2}$$

where $\chi^{*2}_{\alpha/2,(n-1)}$ and $\chi^{*2}_{1-\alpha/2,(n-1)}$ are the $(\frac{\alpha}{2})^{th}$ and $(1-\frac{\alpha}{2})^{th}$ sample quantiles of $\chi^{*2} = \frac{(n-1)s^{*2}}{n\hat{\sigma}^2_B}$, and $\hat{\sigma}_B = \sqrt{\frac{1}{B-1}\sum_{i=1}^{B}(\bar{x}_i{}^* - \bar{\bar{x}})^2}$, where $\bar{x}_i{}^*$ is the i^{th} bootstrap sample mean, $\bar{\bar{x}}$ is the overall bootstrap mean, s^{*2}_i is the i^{th} bootstrap sample variance and $\hat{\sigma}_B$ is the overall bootstrap standard deviation of the bootstrap means.

Robust Bootstrap Confidence Interval

The $(1-\alpha)100\%$ bootstrap CI for the population σ analogous to the robust estimator in (2.19) is given by

$$LCL = \frac{1.28\sqrt{n} \cdot d_n Q_n}{Z^*_{\alpha/2} + 1.28\sqrt{n}} \text{ and } UCL = \frac{1.28\sqrt{n} \cdot d_n Q_n}{Z^*_{1-\alpha/2} + 1.28\sqrt{n}} \tag{3.3}$$

where $Z^*_{\alpha/2}$ and $Z^*_{1-\alpha/2}$ are the $(\frac{\alpha}{2})^{th}$ and $(1-\frac{\alpha}{2})^{th}$ sample quantiles of the bootstrap test statistics $Z^*_i = \frac{\bar{x_i}^* - \bar{\bar{x}}}{\hat{\sigma}_B}$, with $\bar{x_i}^*$, $\bar{\bar{x}}$, and $\hat{\sigma}_B$ as defined above.

Cojbasic and Tomovic (CT) Confidence Interval

Another approach investigated was the nonparametric bootstrap confidence interval proposed by Cojbasic and Tomovic [5]. This method is based on the t-statistic and aims to construct a robust interval without making distributional assumptions about the data. The CT confidence interval is defined as:

$$I_{boot} = s^2 - \hat{t}^{(\alpha)}\sqrt{v\hat{a}r(s^2)} \tag{3.4}$$

where $s^2 = \frac{1}{n-1}\sum_i (x_i - \bar{x})^2$ is the sample variance, $\hat{t}^{(\alpha)}$ is the α percentile of T^* defined as $T^* = \frac{s_i^{2*} - s^2}{v\hat{a}r(s^{2*})}$, s_i^{2*} is the i^{th} bootstrap sample variance, $i = 1, 2, 3, ..., B$, and $v\hat{a}r(s^{2*})$ is a consistent estimator of the variance, defined as $\frac{2\hat{\sigma}_B^4}{n-1}$, $\hat{\sigma}_B$ defined in (3.2).

3.3 Simulation Results

Estimates of the coverage probabilities, average widths, median widths, and standard deviation (SD) widths were obtained using 1000 simulation replications for a given sample size from various distributions, and for methods involving bootstrap methods, 1000 bootstrap samples for each sample size were considered. The coverage probability is found by the sum of the total number of times the population standard deviation is found in the constructed intervals divided by the simulation size of 1000. The under and over coverage of a confidence interval is the fraction of 1000 samples that resulted in intervals that lie entirely below and entirely above the population standard deviation. The simulation result or the performance of each method is tabulated in Tables (3.1), (3.2), and (3.3) for normal, chi-square, and log-normal distributions respectively.

The results in Table (3.1) show that when sampling from a normal distribution, the exact method performs better than the other methods as expected. Also, Bonett's method performed well compared to the different methods. It can be noticed that for a small sample size, that is, $n = 5$, Bonett's method was conservative, that is, had higher coverage than the target. The average width of the exact method is shorter than all the other methods, again, as expected. Bonett's method is the only method that could compete with the exact method when sampling from a normal distribution. The results of this simulation support the findings and work presented by Cohen [4], confirming that no other confidence interval based on s is shorter than the exact interval (1.2). So, given that the samples or the data at our disposal are normally distributed, (1.2) should be used instead of considering other techniques.

		Sample Sizes						
Methods	Measuring Criteria	5	10	20	30	50	70	100
Exact	Coverage	0.964	0.965	0.954	0.949	0.937	0.949	0.946
	Under Coverage	0.018	0.020	0.021	0.029	0.028	0.031	0.024
	Over Coverage	0.018	0.015	0.025	0.022	0.035	0.020	0.030
	Mean Width	2.1267	1.0987	0.6956	0.5427	0.4079	0.34033	0.2833
	Median Width	2.0543	1.0835	0.6951	0.5453	0.4065	0.3406	0.2829
	SD Width	0.7471	0.2559	0.1103	0.07195	0.04273	0.0288	0.0205
Bonett	Coverage	0.993	0.968	0.948	0.949	0.933	0.943	0.946
	Under Coverage	0.006	0.019	0.025	0.033	0.031	0.033	0.026
	Over Coverage	0.001	0.013	0.027	0.018	0.036	0.024	0.028
	Mean Width	4.4779	1.3143	0.7303	0.5566	0.4176	0.3436	0.2858
	Median Width	3.8027	1.2317	0.7014	0.5392	0.4095	0.3376	0.2815
	SD Width	2.6289	0.4994	0.1928	0.1306	0.0755	0.0540	0.0392
Robust	Coverage	0.831	0.913	0.925	0.922	0.922	0.932	0.939
	Under Coverage	0.071	0.035	0.028	0.042	0.038	0.038	0.035
	Over Coverage	0.098	0.052	0.047	0.036	0.040	0.030	0.026
	Mean Width	2.5933	1.2529	0.7854	0.6054	0.4518	0.3767	0.3139
	Median Width	2.4201	1.2309	0.7832	0.6037	0.4487	0.3764	0.3136
	SD Width	1.3374	0.3611	0.1547	0.0938	0.0547	0.0363	0.0251
Non-Parametric Bootstrap	Coverage	0.642	0.779	0.843	0.907	0.904	0.916	0.938
	Under Coverage	0.358	0.221	0.149	0.089	0.088	0.072	0.056
	Over Coverage	0.000	0.000	0.008	0.004	0.008	0.012	0.006
	Mean Width	0.9458	0.7439	0.5489	0.4671	0.3735	0.3166	0.2682
	Median Width	0.9004	0.7174	0.5335	0.4540	0.3670	0.3125	0.2662
	SD Width	0.3711	0.2348	0.1334	0.0984	0.0698	0.0473	0.0350
Parametric Bootstrap	Coverage	0.885	0.899	0.915	0.945	0.928	0.942	0.931
	Under Coverage	0.029	0.033	0.034	0.026	0.037	0.030	0.037
	Over Coverage	0.086	0.068	0.051	0.029	0.035	0.028	0.032
	Mean Width	5.1377	1.1600	0.6497	0.5183	0.3960	0.3297	0.2753
	Median Width	2.4616	1.0274	0.6154	0.5006	0.3892	0.3245	0.2720
	SD Width	34.4886	0.5795	0.1875	0.1216	0.0785	0.0514	0.0371
Robust Bootstrap	Coverage	0.788	0.885	0.916	0.933	0.923	0.942	0.952
	Under Coverage	0.090	0.045	0.041	0.030	0.036	0.027	0.024
	Over Coverage	0.122	0.070	0.043	0.037	0.041	0.031	0.024
	Mean Width	2.5363	1.2445	0.7656	0.5994	0.4509	0.3773	0.3123
	Median Width	2.3288	1.2159	0.7616	0.5984	0.4511	0.3759	0.3122
	SD Width	1.4980	0.3935	0.1563	0.0910	0.0557	0.0370	0.0254
CT Bootstrap	Coverage	0.964	0.948	0.944	0.960	0.948	0.950	0.955
	Under Coverage	0.036	0.052	0.056	0.040	0.052	0.050	0.045
	Over Coverage	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Mean Width	59.4791	3.2817	1.9269	1.6657	1.4607	1.3652	1.2922
	Median Width	8.9154	2.7655	1.8316	1.6066	1.4408	1.3530	1.2762
	SD Width	541.8989	2.3585	0.6790	0.4412	0.3144	0.2365	0.1825

Table 3.1: Coverage Properties for N(3, 1)

The next simulation results in the performance of the methods on non-normal distributions. Tables (3.2) and (3.3) show the performance of the methods for chi-square and log-normal distributions respectively.

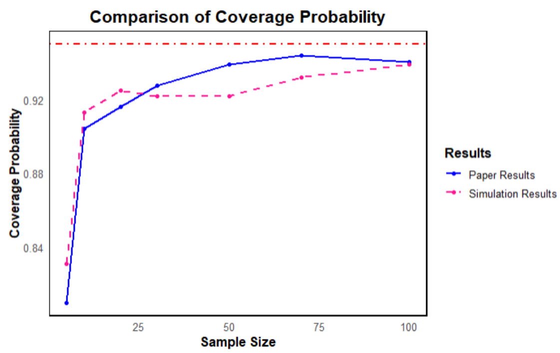
		Sample Sizes						
Methods	Measuring Criteria	5	10	20	30	50	70	100
Exact	Coverage	0.708	0.651	0.587	0.560	0.548	0.558	0.565
	Under Coverage	0.199	0.238	0.272	0.279	0.271	0.250	0.254
	Over Coverage	0.093	0.111	0.141	0.161	0.181	0.192	0.181
	Mean Width	2.6218	1.4370	0.9293	0.7442	0.5615	0.4789	0.3962
	Median Width	2.1508	1.3040	0.8800	0.7157	0.5484	0.4663	0.3893
	SD Width	1.8872	0.7443	0.3653	0.2404	0.1442	0.1082	0.0732
Bonett	Coverage	0.922	0.851	0.826	0.851	0.872	0.889	0.914
	Under Coverage	0.076	0.141	0.158	0.128	0.119	0.096	0.068
	Over Coverage	0.002	0.008	0.016	0.021	0.009	0.015	0.018
	Mean Width	8.9043	3.1929	1.9636	1.6118	1.2697	1.1303	0.9661
	Median Width	5.5727	2.4327	1.6315	1.3813	1.1032	1.0103	0.8834
	SD Width	8.9684	2.5638	1.3286	0.9577	0.6567	0.5520	0.4056
Robust	Coverage	0.561	0.397	0.127	0.032	0.002	0.000	0.000
	Under Coverage	0.410	0.594	0.873	0.968	0.998	1.000	1.000
	Over Coverage	0.029	0.009	0.000	0.000	0.000	0.000	0.000
	Mean Width	1.9007	0.8936	0.4849	0.3624	0.2622	0.2147	0.1721
	Median Width	1.4196	0.7885	0.4601	0.3490	0.2554	0.2107	0.1707
	SD Width	1.6677	0.5271	0.1998	0.1246	0.0716	0.0471	0.0314
Non-Parametric Bootstrap	Coverage	0.387	0.539	0.655	0.707	0.782	0.817	0.850
	Under Coverage	0.613	0.458	0.344	0.292	0.217	0.182	0.142
	Over Coverage	0.000	0.003	0.001	0.001	0.001	0.001	0.008
	Mean Width	1.2350	1.3242	1.2601	1.1368	0.9825	0.9248	0.8188
	Median Width	0.9665	1.1106	1.0884	1.0209	0.9071	0.8495	0.7558
	SD Width	0.9706	0.8581	0.7408	0.5940	0.4233	0.4025	0.3177
Parametric Bootstrap	Coverage	0.738	0.738	0.765	0.802	0.859	0.861	0.874
	Under Coverage	0.128	0.150	0.132	0.126	0.099	0.082	0.076
	Over Coverage	0.134	0.112	0.103	0.072	0.042	0.057	0.050
	Mean Width	19.8398	5.0251	2.4548	1.7726	1.2760	1.1277	0.9413
	Median Width	4.1655	2.6204	1.7145	1.3889	1.1011	0.9660	0.8464
	SD Width	59.9651	8.1556	2.4908	1.5849	0.7392	0.6313	0.4457
Robust Bootstrap	Coverage	0.457	0.316	0.114	0.027	0.000	0.000	0.000
	Under Coverage	0.511	0.672	0.886	0.973	1.000	1.000	1.000
	Over Coverage	0.032	0.012	0.000	0.000	0.000	0.000	0.000
	Mean Width	1.5342	0.7742	0.4488	0.3425	0.2459	0.2070	0.1708
	Median Width	1.0535	0.6761	0.4141	0.3243	0.2427	0.2045	0.1677
	SD Width	1.5596	0.5103	0.2010	0.1190	0.0658	0.0486	0.0330
CT Bootstrap	Coverage	0.887	0.876	0.907	0.928	0.956	0.975	0.983
	Under Coverage	0.113	0.124	0.093	0.072	0.044	0.025	0.017
	Over Coverage	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Mean Width	3672.5178	57.2263	14.0052	7.9025	5.1196	4.7083	3.9389
	Median Width	20.4974	8.4713	5.4765	4.6265	3.9086	3.6960	3.2994
	SD Width	32280.7581	301.6586	33.5974	13.3493	5.0467	3.9748	2.4160

Table 3.2: Coverage Properties for $\chi^2_{(1)}$

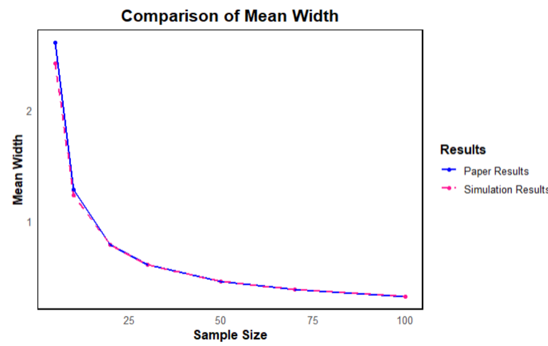
The results tabulated in Tables (3.2) and (3.3) reveal that when sampling from a non-normal distribution, the exact method performs poorly compared to the other methods. Among all the two-sided confidence intervals, Bonett's method had a better performance, and even with that, the coverage probabilities are consistently below the target of 95%. The most interesting or shocking revelation from the results tabulated in Tables (3.2) and (3.3) is that the robust method proposed by Abu-Shawiesh et al. [1] has coverage probabilities dying off to 0 for both chi-square and log-normal distributions.

		Sample Sizes						
Methods	Measuring Criteria	5	10	20	30	50	70	100
Exact	Coverage	0.801	0.701	0.659	0.582	0.570	0.566	0.529
	Under Coverage	0.128	0.197	0.215	0.266	0.269	0.272	0.265
	Over Coverage	0.071	0.102	0.126	0.152	0.161	0.162	0.206
	Mean Width	1.4756	0.8120	0.5274	0.4170	0.3203	0.2671	0.2255
	Median Width	1.2145	0.6911	0.4890	0.3875	0.3070	0.2554	0.2188
	SD Width	0.9379	0.4276	0.1987	0.1376	0.0815	0.0611	0.0456
Bonett	Coverage	0.965	0.831	0.840	0.818	0.856	0.861	0.871
	Under Coverage	0.035	0.158	0.147	0.176	0.128	0.130	0.119
	Over Coverage	0.000	0.011	0.013	0.006	0.016	0.009	0.010
	Mean Width	4.0860	1.5396	1.0019	0.8325	0.6926	0.5950	0.5360
	Median Width	2.4769	1.0376	0.7725	0.6516	0.5699	0.4973	0.4564
	SD Width	4.2641	1.4750	0.7801	0.6174	0.4206	0.3707	0.3182
Robust	Coverage	0.778	0.707	0.442	0.240	0.060	0.011	0.000
	Under Coverage	0.186	0.288	0.558	0.759	0.940	0.989	1.000
	Over Coverage	0.036	0.005	0.000	0.001	0.000	0.000	0.000
	Mean Width	1.4251	0.6758	0.4042	0.3073	0.2287	0.1893	0.1573
	Median Width	1.2089	0.6479	0.3927	0.3037	0.2262	0.1877	0.1569
	SD Width	0.9212	0.2478	0.1026	0.0610	0.0351	0.03512	0.0165
Non-Parametric Bootstrap	Coverage	0.393	0.525	0.611	0.673	0.716	0.761	0.780
	Under Coverage	0.606	0.475	0.389	0.326	0.282	0.236	0.219
	Over Coverage	0.001	0.000	0.000	0.001	0.002	0.003	0.001
	Mean Width	0.6913	0.6916	0.6184	0.5790	0.5315	0.4780	0.4288
	Median Width	0.5507	0.5410	0.5106	0.5001	0.4535	0.4167	0.3746
	SD Width	0.5036	0.5400	0.4029	0.3467	0.3311	0.2537	0.2113
Parametric Bootstrap	Coverage	0.771	0.697	0.741	0.780	0.789	0.812	0.822
	Under Coverage	0.118	0.201	0.185	0.164	0.166	0.142	0.129
	Over Coverage	0.111	0.102	0.074	0.056	0.045	0.046	0.049
	Mean Width	7.0879	1.8392	1.0348	0.8346	0.6936	0.5665	0.4843
	Median Width	1.9187	0.9468	0.6959	0.6161	0.5319	0.4632	0.4045
	SD Width	52.3355	3.6109	1.2346	0.8267	0.7371	0.4014	0.3007
Robust Bootstrap	Coverage	0.686	0.599	0.355	0.207	0.053	0.005	0.0000
	Under Coverage	0.284	0.397	0.645	0.793	0.947	0.995	1.000
	Over Coverage	0.030	0.004	0.000	0.000	0.000	0.000	0.000
	Mean Width	1.1319	0.5907	0.3689	0.2938	0.22039	0.1853	0.1538
	Median Width	0.9440	0.5553	0.3619	0.2878	0.2179	0.1849	0.1537
	SD Width	0.8628	0.2418	0.0981	0.0610	0.0355	0.0237	0.0166
CT Bootstrap	Coverage	0.826	0.701	0.694	0.699	0.678	0.672	0.627
	Under Coverage	0.174	0.299	0.306	0.301	0.322	0.328	0.373
	Over Coverage	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Mean Width	203.7669	14.505	3.6398	2.3969	2.0420	1.4163	1.2153
	Median Width	4.6323	1.6187	1.2597	1.1945	1.0897	0.9992	0.9581
	SD Width	2396.5331	116.2155	17.4285	8.7680	6.6310	1.9521	1.2158

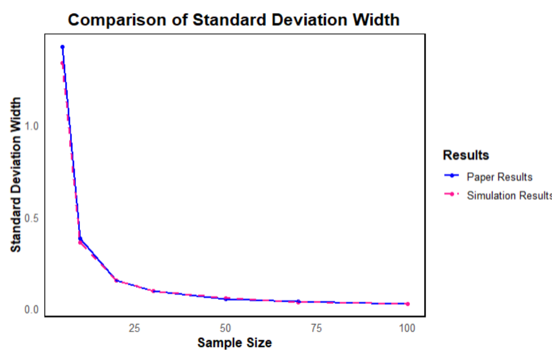
Table 3.3: Coverage Properties for Lognormal(1, 0.8)



(a) Coverage Probability



(b) Mean Width

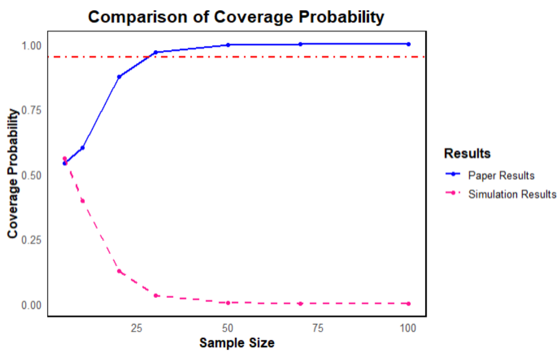


(c) Standard Deviation Width

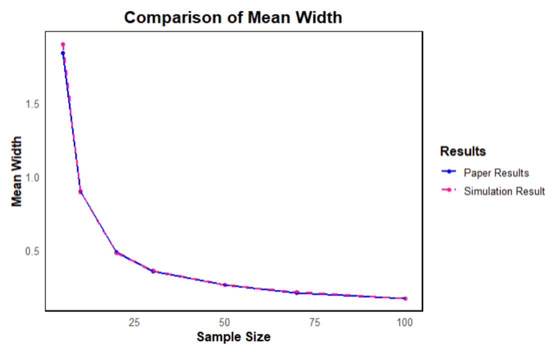
Figure 3.4: Normal(3,1)

Figures (3.4),(3.5), and (3.6) present a comparison between the results from the literature by Abu-Shawiesh et al. [1] and our simulation outcomes. The dot-dash line represents the target of 95%. Figure (3.4) specifically compares the coverage probability, the mean width, and the standard deviation width for $N(3,1)$. As depicted in Figure 3.4a the coverage probability from the published literature and our simulation results are consistent, accounting for simulation error. In Figures 3.4b and 3.4c, we observe that the mean and standard deviation of the widths are very close to each other, with one line effectively overlapping the other.

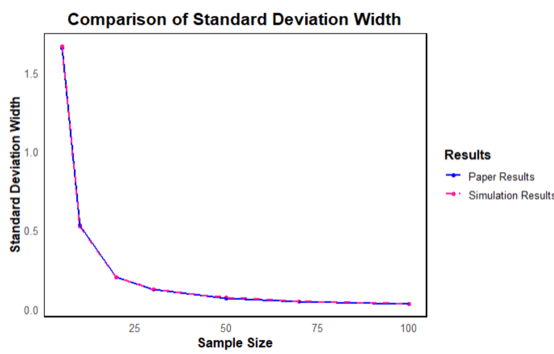
Figure (3.5) compares the coverage probability, the mean width, and the standard deviation width for $\chi^2_{(1)}$. In Figure 3.5a the coverage probability from the literature is conservative, that is the results exceed the target of 95%, as sample size increases, whereas our simulation results diminish towards 0. Figures 3.5b and 3.5c show that the mean width and standard deviation widths are closely aligned, with one line overlapping the other.



(a) Coverage Probability



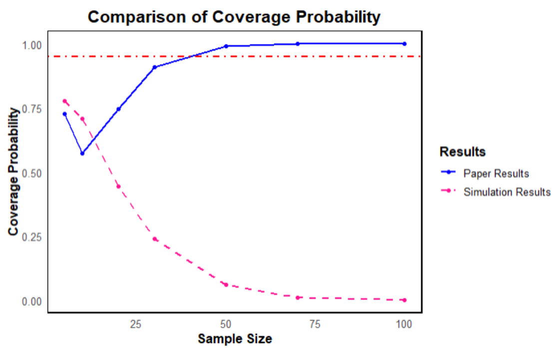
(b) Mean Width



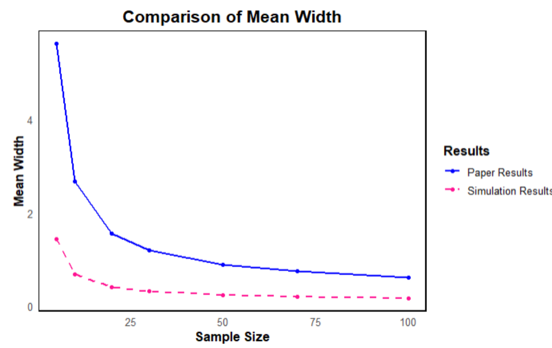
(c) Standard Deviation Width

Figure 3.5: Chi-square with $df = 1$ ($\chi^2_{(1)}$)

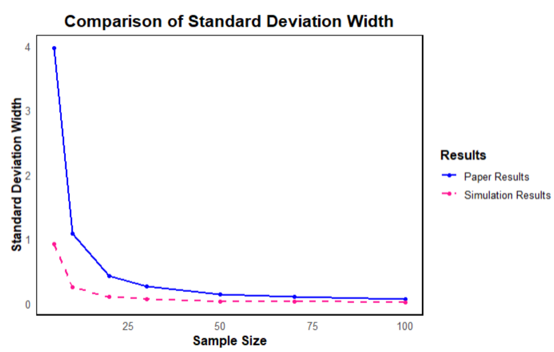
Figure (3.6) compares the coverage probability, the mean width, and the standard deviation width for Lognormal(1,0.8). For the Lognormal (1, 0.8) distribution the difference in the coverage behavior, Figure 3.6a, is similar to that observed for the chi-square distribution with one degree of freedom. However, the mean and standard deviation width, figures Figure 3.6b and Figure 3.6c, are not as closely aligned. This suggests that perhaps the difference in coverage here may have narrower width intervals in our results. Yet, where the largest differences occur in the widths (at $n = 5$ and $n = 10$), our coverages are higher than Abu-Shawiesh et al. [1].



(a) Coverage Probability



(b) Mean Width



(c) Standard Deviation Width

Figure 3.6: Lognormal(1,0.8)

These observations led us to explore possible ways to improve upon the performance of Abu-Shawiesh et al.'s robust confidence interval for skewed distributions, and still retain its performance for symmetric distributions.

4 Simulation Results

4.1 Modification on Robust Method

This section considers a modification of Abu-Shawiesh et al.'s robust confidence interval, 2011, to improve its performance. By customizing this approach, we aim to enhance the method's ability to provide more reasonable confidence estimates for the population standard deviation for skewed distributions while retaining its acceptable performance for symmetric distributions.

The $100(1-\alpha)\%$ for the modified robust confidence interval for σ is defined as

$$\left(\frac{1.28\sqrt{n}(d_n Q_n)(1+[|\hat{\gamma}_3|])}{z_{1-\frac{9\alpha}{10}} + 1.28\sqrt{n}} < \sigma < \frac{1.28\sqrt{n}(d_n Q_n)(1+[|\hat{\gamma}_3|])}{z_{\frac{\alpha}{10}} + 1.28\sqrt{n}} \right) \tag{4.1}$$

where $z_{\frac{\alpha}{10}}$ and $z_{1-\frac{9\alpha}{10}}$ are the $(\frac{\alpha}{10})^{th}$ and $(1-\frac{9\alpha}{10})^{th}$ percentiles of the standard normal distribution. Also, $\hat{\gamma}_3$ is a sample skew defined as $\frac{n}{(n-1)(n-2)} \sum_{i=1}^{n} (\frac{x_i - \bar{x}}{s})^3$ with $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$, $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$. In addition, $[x]$ is the greatest integer function, and $|x|$ is an absolute value.

The same adjustment was made for Abu-Shawiesh et al.'s robust bootstrap method, in which our results indicated failure for the skewed distributions. Our objective was to compare the performance of the modified robust method to Abu-Shawiesh et al.'s robust confidence interval (2.19), while keeping track of how the modified robust method performs compared to the other techniques we looked at in Section 3.2.

The simulation framework follows the same simulation framework from Section 3.1, however, more distributions were considered. The choice of distributions captures some of the real-world datasets' different characteristics and complexities. The distributions we looked at in addition to those considered in Section 3.1 are as follows:

1. Gamma distribution with shape 5 and scale 0.5.

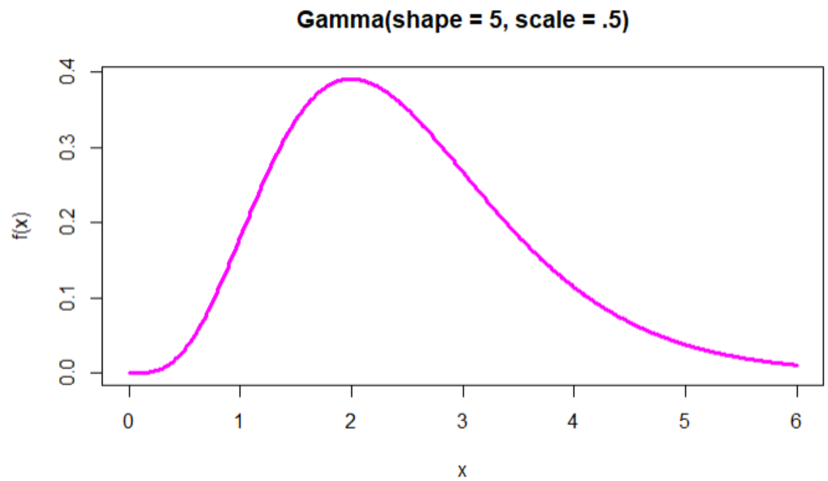


Figure 4.1: Gamma Distribution

2. Exponential distribution with a rate of 1.5.

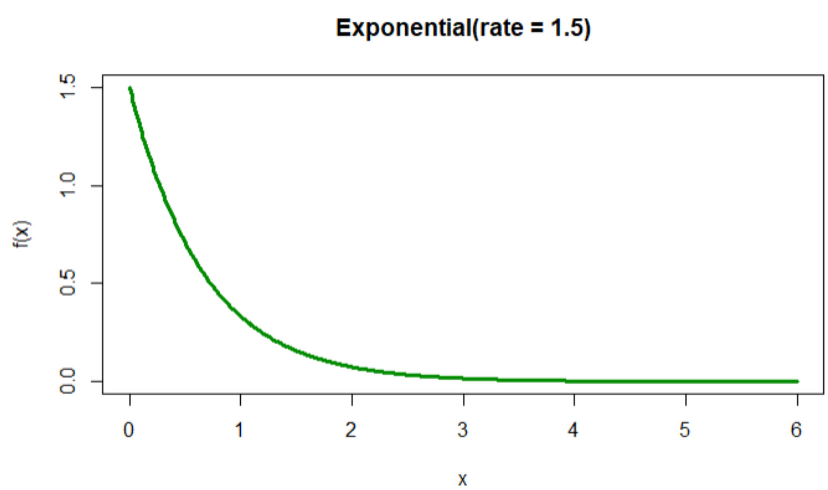


Figure 4.2: Exponential Distribution

3. Beta distribution with both shapes 0.5.

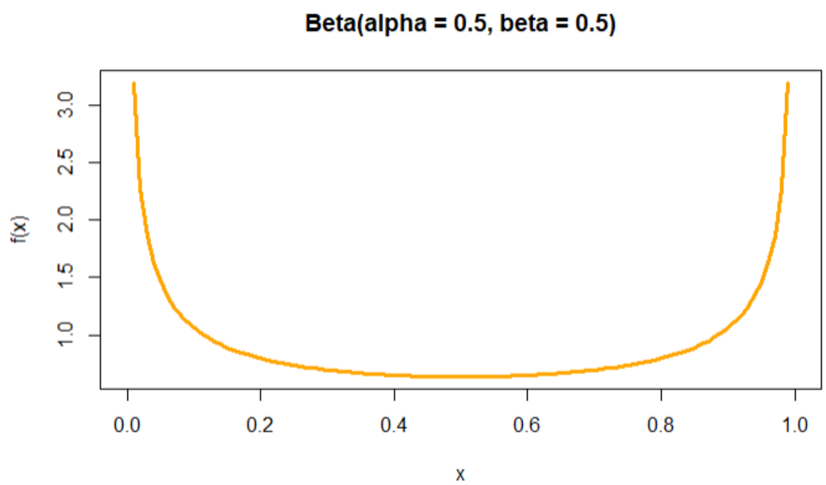


Figure 4.3: Beta Distribution

4. Laplace distribution with location 0 and scale 4.

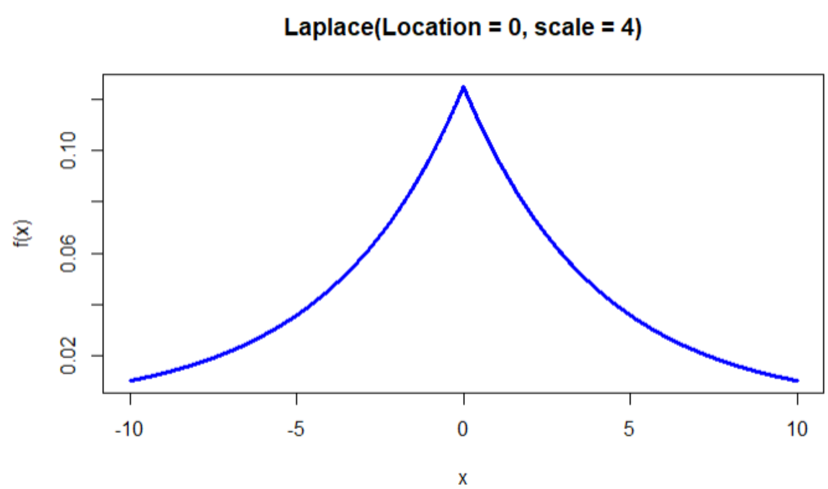


Figure 4.4: Laplace Distribution

5. Beta distribution with shapes 20 and 1.

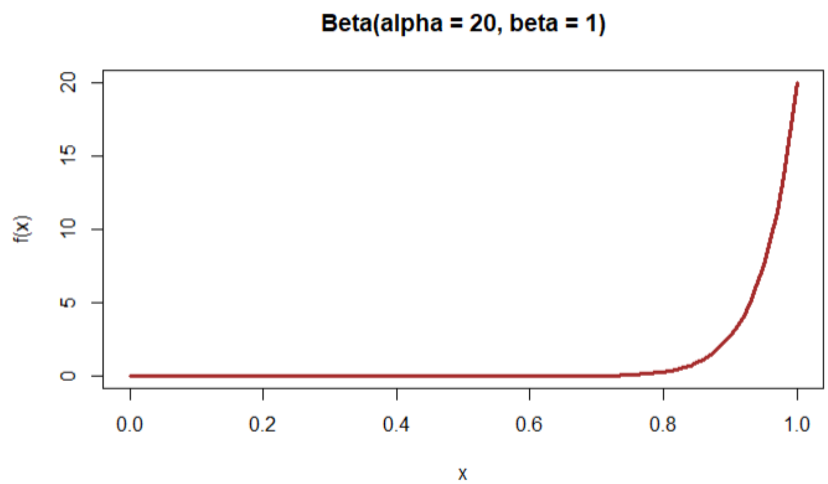


Figure 4.5: Beta Distribution

6. Beta distribution with shapes 10 and 4.

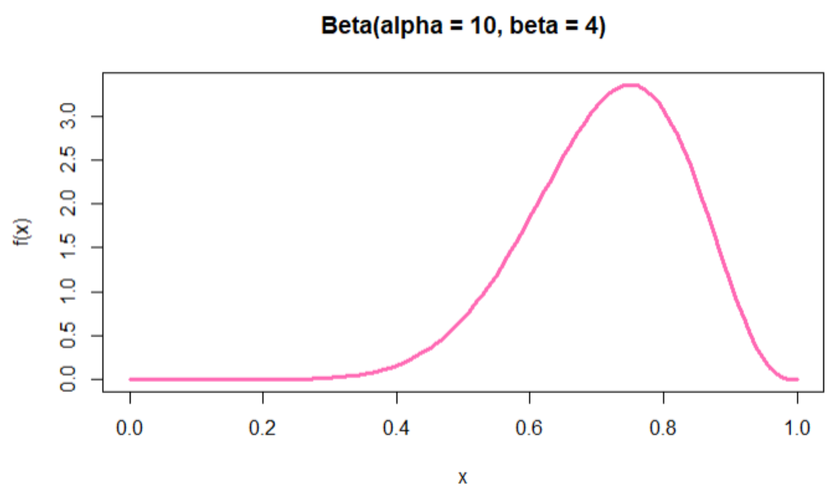
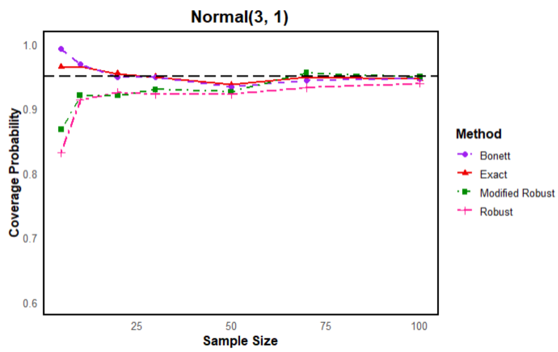


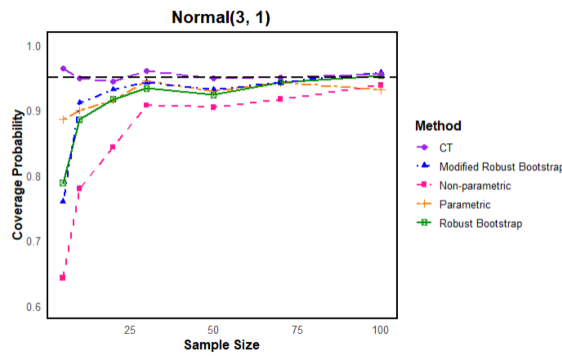
Figure 4.6: Beta Distribution

4.2 Simulation Results

Figures (4.7) - (4.15) illustrate the relationship between coverage probability and sample size with a simulation error of ± 0.0316. The graphs on the left side display results from non-bootstrap techniques, while the graphs on the right side exclusively show results from bootstrap techniques. This decision was made to ensure clarity and focus on the more effective methods in our analysis.

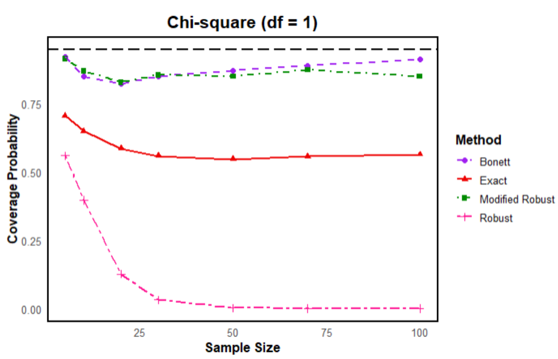


(a) Coverage Probability

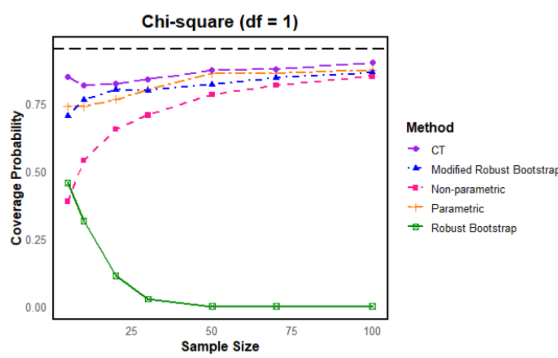


(b) Coverage Probability

Figure 4.7: Simulation Results for N(3, 1)

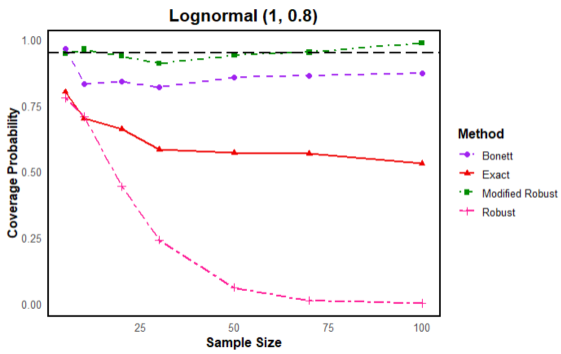


(a) Coverage Probability

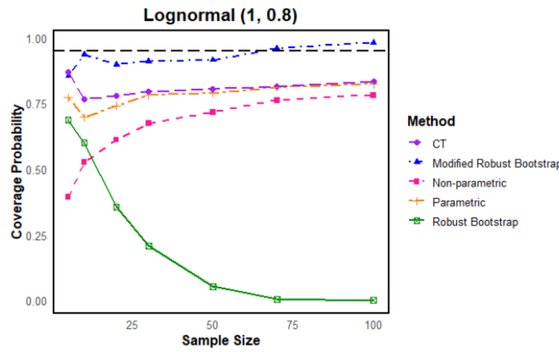


(b) Coverage Probability

Figure 4.8: Simulation Results for $\chi^2_{(1)}$

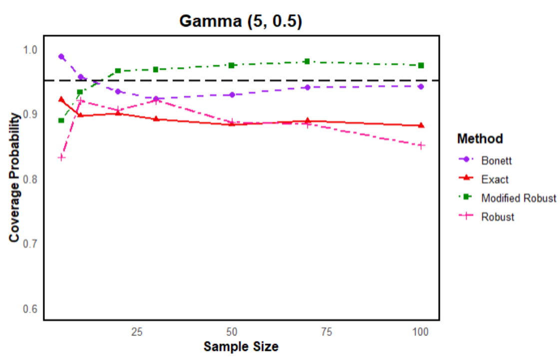


(a) Coverage Probability

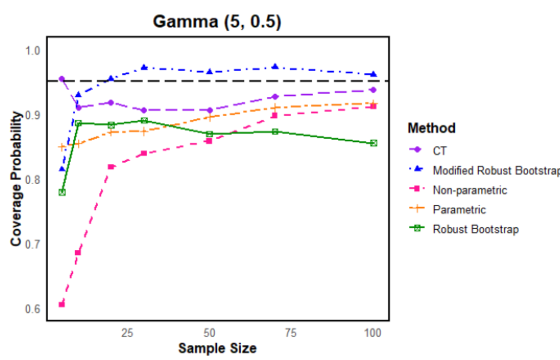


(b) Coverage Probability

Figure 4.9: Simulation Results for Lognormal (1, 0.8)

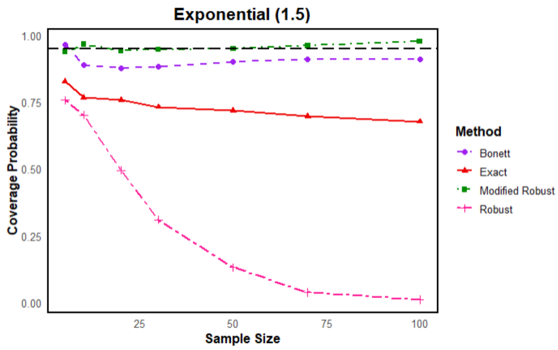


(a) Coverage Probability

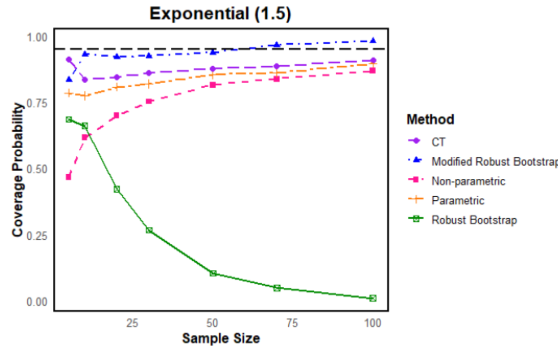


(b) Coverage Probability

Figure 4.10: Simulation Results for Gamma (5, 0.5)

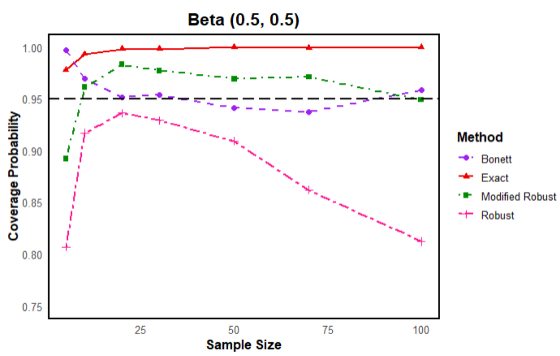


(a) Coverage Probability

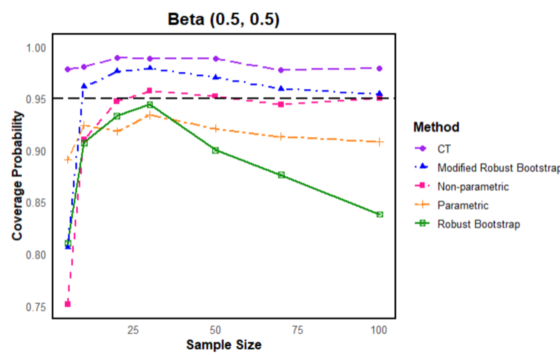


(b) Coverage Probability

Figure 4.11: Simulation Results for Exponential (1.5)

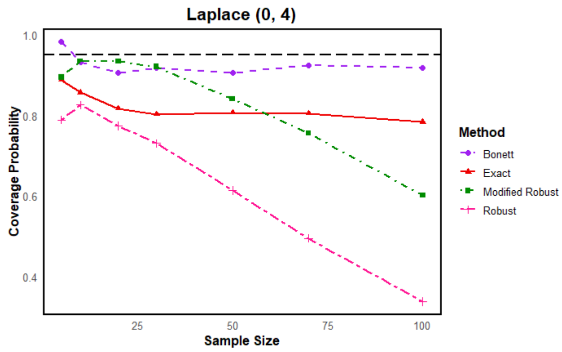


(a) Coverage Probability

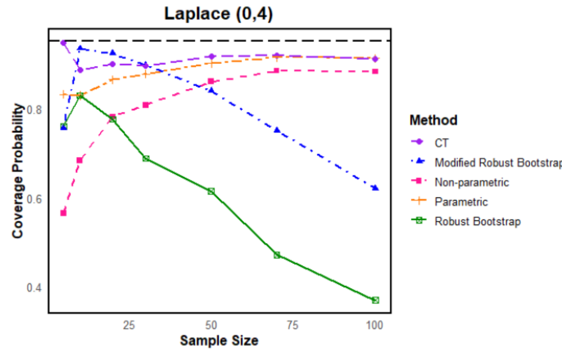


(b) Coverage Probability

Figure 4.12: Simulation Results for Beta (0.5, 0.5)

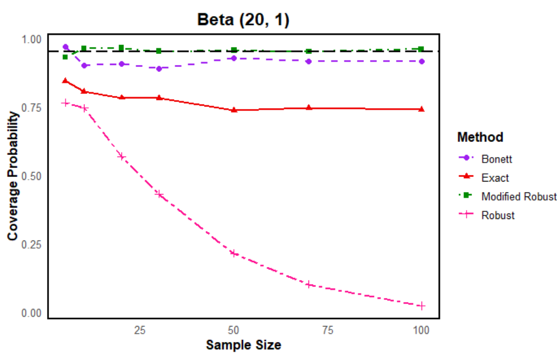


(a) Coverage Probability

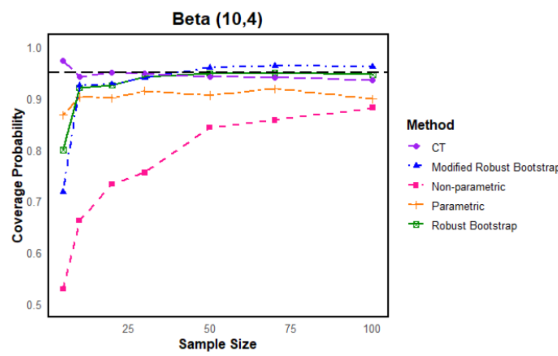


(b) Coverage Probability

Figure 4.13: Simulation Results for Laplace (0, 4)

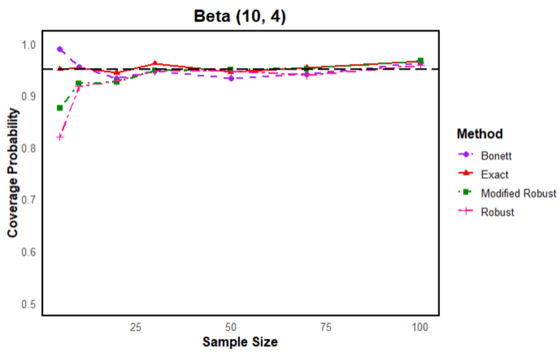


(a) Coverage Probability

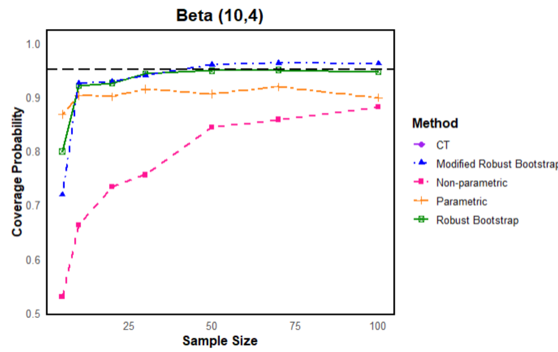


(b) Coverage Probability

Figure 4.14: Simulation Results for Beta (20, 1)



(a) Coverage Probability

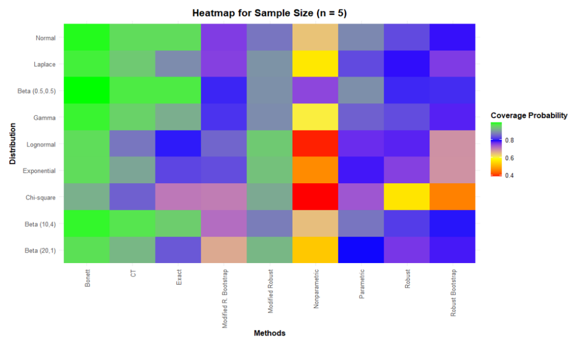


(b) Coverage Probability

Figure 4.15: Simulation Results for Beta (10, 4)

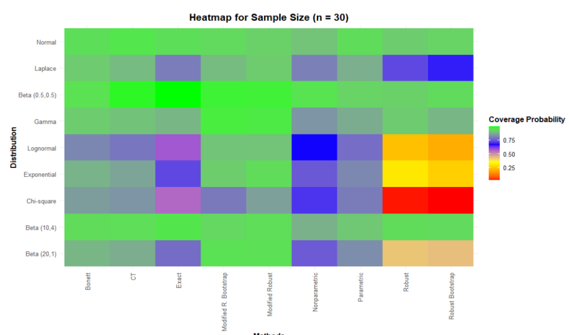
Our exploration revealed that the modifications we made significantly enhanced the performance of Abu-Shawiesh et al.'s robust confidence interval method (2.19). Comparative analysis shows that the modified robust method outperforms the original robust confidence interval method across symmetric, skewed, and heavy-tailed distributions. This improvement is evident even when considering bootstrap techniques. Furthermore, we found that the modified robust method outperformed Bonett's method on skewed distributions. However, Bonett's method showed superior performance compared to the modified method on symmetric and heavy-tailed distributions.

Figures (4.16) - (4.18) offer a comparative analysis of various methods across different distributions while maintaining a constant sample size. This analysis serves as a valuable tool for selecting the most appropriate interval method when the sample size is known, however, the exact distribution of the data is not, though a general understanding of the distribution's characteristics (symmetric, skewed, or heavy-tailed) is available. For instance, consider a scenario with a sample size of 100 and a histogram suggesting a skewed distribution. In this case, figure (4.18) becomes particularly informative, revealing that employing the robust method may not be the optimal choice for highly skewed distributions with this sample size. By examining these figures, researchers and analysts can make more informed decisions about which technique to apply, based on their sample size and the observed characteristics of their data distribution, thereby enhancing the accuracy and reliability of their statistical analyses, especially when dealing with varied and complex datasets.



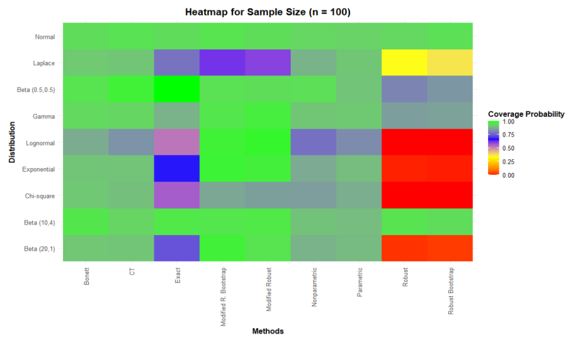
(a) Coverage Probability

Figure 4.16: Heatmap for $n = 5$



(a) Coverage Probability

Figure 4.17: Heatmap for $n = 30$



(a) Coverage Probability

Figure 4.18: Heatmap for $n = 100$

5 Conclusion Remark and Future Work

In this study, we evaluated the performance of several confidence interval methods for estimating population standard deviation, including proposed robust methods and the exact confidence interval. We also introduced a modification to the robust method (2.19) proposed by Abu-Shawiesh et al. [1]. Our findings revealed that:

1. The exact confidence interval (1.2) demonstrated superior coverage performance and narrower width when applied to normally distributed data, as anticipated.

2. Bonett's method (2.8) performed well with heavy-tailed distributions but showed limitations when applied to highly skewed distributions.

3. The robust method (2.19) proposed by Abu-Shawiesh et al. exhibited good coverage performance for symmetric distributions. However, despite its name suggesting otherwise, it performed poorly with skewed and heavy-tailed distributions.

4. Our modified robust method (3.4) showed improved performance for skewed distributions but still exhibited limitations when applied to heavy-tailed distributions.

These results highlight the varying effectiveness of different confidence interval methods across different types of distributions, underscoring the importance of selecting appropriate methods based on the characteristics of the data being analyzed. Some bootstrap procedures were also examined. The robust (3.2), non-parametric (2.20), and parametric (3.1) bootstrap procedures were not as promising compared to our modified robust bootstrap and Cojbasic and Tomovic confidence interval (3.3).

Our analysis revealed that the robust (3.2), non-parametric (2.20), and parametric (3.1) bootstrap procedures did not perform as well as initially anticipated. In contrast, our modified robust bootstrap method and the Cojbasic and Tomovic confidence interval (3.3) demonstrated superior performance. Specifically, the robust (3.2), non-parametric (2.20), and parametric (3.1) bootstrap procedures showed limitations in their effectiveness across various distribution types. Our modified robust bootstrap method exhibited notably better performance, providing more reliable confidence intervals for the population standard deviation. The Cojbasic and Tomovic confidence interval (3.3) demonstrated superior performance compared to the aforementioned bootstrap procedures in most scenarios. It offered a robust alternative for estimating the population standard deviation across various distribution types.

In future research, we aim to evaluate the performance of our proposed confidence interval (3.4) and other proposed intervals across a diverse range of probability distributions. This includes examining distributions with varying shapes and characteristics, such as multimodal distributions and those affected by contamination. Furthermore, we intend to explore modifications to our robust method (3.4) to enhance its accuracy and get reasonable confidence estimates when estimating population standard deviations for heavy-tailed distributions. These extensions will broaden the applicability of our approach and provide more comprehensive guidance for practitioners dealing with non-standard data scenarios.

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APPENDIX

		Sample Sizes						
Methods	Measuring Criteria	5	10	20	30	50	70	100
Exact	Coverage	0.964	0.965	0.954	0.949	0.937	0.949	0.946
	Under Coverage	0.018	0.020	0.021	0.029	0.028	0.031	0.024
	Over Coverage	0.018	0.015	0.025	0.022	0.035	0.020	0.030
	Mean Width	2.1267	1.0987	0.6956	0.5427	0.4079	0.34033	0.2833
	Median Width	2.0543	1.0835	0.6951	0.5453	0.4065	0.3406	0.2829
	SD Width	0.7471	0.2559	0.1103	0.07195	0.04273	0.0288	0.0205
Bonett	Coverage	0.993	0.968	0.948	0.949	0.933	0.943	0.946
	Under Coverage	0.006	0.019	0.025	0.033	0.031	0.033	0.026
	Over Coverage	0.001	0.013	0.027	0.018	0.036	0.024	0.028
	Mean Width	4.4779	1.3143	0.7303	0.5566	0.4176	0.3436	0.2858
	Median Width	3.8027	1.2317	0.7014	0.5392	0.4095	0.3376	0.2815
	SD Width	2.6289	0.4994	0.1928	0.1306	0.0755	0.0540	0.0392
Robust	Coverage	0.831	0.913	0.925	0.922	0.922	0.932	0.939
	Under Coverage	0.071	0.035	0.028	0.042	0.038	0.038	0.035
	Over Coverage	0.098	0.052	0.047	0.036	0.040	0.030	0.026
	Mean Width	2.5933	1.2529	0.7854	0.6054	0.4518	0.3767	0.3139
	Median Width	2.4201	1.2309	0.7832	0.6037	0.4487	0.3764	0.3136
	SD Width	1.3374	0.3611	0.1547	0.0938	0.0547	0.0363	0.0251
Modified Robust	Coverage	0.867	0.920	0.920	0.929	0.926	0.955	0.949
	Under Coverage	0.005	0.003	0.003	0.008	0.009	0.006	0.004
	Over Coverage	0.128	0.077	0.077	0.063	0.065	0.039	0.047
	Mean Width	11.5261	2.3525	1.1373	0.8069	0.5577	0.4547	0.3703
	Median Width	10.4731	2.1834	1.0846	0.7818	0.5489	0.4508	0.3690
	SD Width	5.9412	0.9617	0.3615	0.2330	0.1051	0.0786	0.0400
Non-Parametric Bootstrap	Coverage	0.642	0.779	0.843	0.907	0.904	0.916	0.938
	Under Coverage	0.358	0.221	0.149	0.089	0.088	0.072	0.056
	Over Coverage	0.000	0.000	0.008	0.004	0.008	0.012	0.006
	Mean Width	0.9458	0.7439	0.5489	0.4671	0.3735	0.3166	0.2682
	Median Width	0.9004	0.7174	0.5335	0.4540	0.3670	0.3125	0.2662
	SD Width	0.3711	0.2348	0.1334	0.0984	0.0698	0.0473	0.0350
Parametric Bootstrap	Coverage	0.885	0.899	0.915	0.945	0.928	0.942	0.931
	Under Coverage	0.029	0.033	0.034	0.026	0.037	0.030	0.037
	Over Coverage	0.086	0.068	0.051	0.029	0.035	0.028	0.032
	Mean Width	5.1377	1.1600	0.6497	0.5183	0.3960	0.3297	0.2753
	Median Width	2.4616	1.0274	0.6154	0.5006	0.3892	0.3245	0.2720
	SD Width	34.4886	0.5795	0.1875	0.1216	0.0785	0.0514	0.0371
Robust Bootstrap	Coverage	0.788	0.885	0.916	0.933	0.923	0.942	0.952
	Under Coverage	0.090	0.045	0.041	0.030	0.036	0.027	0.024
	Over Coverage	0.122	0.070	0.043	0.037	0.041	0.031	0.024
	Mean Width	2.5363	1.2445	0.7656	0.5994	0.4509	0.3773	0.3123
	Median Width	2.3288	1.2159	0.7616	0.5984	0.4511	0.3759	0.3122
	SD Width	1.4980	0.3935	0.1563	0.0910	0.0557	0.0370	0.0254
Modified Robust Bootstrap	Coverage	0.759	0.911	0.931	0.942	0.931	0.942	0.957
	Under Coverage	0.101	0.004	0.009	0.006	0.006	0.004	0.003
	Over Coverage	0.144	0.085	0.060	0.052	0.063	0.054	0.040
	Mean Width	3.9052	2.2854	1.1071	0.7869	0.5581	0.4480	0.3659
	Median Width	5.6022	2.0602	1.0362	0.7579	0.5451	0.4448	0.3632
	SD Width	70.8065	1.0845	0.4018	0.2260	0.1339	0.0480	0.0463
CT Bootstrap	Coverage	0.964	0.948	0.944	0.960	0.948	0.950	0.955
	Under Coverage	0.036	0.052	0.056	0.040	0.052	0.050	0.045
	Over Coverage	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Mean Width	4.3076	1.7267	1.3673	1.2794	1.2016	1.1640	1.1339
	Median Width	2.9858	1.6630	1.3533	1.2675	1.2003	1.1632	1.1296
	SD Width	6.4003	0.5481	0.2393	0.1692	0.1294	0.1007	0.0802

Table 1: Coverage Properties for N(3, 1)

		Sample Sizes						
Methods	Measuring Criteria	5	10	20	30	50	70	100
Exact	Coverage	0.708	0.651	0.587	0.560	0.548	0.558	0.565
	Under Coverage	0.199	0.238	0.272	0.279	0.271	0.250	0.254
	Over Coverage	0.093	0.111	0.141	0.161	0.181	0.192	0.181
	Mean Width	2.6218	1.4370	0.9293	0.7442	0.5615	0.4789	0.3962
	Median Width	2.1508	1.3040	0.8800	0.7157	0.5484	0.4663	0.3893
	SD Width	1.8872	0.7443	0.3653	0.2404	0.1442	0.1082	0.0732
Bonett	Coverage	0.922	0.851	0.826	0.851	0.872	0.889	0.914
	Under Coverage	0.076	0.141	0.158	0.128	0.119	0.096	0.068
	Over Coverage	0.002	0.008	0.016	0.021	0.009	0.015	0.018
	Mean Width	8.9043	3.1929	1.9636	1.6118	1.2697	1.1303	0.9661
	Median Width	5.5727	2.4327	1.6315	1.3813	1.1032	1.0103	0.8834
	SD Width	8.9684	2.5638	1.3286	0.9577	0.6567	0.5520	0.4056
Robust	Coverage	0.561	0.397	0.127	0.032	0.002	0.000	0.000
	Under Coverage	0.410	0.594	0.873	0.968	0.998	1.000	1.000
	Over Coverage	0.029	0.009	0.000	0.000	0.000	0.000	0.000
	Mean Width	1.9007	0.8936	0.4849	0.3624	0.2622	0.2147	0.1721
	Median Width	1.4196	0.7885	0.4601	0.3490	0.2554	0.2107	0.1707
	SD Width	1.6677	0.5271	0.1998	0.1246	0.0716	0.0471	0.0314
Modified Robust	Coverage	0.916	0.871	0.830	0.856	0.852	0.875	0.851
	Under Coverage	0.050	0.119	0.170	0.143	0.148	0.125	0.149
	Over Coverage	0.034	0.010	0.000	0.001	0.000	0.000	0.000
	Mean Width	11.2832	3.1997	2.1040	1.9027	1.6865	1.6259	1.5625
	Median Width	8.2279	2.6437	1.9109	1.7416	1.5354	1.5150	1.5114
	SD Width	10.1858	2.0980	1.1359	0.8909	0.7283	0.6430	0.5935
Non-Parametric Bootstrap	Coverage	0.387	0.539	0.655	0.707	0.782	0.817	0.850
	Under Coverage	0.613	0.458	0.344	0.292	0.217	0.182	0.142
	Over Coverage	0.000	0.003	0.001	0.001	0.001	0.001	0.008
	Mean Width	1.2350	1.3242	1.2601	1.1368	0.9825	0.9248	0.8188
	Median Width	0.9665	1.1106	1.0884	1.0209	0.9071	0.8495	0.7558
	SD Width	0.9706	0.8581	0.7408	0.5940	0.4233	0.4025	0.3177
Parametric Bootstrap	Coverage	0.738	0.738	0.765	0.802	0.859	0.861	0.874
	Under Coverage	0.128	0.150	0.132	0.126	0.099	0.082	0.076
	Over Coverage	0.134	0.112	0.103	0.072	0.042	0.057	0.050
	Mean Width	19.8398	5.0251	2.4548	1.7726	1.2760	1.1277	0.9413
	Median Width	4.1655	2.6204	1.7145	1.3889	1.1011	0.9660	0.8464
	SD Width	59.9651	8.1556	2.4908	1.5849	0.7392	0.6313	0.4457
Robust Bootstrap	Coverage	0.457	0.316	0.114	0.027	0.000	0.000	0.000
	Under Coverage	0.511	0.672	0.886	0.973	1.000	1.000	1.000
	Over Coverage	0.032	0.012	0.000	0.000	0.000	0.000	0.000
	Mean Width	1.5342	0.7742	0.4488	0.3425	0.2459	0.2070	0.1708
	Median Width	1.0535	0.6761	0.4141	0.3243	0.2427	0.2045	0.1677
	SD Width	1.5596	0.5103	0.2010	0.1190	0.0658	0.0486	0.0330
Modified Robust Bootstrap	Coverage	0.704	0.764	0.800	0.798	0.820	0.845	0.864
	Under Coverage	0.262	0.222	0.200	0.202	0.180	0.155	0.136
	Over Coverage	0.034	0.014	0.000	0.000	0.000	0.000	0.000
	Mean Width	4.3381	2.1944	1.8237	1.6898	1.5396	1.5540	1.5284
	Median Width	2.6086	1.9048	1.6585	1.5916	1.4281	1.4590	1.48371
	SD Width	8.6559	1.4005	0.9192	0.7568	0.6046	0.6091	0.5463
CT Bootstrap	Coverage	0.849	0.817	0.823	0.839	0.872	0.876	0.900
	Under Coverage	0.151	0.183	0.177	0.161	0.128	0.124	0.100
	Over Coverage	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Mean Width	19.2609	4.6469	2.9837	2.4675	2.1211	2.0565	1.9198
	Median Width	4.5274	2.9105	2.3402	2.1509	1.9770	1.9225	1.8164
	SD Width	57.4877	5.9722	2.2600	1.3473	0.7879	0.6924	0.5033

Table 2: Coverage Properties for $\chi^2_{(1)}$

		Sample Sizes						
Methods	Measuring Criteria	5	10	20	30	50	70	100
Exact	Coverage	0.801	0.701	0.659	0.582	0.570	0.566	0.529
	Under Coverage	0.128	0.197	0.215	0.266	0.269	0.272	0.265
	Over Coverage	0.071	0.102	0.126	0.152	0.161	0.162	0.206
	Mean Width	1.4756	0.8120	0.5274	0.4170	0.3203	0.2671	0.2255
	Median Width	1.2145	0.6911	0.4890	0.3875	0.3070	0.2554	0.2188
	SD Width	0.9379	0.4276	0.1987	0.1376	0.0815	0.0611	0.0456
Bonett	Coverage	0.965	0.831	0.840	0.818	0.856	0.861	0.871
	Under Coverage	0.035	0.158	0.147	0.176	0.128	0.130	0.119
	Over Coverage	0.000	0.011	0.013	0.006	0.016	0.009	0.010
	Mean Width	4.0860	1.5396	1.0019	0.8325	0.6926	0.5950	0.5360
	Median Width	2.4769	1.0376	0.7725	0.6516	0.5699	0.4973	0.4564
	SD Width	4.2641	1.4750	0.7801	0.6174	0.4206	0.3707	0.3182
Robust	Coverage	0.778	0.707	0.442	0.240	0.060	0.011	0.000
	Under Coverage	0.186	0.288	0.558	0.759	0.940	0.989	1.000
	Over Coverage	0.036	0.005	0.000	0.001	0.000	0.000	0.000
	Mean Width	1.4251	0.6758	0.4042	0.3073	0.2287	0.1893	0.1573
	Median Width	1.2089	0.6479	0.3927	0.3037	0.2262	0.1877	0.1569
	SD Width	0.9212	0.2478	0.1026	0.0610	0.0351	0.03512	0.0165
Modified Robust	Coverage	0.946	0.962	0.937	0.908	0.940	0.951	0.986
	Under Coverage	0.011	0.029	0.063	0.091	0.060	0.049	0.014
	Over Coverage	0.043	0.009	0.000	0.001	0.000	0.000	0.000
	Mean Width	7.6192	2.0728	1.5100	1.3606	1.3238	1.2543	1.2931
	Median Width	6.2060	1.7027	1.4440	1.2641	1.1399	1.0270	1.2585
	SD Width	5.6904	1.2770	0.8261	0.7668	0.6472	0.6240	0.6082
Non-Parametric Bootstrap	Coverage	0.393	0.525	0.611	0.673	0.716	0.761	0.780
	Under Coverage	0.606	0.475	0.389	0.326	0.282	0.236	0.219
	Over Coverage	0.001	0.000	0.000	0.001	0.002	0.003	0.001
	Mean Width	0.6913	0.6916	0.6184	0.5790	0.5315	0.4780	0.4288
	Median Width	0.5507	0.5410	0.5106	0.5001	0.4535	0.4167	0.3746
	SD Width	0.5036	0.5400	0.4029	0.3467	0.3311	0.2537	0.2113
Parametric Bootstrap	Coverage	0.771	0.697	0.741	0.780	0.789	0.812	0.822
	Under Coverage	0.118	0.201	0.185	0.164	0.166	0.142	0.129
	Over Coverage	0.111	0.102	0.074	0.056	0.045	0.046	0.049
	Mean Width	7.0879	1.8392	1.0348	0.8346	0.6936	0.5665	0.4843
	Median Width	1.9187	0.9468	0.6959	0.6161	0.5319	0.4632	0.4045
	SD Width	52.3355	3.6109	1.2346	0.8267	0.7371	0.4014	0.3007
Robust Bootstrap	Coverage	0.686	0.599	0.355	0.207	0.053	0.005	0.0000
	Under Coverage	0.284	0.397	0.645	0.793	0.947	0.995	1.000
	Over Coverage	0.030	0.004	0.000	0.000	0.000	0.000	0.000
	Mean Width	1.1319	0.5907	0.3689	0.2938	0.22039	0.1853	0.1538
	Median Width	0.9440	0.5553	0.3619	0.2878	0.2179	0.1849	0.1537
	SD Width	0.8628	0.2418	0.0981	0.0610	0.0355	0.0237	0.0166
Modified Robust Bootstrap	Coverage	0.853	0.935	0.897	0.909	0.914	0.958	0.980
	Under Coverage	0.109	0.059	0.103	0.091	0.086	0.042	0.020
	Over Coverage	0.038	0.006	0.000	0.000	0.000	0.000	0.000
	Mean Width	3.3997	1.4963	1.2116	1.1983	1.1948	1.1864	1.1655
	Median Width	2.3643	1.2335	1.1876	1.1547	1.0731	0.9809	0.9231
	SD Width	13.3384	0.8560	0.6610	0.5917	0.5895	0.5585	0.5261
CT Bootstrap	Coverage	0.868	0.766	0.779	0.793	0.803	0.813	0.832
	Under Coverage	0.132	0.234	0.221	0.207	0.197	0.187	0.168
	Over Coverage	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Mean Width	5.2802	2.0590	1.4387	1.2919	1.2028	1.1043	1.0499
	Median Width	2.1522	1.2723	1.1223	1.0929	1.0439	0.9996	0.9788
	SD Width	13.2688	3.2055	1.2535	0.8536	0.7718	0.4436	0.3363

Table 3: Coverage Properties for Lognormal(1, 0.8)

		Sample Sizes						
Methods	Measuring Criteria	5	10	20	30	50	70	100
Exact	Coverage	0.920	0.896	0.899	0.890	0.882	0.888	0.880
	Under Coverage	0.032	0.055	0.040	0.053	0.057	0.045	0.051
	Over Coverage	0.048	0.049	0.061	0.057	0.061	0.067	0.069
	Mean Width	2.3346	1.2060	0.7714	0.6059	0.4543	0.3827	0.3174
	Median Width	2.1826	1.1781	0.7616	0.5975	0.4507	0.3804	0.3162
	SD Width	0.9604	0.3447	0.1519	0.1001	0.0581	0.0410	0.0292
Bonett	Coverage	0.987	0.956	0.933	0.922	0.928	0.940	0.941
	Under Coverage	0.010	0.033	0.042	0.056	0.056	0.039	0.043
	Over Coverage	0.003	0.011	0.025	0.022	0.016	0.021	0.016
	Mean Width	5.2367	1.5874	0.9463	0.7369	0.5641	0.4718	0.3986
	Median Width	4.2100	1.3541	0.8115	0.6572	0.5109	0.4395	0.3760
	SD Width	3.5750	0.8478	0.4311	0.3010	0.2103	0.1482	0.1115
Robust	Coverage	0.832	0.919	0.905	0.919	0.886	0.883	0.850
	Under Coverage	0.086	0.051	0.076	0.070	0.110	0.109	0.150
	Over Coverage	0.082	0.030	0.019	0.011	0.004	0.008	0.000
	Mean Width	2.6861	1.3162	0.8089	0.6298	0.4673	0.3940	0.3264
	Median Width	2.4385	1.2636	0.8064	0.6241	0.4681	0.3913	0.3257
	SD Width	1.5199	0.4065	0.1692	0.0992	0.0562	0.0401	0.0273
Modified Robust	Coverage	0.888	0.932	0.965	0.967	0.973	0.979	0.979
	Under Coverage	0.002	0.007	0.003	0.011	0.016	0.011	0.022
	Over Coverage	0.110	0.061	0.032	0.022	0.011	0.010	0.005
	Mean Width	12.3828	2.7490	1.5685	1.2435	1.0052	0.8513	0.7507
	Median Width	10.9895	2.3726	1.2031	0.8703	0.6151	0.5017	0.4040
	SD Width	7.2870	1.40793	0.8776	0.7858	0.7106	0.6302	0.5874
Non-Parametric Bootstrap	Coverage	0.605	0.685	0.818	0.839	0.858	0.897	0.911
	Under Coverage	0.395	0.310	0.178	0.158	0.138	0.098	0.073
	Over Coverage	0.000	0.005	0.004	0.003	0.004	0.005	0.016
	Mean Width	1.0733	0.8332	0.6830	0.5938	0.4728	0.4172	0.3657
	Median Width	0.9923	0.7588	0.6228	0.5463	0.4474	0.3948	0.3473
	SD Width	0.5019	0.3485	0.2568	0.2128	0.1378	0.1166	0.0953
Parametric Bootstrap	Coverage	0.849	0.854	0.872	0.873	0.895	0.910	0.916
	Under Coverage	0.034	0.067	0.066	0.071	0.075	0.054	0.048
	Over Coverage	0.117	0.079	0.062	0.056	0.030	0.036	0.036
	Mean Width	5.3405	1.3616	0.8572	0.6865	0.5117	0.4402	0.3792
	Median Width	2.8291	1.0915	0.7437	0.6110	0.4797	0.4100	0.3589
	SD Width	17.5285	0.9215	0.4276	0.2960	0.1643	0.1299	0.1024
Robust Bootstrap	Coverage	0.779	0.886	0.883	0.890	0.869	0.873	0.855
	Under Coverage	0.120	0.067	0.095	0.098	0.124	0.126	0.142
	Over Coverage	0.101	0.047	0.022	0.012	0.007	0.001	0.003
	Mean Width	2.5048	1.2683	0.7800	0.6147	0.4598	0.3876	0.3219
	Median Width	2.3415	1.2306	0.7801	0.6122	0.4602	0.3865	0.3208
	SD Width	1.4578	0.4079	0.1669	0.1019	0.0583	0.0404	0.0275
Modified Robust Bootstrap	Coverage	0.814	0.929	0.954	0.971	0.965	0.972	0.961
	Under Coverage	0.074	0.005	0.010	0.009	0.024	0.021	0.032
	Over Coverage	0.113	0.066	0.036	0.020	0.011	0.007	0.007
	Mean Width	3.3639	2.3315	1.4499	1.1835	0.9125	0.7999	0.7540
	Median Width	5.3021	2.0723	1.1426	0.8330	0.5878	0.4799	0.3939
	SD Width	73.8640	1.0353	0.7618	0.7266	0.6129	0.5938	0.5916
CT Bootstrap	Coverage	0.954	0.910	0.918	0.906	0.906	0.927	0.937
	Under Coverage	0.046	0.090	0.082	0.094	0.094	0.073	0.063
	Over Coverage	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Mean Width	5.3934	1.9827	1.6259	1.5071	1.3774	1.3405	1.3090
	Median Width	3.2311	1.7682	1.5425	1.4559	1.3529	1.3254	1.2939
	SD Width	13.4404	0.9395	0.4694	0.3430	0.2147	0.1744	0.1392

Table 4: Coverage Properties for Gamma (5, .5)

		Sample Sizes						
Methods	Measuring Criteria	5	10	20	30	50	70	100
Exact	Coverage	0.827	0.765	0.756	0.731	0.717	0.696	0.676
	Under Coverage	0.100	0.158	0.139	0.156	0.159	0.186	0.184
	Over Coverage	0.073	0.077	0.105	0.113	0.124	0.118	0.140
	Mean Width	1.2992	0.6809	0.4460	0.3527	0.2699	0.2242	0.1874
	Median Width	1.1462	0.6359	0.4327	0.3455	0.2654	0.2232	0.1860
	SD Width	0.7423	0.2770	0.1281	0.0858	0.0510	0.0363	0.0267
Bonett	Coverage	0.964	0.888	0.875	0.883	0.901	0.909	0.910
	Under Coverage	0.034	0.105	0.115	0.109	0.082	0.082	0.076
	Over Coverage	0.002	0.007	0.010	0.008	0.017	0.009	0.014
	Mean Width	3.6834	1.1995	0.7654	0.6100	0.4849	0.4175	0.3564
	Median Width	2.5198	0.9185	0.6548	0.5512	0.4402	0.3784	0.3307
	SD Width	3.3881	0.8674	0.4637	0.3256	0.2204	0.1743	0.1317
Robust	Coverage	0.756	0.701	0.494	0.309	0.131	0.037	0.010
	Under Coverage	0.200	0.291	0.505	0.690	0.869	0.963	0.990
	Over Coverage	0.044	0.008	0.001	0.001	0.000	0.000	0.000
	Mean Width	1.2232	0.5873	0.3467	0.2649	0.1985	0.1638	0.1345
	Median Width	1.0469	0.5590	0.3390	0.2623	0.1965	0.1633	0.1338
	SD Width	0.8226	0.2358	0.0941	0.0585	0.0333	0.0234	0.0164
Modified Robust	Coverage	0.938	0.965	0.942	0.946	0.949	0.960	0.976
	Under Coverage	0.013	0.025	0.051	0.053	0.051	0.040	0.024
	Over Coverage	0.049	0.010	0.007	0.001	0.000	0.000	0.000
	Mean Width	6.4381	1.7275	1.1737	1.0421	0.9451	0.9088	0.8548
	Median Width	5.3748	1.4288	1.1505	1.0251	0.8914	0.8194	0.7460
	SD Width	4.5251	1.0126	0.6195	0.4835	0.3786	0.3483	0.3098
Non-Parametric Bootstrap	Coverage	0.466	0.616	0.699	0.752	0.815	0.837	0.868
	Under Coverage	0.534	0.383	0.296	0.245	0.180	0.156	0.125
	Over Coverage	0.000	0.001	0.005	0.003	0.005	0.007	0.007
	Mean Width	0.6132	0.5606	0.4918	0.4626	0.3933	0.3498	0.3111
	Median Width	0.5219	0.4968	0.4372	0.4153	0.3602	0.3262	0.2925
	SD Width	0.4054	0.3114	0.2534	0.2205	0.1638	0.1279	0.1058
Parametric Bootstrap	Coverage	0.785	0.774	0.807	0.819	0.853	0.860	0.897
	Under Coverage	0.080	0.134	0.126	0.119	0.106	0.094	0.072
	Over Coverage	0.135	0.092	0.067	0.062	0.041	0.046	0.031
	Mean Width	5.6355	1.2829	0.7553	0.6191	0.4665	0.3931	0.3374
	Median Width	1.7538	0.8473	0.5893	0.5063	0.4093	0.3566	0.3128
	SD Width	26.9955	1.3754	0.6210	0.4188	0.2390	0.1665	0.1276
Robust Bootstrap	Coverage	0.685	0.660	0.422	0.266	0.103	0.048	0.008
	Under Coverage	0.274	0.326	0.573	0.734	0.897	0.952	0.992
	Over Coverage	0.041	0.014	0.005	0.000	0.000	0.000	0.000
	Mean Width	1.0330	0.5389	0.3241	0.2525	0.1919	0.1607	0.1316
	Median Width	0.8563	0.5039	0.5039	0.2481	0.1896	0.1586	0.1309
	SD Width	0.7689	0.2311	0.0951	0.0587	0.0336	0.0236	0.0161
Modified Robust Bootstrap	Coverage	0.833	0.931	0.921	0.923	0.936	0.965	0.981
	Under Coverage	0.116	0.047	0.074	0.077	0.064	0.035	0.019
	Over Coverage	0.051	0.022	0.005	0.000	0.000	0.000	0.000
	Mean Width	2.9432	1.3130	0.9902	0.9712	0.8908	0.8703	0.8449
	Median Width	2.0720	1.1499	0.9474	0.9626	0.8476	0.7926	0.7299
	SD Width	9.1845	0.6685	0.4986	0.4399	0.3717	0.3245	0.3073
CT Bootstrap	Coverage	0.912	0.836	0.845	0.862	0.875	0.887	0.907
	Under Coverage	0.088	0.164	0.155	0.138	0.125	0.113	0.093
	Over Coverage	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Mean Width	5.6364	1.4989	1.1162	1.0328	0.9261	0.8766	0.8411
	Median Width	1.9226	1.1587	0.9760	0.9453	0.8776	0.8460	0.8197
	SD Width	27.0620	1.1974	0.5795	0.4393	0.2632	0.1949	0.1568

Table 5: Coverage Properties for Exponential (1.5)

		Sample Sizes						
Methods	Measuring Criteria	5	10	20	30	50	70	100
Exact	Coverage	0.978	0.993	0.998	0.998	1.000	0.999	1.000
	Under Coverage	0.022	0.007	0.002	0.002	0.000	0.001	0.000
	Over Coverage	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Mean Width	0.7698	0.3996	0.2472	0.1931	0.1447	0.1209	0.1003
	Median Width	0.7984	0.4023	0.2485	0.1934	0.1446	0.1208	0.1005
	SD Width	0.1980	0.0559	0.0218	0.0129	0.0077	0.0053	0.0035
Bonett	Coverage	0.997	0.969	0.951	0.954	0.941	0.937	0.958
	Under Coverage	0.003	0.006	0.001	0.002	0.005	0.009	0.008
	Over Coverage	0.000	0.025	0.048	0.044	0.054	0.054	0.034
	Mean Width	1.8220	0.3832	0.1722	0.1212	0.0835	0.0668	0.0539
	Median Width	1.5370	0.3482	0.1655	0.1188	0.0826	0.0663	0.0535
	SD Width	1.0248	0.1136	0.0297	0.0153	0.0077	0.0048	0.0032
Robust	Coverage	0.807	0.917	0.936	0.929	0.909	0.862	0.812
	Under Coverage	0.125	0.065	0.064	0.071	0.091	0.138	0.188
	Over Coverage	0.068	0.018	0.000	0.000	0.000	0.000	0.000
	Mean Width	0.8159	0.4235	0.2475	0.1933	0.1441	0.1194	0.0988
	Median Width	0.7766	0.4322	0.2521	0.1970	0.1457	0.1204	0.0995
	SD Width	0.4469	0.1165	0.0415	0.0249	0.0133	0.0084	0.0058
Modified Robust	Coverage	0.892	0.961	0.983	0.977	0.969	0.971	0.949
	Under Coverage	0.015	0.008	0.017	0.023	0.031	0.029	0.051
	Over Coverage	0.093	0.031	0.000	0.000	0.000	0.000	0.000
	Mean Width	3.6143	0.7401	0.3400	0.2486	0.1765	0.1429	0.1162
	Median Width	3.4831	0.7295	0.3431	0.2520	0.1782	0.1441	0.1171
	SD Width	1.9001	0.2113	0.0592	0.0340	0.0176	0.0101	0.0069
Non-Parametric Bootstrap	Coverage	0.751	0.910	0.947	0.957	0.952	0.944	0.950
	Under Coverage	0.249	0.089	0.048	0.036	0.037	0.040	0.028
	Over Coverage	0.000	0.001	0.005	0.007	0.011	0.016	0.022
	Mean Width	0.3374	0.2151	0.1317	0.1013	0.0745	0.0620	0.0508
	Median Width	0.3461	0.2037	0.1285	0.1002	0.0742	0.0618	0.0506
	SD Width	0.0942	0.0485	0.0192	0.0115	0.0062	0.0047	0.0032
Parametric Bootstrap	Coverage	0.891	0.924	0.918	0.934	0.921	0.913	0.908
	Under Coverage	0.020	0.013	0.007	0.008	0.009	0.016	0.028
	Over Coverage	0.089	0.063	0.075	0.058	0.070	0.071	0.064
	Mean Width	2.5883	0.3450	0.1492	0.1087	0.0773	0.0773	0.0516
	Median Width	1.0444	0.2691	0.1434	0.1074	0.0769	0.0634	0.0512
	SD Width	8.7341	0.2581	0.0296	0.0146	0.0073	0.0054	0.0036
Robust Bootstrap	Coverage	0.810	0.907	0.933	0.944	0.900	0.876	0.838
	Under Coverage	0.126	0.069	0.067	0.056	0.100	0.124	0.162
	Over Coverage	0.064	0.024	0.000	0.000	0.000	0.000	0.000
	Mean Width	0.7889	0.4140	0.2469	0.1929	0.1433	0.1189	0.0990
	Median Width	0.7649	0.4203	0.2515	0.1951	0.1447	0.1197	0.0994
	SD Width	0.4265	0.1191	0.0419	0.0236	0.0133	0.0093	0.0061
Modified Robust Bootstrap	Coverage	0.806	0.961	0.976	0.979	0.970	0.959	0.954
	Under Coverage	0.109	0.010	0.023	0.021	0.030	0.041	0.046
	Over Coverage	0.085	0.029	0.001	0.000	0.000	0.000	0.000
	Mean Width	2.9969	0.6931	0.3290	0.2427	0.1733	0.1408	0.1153
	Median Width	1.8469	0.6675	0.3319	0.2448	0.1737	0.1413	0.1155
	SD Width	38.9290	0.2633	0.0675	0.0338	0.0188	0.0124	0.0081
CT Bootstrap	Coverage	0.978	0.980	0.989	0.988	0.988	0.977	0.979
	Under Coverage	0.022	0.020	0.011	0.012	0.012	0.023	0.021
	Over Coverage	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Mean Width	2.5559	0.5512	0.4399	0.4130	0.3946	0.3855	0.3784
	Median Width	1.1392	0.5396	0.4406	0.4132	0.3953	0.3856	0.3785
	SD Width	7.6558	0.1281	0.0305	0.0225	0.0172	0.0147	0.0120

Table 6: Coverage Properties for Beta (0.5, 0.5)

		Sample Sizes						
Methods	Measuring Criteria	5	10	20	30	50	70	100
Exact	Coverage	0.888	0.857	0.817	0.803	0.807	0.805	0.784
	Under Coverage	0.057	0.085	0.105	0.097	0.104	0.103	0.118
	Over Coverage	0.055	0.058	0.078	0.100	0.089	0.092	0.098
	Mean Width	11.6299	5.9184	3.8433	3.0692	2.2963	1.9217	1.5861
	Median Width	10.5004	5.7345	3.7939	3.0272	2.2714	1.9156	1.5722
	SD Width	5.5585	1.9788	0.9300	0.6012	0.3677	0.2481	0.1814
Bonett	Coverage	0.983	0.933	0.906	0.919	0.906	0.924	0.918
	Under Coverage	0.013	0.013	0.069	0.062	0.066	0.056	0.061
	Over Coverage	0.004	0.012	0.025	0.019	0.028	0.020	0.021
	Mean Width	27.5871	8.7643	5.4126	4.3964	3.3564	2.8054	2.3737
	Median Width	21.7357	7.5282	4.9363	3.9939	3.0835	2.6488	2.2424
	SD Width	19.6564	4.7922	2.3289	1.7659	1.2173	0.8334	0.6867
Robust	Coverage	0.789	0.826	0.774	0.731	0.614	0.495	0.338
	Under Coverage	0.132	0.131	0.211	0.258	0.380	0.503	0.660
	Over Coverage	0.079	0.043	0.015	0.011	0.006	0.002	0.002
	Mean Width	12.7354	6.0434	3.6553	2.8810	2.1196	1.7584	1.4461
	Median Width	11.2923	5.7700	3.5703	2.8483	2.1176	1.7471	1.4424
	SD Width	8.0040	2.2284	0.9285	0.5867	0.3381	0.2368	0.1650
Modified Robust	Coverage	0.895	0.934	0.934	0.921	0.841	0.756	0.602
	Under Coverage	0.006	0.013	0.038	0.060	0.153	0.240	0.395
	Over Coverage	0.099	0.053	0.028	0.019	0.006	0.004	0.003
	Mean Width	63.9305	14.0281	7.2678	5.4795	3.9074	2.9832	2.2832
	Median Width	54.4886	11.6357	5.5640	3.9666	2.7366	2.1791	1.7262
	SD Width	41.5528	7.9647	4.3396	3.5780	2.9032	2.2536	1.9315
Non-Parametric Bootstrap	Coverage	0.565	0.683	0.781	0.807	0.860	0.884	0.882
	Under Coverage	0.435	0.314	0.215	0.190	0.131	0.113	0.109
	Over Coverage	0.000	0.003	0.004	0.003	0.009	0.003	0.009
	Mean Width	5.4679	4.7345	4.0405	3.5692	2.9075	2.5746	2.2313
	Median Width	4.9299	4.3200	3.7340	3.3272	2.7551	2.4208	2.0976
	SD Width	2.8450	2.2061	1.5848	1.3418	0.8700	0.7610	0.6009
Parametric Bootstrap	Coverage	0.831	0.829	0.864	0.877	0.901	0.914	0.912
	Under Coverage	0.046	0.084	0.074	0.083	0.068	0.058	0.060
	Over Coverage	0.123	0.087	0.062	0.040	0.031	0.028	0.028
	Mean Width	27.0648	8.7646	5.3882	4.3233	3.2327	2.7760	2.3457
	Median Width	14.4710	6.8922	4.6815	3.8866	3.0270	2.5972	2.1848
	SD Width	37.5785	6.8452	2.9141	2.0101	1.1002	0.9104	0.6669
Robust Bootstrap	Coverage	0.760	0.829	0.775	0.688	0.614	0.472	0.370
	Under Coverage	0.146	0.134	0.197	0.302	0.381	0.526	0.629
	Over Coverage	0.094	0.037	0.028	0.010	0.005	0.002	0.001
	Mean Width	12.6053	6.0217	3.6816	2.8072	2.1155	1.7393	1.4488
	Median Width	10.8003	5.7114	3.5990	2.7446	2.1031	1.7231	1.4445
	SD Width	8.4953	2.2843	0.9796	0.6031	0.3250	0.2331	0.1698
Modified Robust Bootstrap	Coverage	0.756	0.934	0.923	0.897	0.839	0.750	0.621
	Under Coverage	0.144	0.017	0.041	0.088	0.155	0.248	0.378
	Over Coverage	0.112	0.049	0.036	0.015	0.006	0.002	0.001
	Mean Width	84.0042	13.8991	7.2076	5.5766	3.7978	3.0516	2.3450
	Median Width	24.9012	10.9809	5.5705	3.8993	2.6922	2.1469	1.7174
	SD Width	1421.9406	9.5654	4.4767	3.9617	2.6821	2.4471	2.0382
CT Bootstrap	Coverage	0.946	0.886	0.899	0.896	0.916	0.918	0.911
	Under Coverage	0.054	0.114	0.101	0.104	0.084	0.082	0.089
	Over Coverage	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Mean Width	28.1224	11.3130	8.8942	8.1127	7.3617	7.0548	6.8343
	Median Width	17.0455	9.6400	8.3663	7.8070	7.1914	6.9208	6.7683
	SD Width	35.1651	6.5993	3.0708	2.2638	1.4107	1.1551	0.9356

Table 7: Coverage Properties for Laplace (0, 4)

		Sample Sizes						
Methods	Measuring Criteria	5	10	20	30	50	70	100
Exact	Coverage	0.842	0.804	0.781	0.779	0.735	0.744	0.739
	Under Coverage	0.104	0.120	0.126	0.128	0.127	0.148	0.133
	Over Coverage	0.054	0.076	0.093	0.093	0.138	0.108	0.128
	Mean Width	0.0904	0.0477	0.0306	0.0241	0.0185	0.0153	0.0127
	Median Width	0.0815	0.0449	0.0296	0.0237	0.0181	0.0152	0.0126
	SD Width	0.0494	0.0182	0.0082	0.0053	0.0032	0.0022	0.0016
Bonett	Coverage	0.968	0.901	0.906	0.889	0.925	0.925	0.913
	Under Coverage	0.030	0.088	0.085	0.096	0.055	0.077	0.063
	Over Coverage	0.002	0.011	0.009	0.015	0.020	0.009	0.024
	Mean Width	0.2474	0.0822	0.0495	0.0390	0.0312	0.0257	0.0218
	Median Width	0.1753	0.0655	0.0424	0.0354	0.0284	0.0242	0.0207
	SD Width	0.2263	0.0571	0.0276	0.0185	0.0129	0.0092	0.0068
Robust	Coverage	0.764	0.744	0.567	0.430	0.213	0.100	0.021
	Under Coverage	0.181	0.240	0.433	0.570	0.786	0.900	0.979
	Over Coverage	0.055	0.016	0.000	0.000	0.001	0.000	0.000
	Mean Width	0.0884	0.0419	0.0246	0.0192	0.0143	0.0118	0.0097
	Median Width	0.0748	0.0396	0.0242	0.0189	0.0141	0.0117	0.0097
	SD Width	0.0603	0.0170	0.0063	0.0039	0.0024	0.0016	0.0010
Modified Robust	Coverage	0.929	0.961	0.963	0.951	0.954	0.950	0.958
	Under Coverage	0.009	0.020	0.035	0.049	0.044	0.050	0.042
	Over Coverage	0.062	0.019	0.002	0.000	0.002	0.000	0.000
	Mean Width	0.4579	0.1189	0.0791	0.0697	0.0636	0.0576	0.0546
	Median Width	0.3887	0.1003	0.0767	0.0714	0.0619	0.0555	0.0515
	SD Width	0.3150	0.0659	0.0415	0.0326	0.0259	0.0216	0.0180
Non-Parametric Bootstrap	Coverage	0.529	0.662	0.733	0.755	0.843	0.857	0.881
	Under Coverage	0.471	0.336	0.265	0.237	0.149	0.139	0.110
	Over Coverage	0.000	0.002	0.002	0.008	0.008	0.004	0.009
	Mean Width	0.0434	0.0395	0.0339	0.0287	0.0251	0.0225	0.0195
	Median Width	0.0386	0.0352	0.0298	0.0261	0.0233	0.0209	0.0185
	SD Width	0.0250	0.0213	0.0166	0.0123	0.0093	0.0077	0.0057
Parametric Bootstrap	Coverage	0.795	0.783	0.828	0.826	0.868	0.881	0.893
	Under Coverage	0.059	0.104	0.090	0.106	0.069	0.078	0.066
	Over Coverage	0.146	0.113	0.082	0.068	0.063	0.041	0.041
	Mean Width	0.2997	0.0915	0.0505	0.0367	0.0291	0.0249	0.0209
	Median Width	0.1277	0.0604	0.0391	0.0312	0.0261	0.0229	0.0197
	SD Width	0.5467	0.0979	0.0364	0.0206	0.0130	0.0097	0.0067
Robust Bootstrap	Coverage	0.784	0.793	0.641	0.440	0.228	0.094	0.028
	Under Coverage	0.140	0.180	0.357	0.558	0.772	0.906	0.972
	Over Coverage	0.076	0.027	0.002	0.002	0.000	0.000	0.000
	Mean Width	0.1027	0.0462	0.0261	0.0197	0.0145	0.0119	0.0097
	Median Width	0.0891	0.0435	0.0256	0.0193	0.0143	0.0118	0.0097
	SD Width	0.0685	0.0181	0.0067	0.0043	0.0023	0.0015	0.0011
Modified Robust Bootstrap	Coverage	0.665	0.949	0.955	0.953	0.945	0.951	0.978
	Under Coverage	0.265	0.018	0.041	0.044	0.055	0.049	0.022
	Over Coverage	0.082	0.033	0.004	0.003	0.000	0.000	0.000
	Mean Width	0.1731	0.1589	0.0881	0.0726	0.0641	0.0606	0.0565
	Median Width	0.2330	0.1326	0.0866	0.0719	0.0634	0.0570	0.0525
	SD Width	6.3671	0.1069	0.0485	0.0358	0.0267	0.0235	0.0177
CT Bootstrap	Coverage	0.929	0.872	0.889	0.875	0.898	0.903	0.910
	Under Coverage	0.071	0.128	0.111	0.125	0.102	0.097	0.090
	Over Coverage	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Mean Width	0.3043	0.1072	0.0769	0.0660	0.0617	0.0586	0.0562
	Median Width	0.1460	0.0835	0.0681	0.0614	0.0591	0.0571	0.0553
	SD Width	0.5169	0.0853	0.0358	0.0221	0.0149	0.0115	0.0087

Table 8: Coverage Properties for Beta (20, 1)

		Sample Sizes						
Methods	Measuring Criteria	5	10	20	30	50	70	100
Exact	Coverage	0.950	0.953	0.943	0.961	0.945	0.953	0.965
	Under Coverage	0.028	0.026	0.029	0.021	0.029	0.024	0.018
	Over Coverage	0.022	0.021	0.028	0.018	0.026	0.023	0.017
	Mean Width	0.2467	0.1280	0.0808	0.0633	0.0476	0.0396	0.0331
	Median Width	0.2428	0.1265	0.0803	0.0630	0.0475	0.0396	0.0331
	SD Width	0.0880	0.0880	0.0132	0.0080	0.0048	0.0034	0.0022
Bonett	Coverage	0.989	0.954	0.932	0.946	0.933	0.941	0.962
	Under Coverage	0.009	0.025	0.031	0.027	0.034	0.031	0.023
	Over Coverage	0.002	0.021	0.037	0.027	0.033	0.028	0.015
	Mean Width	0.5225	0.1519	0.0840	0.0647	0.0481	0.0402	0.0332
	Median Width	0.4330	0.1394	0.0793	0.0616	0.0466	0.0392	0.0325
	SD Width	0.3205	0.0620	0.0245	0.0160	0.0102	0.0077	0.0053
Robust	Coverage	0.820	0.916	0.928	0.946	0.946	0.939	0.958
	Under Coverage	0.083	0.039	0.031	0.025	0.034	0.034	0.027
	Over Coverage	0.097	0.045	0.041	0.029	0.020	0.027	0.015
	Mean Width	0.2998	0.1469	0.0907	0.0704	0.0525	0.0437	0.0365
	Median Width	0.2906	0.1460	0.0901	0.0699	0.0524	0.0436	0.0364
	SD Width	0.1537	0.0419	0.0175	0.0104	0.0059	0.0041	0.0028
Modified Robust	Coverage	0.875	0.923	0.925	0.948	0.949	0.951	0.966
	Under Coverage	0.002	0.000	0.008	0.004	0.003	0.005	0.002
	Over Coverage	0.123	0.077	0.067	0.048	0.048	0.044	0.032
	Mean Width	1.3260	0.2768	0.1378	0.1003	0.0700	0.0556	0.0448
	Median Width	1.2369	0.2531	0.1258	0.0915	0.0647	0.0526	0.0430
	SD Width	0.6945	0.1128	0.0541	0.0403	0.0283	0.0212	0.0153
Non-Parametric Bootstrap	Coverage	0.647	0.752	0.867	0.881	0.899	0.905	0.906
	Under Coverage	0.353	0.246	0.129	0.114	0.092	0.084	0.079
	Over Coverage	0.000	0.002	0.004	0.005	0.009	0.011	0.015
	Mean Width	0.1108	0.0839	0.0655	0.0533	0.0424	0.0367	0.0308
	Median Width	0.1068	0.0812	0.0621	0.0513	0.0412	0.0358	0.0301
	SD Width	0.0424	0.0262	0.0186	0.0126	0.0085	0.0066	0.0046
Parametric Bootstrap	Coverage	0.867	0.902	0.900	0.914	0.905	0.918	0.898
	Under Coverage	0.898	0.039	0.033	0.036	0.052	0.044	0.054
	Over Coverage	0.106	0.059	0.067	0.050	0.043	0.038	0.048
	Mean Width	0.4739	0.1282	0.0786	0.0595	0.0449	0.0382	0.0317
	Median Width	0.2929	0.1138	0.0724	0.0566	0.0434	0.0372	0.0309
	SD Width	1.1959	0.0625	0.0280	0.0163	0.0097	0.0071	0.0049
Robust Bootstrap	Coverage	0.799	0.920	0.925	0.942	0.948	0.949	0.946
	Under Coverage	0.072	0.034	0.034	0.023	0.026	0.032	0.035
	Over Coverage	0.129	0.046	0.041	0.035	0.026	0.019	0.019
	Mean Width	0.3128	0.1482	0.0906	0.0704	0.0525	0.0438	0.0362
	Median Width	0.2932	0.1476	0.0901	0.0707	0.0524	0.0437	0.0363
	SD Width	0.1693	0.0431	0.0174	0.0106	0.0060	0.0041	0.0029
Modified Robust Bootstrap	Coverage	0.718	0.925	0.927	0.939	0.959	0.962	0.961
	Under Coverage	0.130	0.005	0.008	0.004	0.004	0.007	0.006
	Over Coverage	0.158	0.070	0.065	0.057	0.037	0.031	0.033
	Mean Width	1.2266	0.2937	0.1479	0.1021	0.0691	0.0552	0.0436
	Median Width	0.6765	0.2504	0.1266	0.0920	0.0646	0.0526	0.0427
	SD Width	14.8516	0.1635	0.0738	0.0450	0.0264	0.0194	0.0118
CT Bootstrap	Coverage	0.972	0.941	0.949	0.948	0.941	0.939	0.935
	Under Coverage	0.028	0.059	0.051	0.052	0.059	0.061	0.065
	Over Coverage	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Mean Width	0.4891	0.1960	0.1636	0.1486	0.1394	0.1354	0.1319
	Median Width	0.3436	0.1885	0.1591	0.1468	0.1384	0.1355	0.1319
	SD Width	1.0739	0.0604	0.0333	0.0217	0.0153	0.0121	0.0100

Table 9: Coverage Properties for Beta (10, 4)

VITA

Theophilus Oppong Kyeremeh, a Ghanaian student, earned his Bachelor's in Statistics from Kwame Nkrumah University of Science and Technology in 2019. Building on his statistical foundation, he enrolled in Stephen F. Austin State University's Master of Science program in Mathematical Sciences in 2022, specializing in Statistics. Theophilus Oppong Kyeremeh is on track to complete his graduate studies and receive his Master's degree in August 2024.

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The style manual used in this thesis is A Manual For Authors of Mathematical Papers published by the American Mathematical Society.

This thesis was prepared by Your Theophilus Oppong Kyeremeh using LaTeX.