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Exploring the Mandelbrot Set

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Exploring the Mandelbrot Set

by

J. Eli Shirley, B.S.

Presented to the Faculty of the Graduate School of

Stephen F. Austin State University

In Partial Fulfillment

of the Requirements

For the Degree of

Master of Science

STEPHEN F. AUSTIN STATE UNIVERSITY

May 2024

Exploring the Mandelbrot Set

by

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ABSTRACT

The Mandelbrot set is a mathematical mystery. Finding its home somewhere between holomorphic dynamics and complex analysis, the Mandelbrot set showcases its usefulness in fields across the many realms of math—ranging from physics to numerical methods and even biology. While typically defined in terms of its bounded sequences, this thesis intends to illuminate the Mandelbrot set as a type of parameterization of connectivity itself, specifically that of complex-valued rational maps of the form $z \mapsto z^2 + c$. This fully illustrated guide to the Mandelbrot set merges the worlds of intuition and theory with a series of self-contained arguments found in published texts over the years since the Mandelbrot set's conception—all to answer one question: is the Mandelbrot set connected? That is, are there any pieces of the Mandelbrot set just 'hanging off'? To answer this, we will appeal to the proof by the now-famous collaborators Adrien Douady and John Hubbard, whose work deep-dives into topologically grounded ideas and makes use of some functional analysis.

ACKNOWLEDGEMENTS

I first thank my family for their emotional and financial support through the most difficult time in my life (so far). Thank you to my dad, my mom, my brother, and my sisters for encouraging me even when continuing seemed too hard. My family made my dreams possible, and for this, I am eternally in their debt.

I next thank my cabinet of advisors who entrusted me with a task far beyond my ability. Special thanks to Dr. Pratscher for dealing with my various shenanigans. As my advisor, "Dr. P" made my mathematical work sensible, tangible, and then real. I also extend a big thank-you to Dr. Richardson, who endured independent studies with me to ensure my degree could be realized, and entertained my ideas and thoughts even when they were nonsensical. I hope to bring to my future students the same trust, patience, and kindness that I have been shown.

I additionally thank all the professors on the third floor—not only for their incredible insight and problem-solving but for sharing a passion with me on a subject that is typically underappreciated. This page is simply not long enough to individually thank each of you for the depth of information I have learned from every professor calling the math building their home.

Lastly, I thank my friends. Despite our collective struggles, my friends have remained steadfast as a source of much-needed, sanity-providing fun and entertainment throughout my entire degree. Thank you to Mason, Trevor, Tony, and Reeves for laughing at me. Thank you to Quentin, Ron, Johanna, Logan, Theo, and Richard for helping me keep my life together.

I couldn't have done any of this without you. Thank you for making my dreams a reality.

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1 INTRODUCTION

1.1 GOAL OF THE THESIS

Complex dynamics is a field of mathematics dedicated to the dynamical systems given by iterating complex-valued functions. Of particular interest in the field are the infinitely intricate regions of space in which properties of interest occur, usually exhibiting fractal-like behavior. Due to an increase in available computing power in recent years, images and analysis of these fractal regions increased in popularity and gave way to new findings and puzzles which the mathematics community continues to devour to this day. Among perhaps the most iconic and captivating of these discoveries is the Mandelbrot set: a fractal-like set possessing as much intricate detail as is possible in two dimensions [7]. This set, discovered by Benoit Mandelbrot in 1980, exhibits many interesting properties and shows up in strange places across all realms of mathematics. Despite this shape's infinite roughness, mathematicians argued over its connectivity until it was proven to be connected by collaborators Adrien Douady and John Hubbard. This result, at first glance, is strikingly unintuitive and poses a threat to those pursuing the mysteries of complex dynamics. This thesis aims to produce within the reader a general understanding and appreciation of the arguments used to show that the Mandelbrot set is, indeed, connected.

Our discussion starts by presenting underlying principles that all complex dynamics rely upon. After establishing the basics, we dive into what dynamics means, then hone in on some specific examples that broaden well to other topics. Among these specific maps (called rational maps) are maps of particular interest, called the depleted quadratic. We then analyze these maps under the guise of the metric space of $\mathbb{C}$ and its properties that pertain to connectivity, proving theorems and results when possible. Finally, we parameterize connectivity in $\mathbb{C}$ and then shift to defining the Mandelbrot set, all in hopes of proving that it is connected. To do this, we rely on the aforementioned arguments in a paper by Douady and Hubbard: an enlightening view of the Mandelbrot set as an intersection of many closed sets. Finally, we look into the Mandelbrot set's various extensions and problems left unresolved.

1.2 PRELIMINARY RESULTS

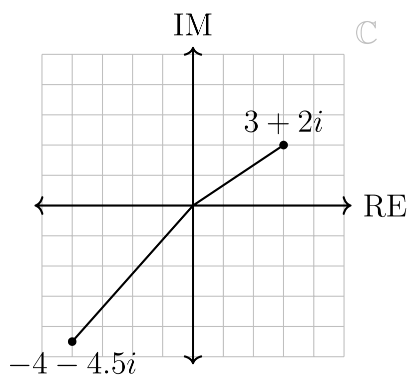
We use this section to introduce fundamental concepts to the study of complex dynamics. While the explanation in this thesis relies upon these concepts, in-depth discussion surrounding them is excluded for time. Those with a further interest are encouraged to view the thesis bibliography, upon which all arguments in this paper are based. In the case of section one, we mostly utilize the comprehensive nature of Conway's *Functions of One Complex Variable* [3], which is an invaluable resource for those interested in complex analysis. The results, definitions, and theorems that follow are found in an introductory analysis course.

Definition 1.1. The imaginary unit is denoted i and is defined by the property that $i^2 = -1$.

Definition 1.2. A *complex number* z is a number with a 'real' part a and an 'imaginary' part b, such that $z = a + bi$, where $a, b \in \mathbb{R}$. In shorthand, we say the real part of z is $\mathrm{RE}(z) = a$, and the imaginary part is $\mathrm{IM}(z) = b$.

Complex numbers are typically called imaginary numbers due to the use of the imaginary unit $i = \sqrt{-1}$. This name, while certainly fun, is outdated and sometimes causes preconceptions that imaginary numbers 'are not real'. This could not be further from the truth. In fact, many real-world applications utilize complex numbers and functions between the complex plane. These applications appear in such fields as electronics, physics, and almost anything involving waves or cyclic information, such as seasonal data sets.

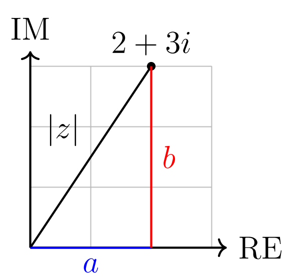
The complex numbers form a field, meaning they can be operated on like any other number, with the additional rule that $i^2 = -1$. For this reason, i is treated algebraically like a variable. Due to the separation enforced by like terms, complex numbers come embedded with two parts: real and imaginary. This makes complex numbers behave a lot like two-dimensional vectors with a real first coordinate and an imaginary second coordinate (demonstrated below).



Definition 1.3. For any complex number $z = a + bi$ $a, b \in \mathbb{R}$, the *modulus* (or *magnitude*) of z, denoted $|z|$, is given by:

$$|z| = \sqrt{a^2 + b^2}.$$

Remark 1.4. The magnitude of a complex number z can be geometrically viewed as the distance from the origin to z.

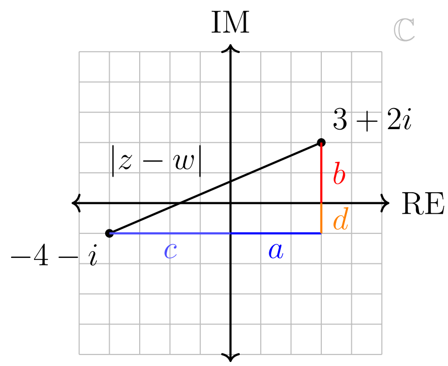


$$|z| = \sqrt{a^2 + b^2} = \sqrt{3^2 + 2^2} \approx 3.317$$

It is sometimes helpful to consider the distance between complex numbers. This is possible by taking the magnitude of their difference (exactly like vectors). Concretely, the distance between complex numbers z and w is given by $|z - w|$. To check this, we break z and w down into their real and imaginary parts. Let $z = a + bi$ and $w = c + di$, where $a, b, c, d \in \mathbb{R}$. It follows that:

$$|z - w| = |a + bi - (c + di)| = \sqrt{(a - c)^2 + (b - d)^2}.$$

This is illustrated more clearly below.



Definition 1.5. The *principal argument* of a complex number z is denoted Arg (z) and is the angle on $[-\pi, \pi)$ such that $\tan(\text{Arg }(z)) = \text{IM}(z)/\text{RE}(z)$, provided $\text{RE}(z) \neq 0$.

Definition 1.6. For any complex number z the *argument* of z is denoted $\arg z$ and given by:

$$\arg z = \text{Arg}(z) + 2k\pi, k \in \mathbb{Z}.$$

Definition 1.7. For any complex number z, the *polar form* of z is given by $z = re^{i\theta}$ for $r = |z|$ and $\theta = \arg(z)$.

Remark 1.8. For most of the discussion, we refer to Arg z, but the problem of dealing with the infinitely many possible angles of z is of relevance later.

Remark 1.9. For some positive n, the n^{th} *roots of unity* are the solutions to $z^n = 1$.

We find the roots of unity a lot behind the scenes throughout the text. We often want to find the roots of a particular polynomial or count the number of solutions to aid in simplifying a problem. The roots of unity tell us how many complex-valued solutions we can expect alongside the fundamental theorem of algebra. Additionally, roots of unity demonstrate a key problem in taking roots of a complex number.

Example 1.10. Suppose $z^7 = 1$. Seven distinct values solve this equation in $\mathbb{C}$. We first note that $z = 1$ is a solution. But, if any time during multiplication z lands on 1, it stays there (since $1^n = 1$ for any n). That is, $e^{2i\pi/7}$ is a solution since $\left(e^{2i\pi/7}\right)^7 = e^{2i\pi} = 1$. Similar solutions include $e^{4i\pi/7}$, $e^{6i\pi/7}$, $e^{8i\pi/7}$, $e^{10i\pi/7}$, and $e^{12i\pi/7}$. As such there are seven distinct solutions to $z^7 = 1$. These are called the seventh roots of unity.

If we were to imagine taking the n^{th} root of z^n, we now note that there are n possible complex numbers to consider.

2 BACKGROUND

2.1 TOPOLOGY ON THE COMPLEX PLANE

Topology is the study of properties and relations that are preserved under continuous deformations. In the realm of complex analysis, topology offers unique insights into the way interesting regions of space behave after being mapped through a function. This is a big part of showing that the Mandelbrot set is connected. Connectivity itself is a topological property we later define. While much of topology deals with generic spaces, we restrict ourselves to the results for metric spaces (spaces where distance has meaning) since we are working in $\mathbb{C}$.

2.1.1 METRIC SPACE TOPOLOGY

Definition 2.1. The *distance function* (or often just *metric*) on $\mathbb{C}$ is given by $d(z_1, z_2) = |z_1 - z_2|$.

The meaning of distance in the complex plane is equivalent to that of vectors in $\mathbb{R}^2$ (as seen in Chapter 1) where the real part of z is the first coordinate, and the second is the imaginary part of z. This is the main idea of metric space topology, where a distance between elements contextualizes them with one another.

We now use the sense of distance to form open sets that are used to lay the groundwork of topology. These sets are called *open balls*, or *epsilon balls*, and are the complex number equivalent to the open intervals or epsilon bands discussed in real analysis.

Definition 2.2. The *epsilon ball* centered at z_0 and radius ε is denoted $B(z_0, \varepsilon)$ and is given by:

$$B(z_0, \varepsilon) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$$

Definition 2.3. A set $C \subset \mathbb{C}$ is *open* if for every point $z \in C$, $B(z, \varepsilon) \subsetneq C$ for some $\varepsilon > 0$.

Every point z_0 in an open set has a neighborhood centered at z_0 that fits entirely inside the set (shown in figure 2.1). Intuitively, this can be imagined as a set without a border or edge. No matter how close z_0 inches towards the edge, there is always some nonzero distance between them.

Definition 2.4. A set $C \subset \mathbb{C}$ is *closed* if and only if $\mathbb{C} - C$ is open.

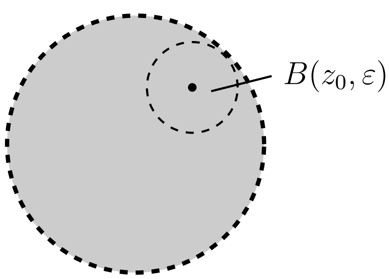


Figure 2.1: An open set containing $B(z_0, \varepsilon)$.

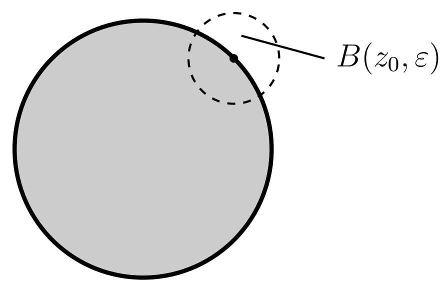


Figure 2.2: A closed set not containing $B(z_0, \varepsilon)$.

Remark 2.5. It is worth noting that open and closed are not perfect opposites; some sets are open and closed (such as $\mathbb{C}$ and $\emptyset$), and some are neither open nor closed.

Example 2.6. Consider the open ball B_1 centered at 1 of radius 2:
$B_1 = \{z \in \mathbb{C} : |z - 1| < 2\}$. Additionally, consider its closed variant across the imaginary axis, given by $B_2 = \{z \in \mathbb{C} : |z + 1| \leq 2\}$. Then $B_1 \cap B_2$ is not open since the point 1 does not admit a neighborhood contained in $B_1 \cap B_2$. Its complement $\mathbb{C} - B_1 \cap B_2$ is similarly not open since the point at -1 now cannot find a neighborhood contained within $\mathbb{C} - B_1 \cap B_2$. As such $B_1 \cap B_2$ is neither open nor closed. This is visually shown in figure 4.20.

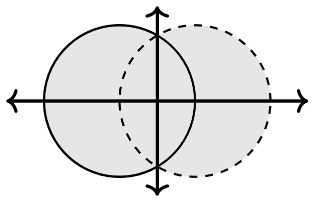


Figure 2.3: $B_1 \cup B_2$

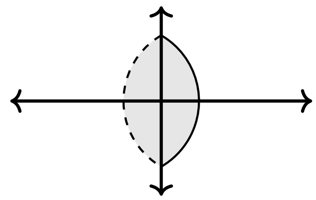


Figure 2.4: $B_1 \cap B_2$

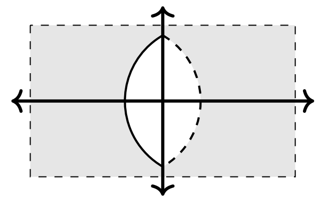


Figure 2.5: $\mathbb{C} - B_1 \cap B_2$

Theorem 2.7. The union of countably many open sets is open. Precisely, for a collection of open sets $\{D_\lambda\}_{\lambda \in \Lambda}$, $\bigcup_{\lambda \in \Lambda} D_\lambda$ is open (Λ countable).

The proof of this is a fun exercise in element chasing should the reader so desire. When completed, the next corollary follows from DeMorgan's laws.

Corollary 2.8. The intersection of countably many closed sets is closed. For a collection of closed sets $\{D_\lambda\}_{\lambda \in \Lambda}$, $\bigcap_{\lambda \in \Lambda} D_\lambda$ is closed (Λ countable).

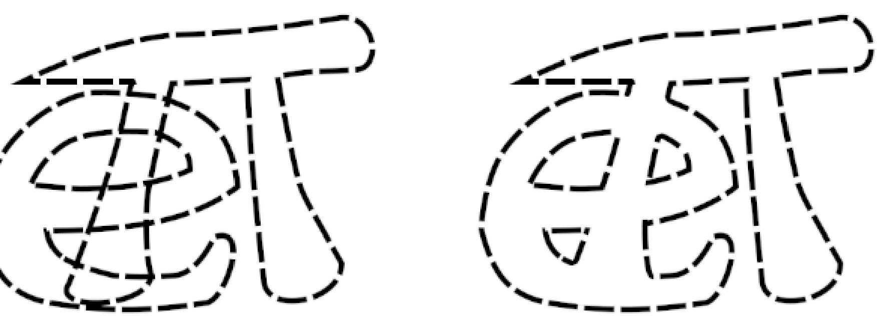


Figure 2.6: A union of two open sets is open.

Definition 2.9. For a set $D \subset \mathbb{C}$, the *interior* of D is denoted int (D) or D° and is given by:

$$\text{int } (D) = \bigcup \{G : G \text{ open }, G \subset D\}.$$

That is, the interior of D is the union of all open subsets of D. If a set is open, then int $(D) = D$. By Theorem 2.7, int (D) is always open. This can also be thought of as the largest open set contained inside of D.

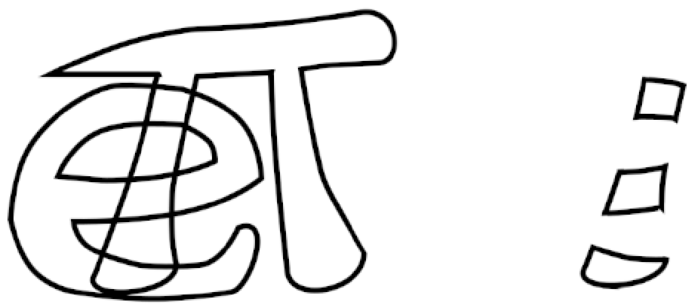


Figure 2.7: An intersection of two closed sets is closed.

Definition 2.10. For a set $D \subset \mathbb{C}$, the *closure* of D, denoted $\overline{D}$ is given by:

$$\overline{D} = \bigcap \{F : F \text{ closed }, D \subset F\}.$$

The closure of D is the intersection of all closed sets containing D. The closure of D is likewise always closed, by Corollary 2.8. The interior and closure of D are also shown in the given figures 2.6 and 2.7. The closure of D is also the smallest closed set containing all of D.

Definition 2.11. The *border* of a set $D \subset \mathbb{C}$ is denoted ∂D and is given by:

$$\partial D = \overline{D} - \text{int } D.$$

The border is a helpful topological tool that comes up in many fields of math. It is especially relevant for our discussion, and its definition is quite intuitive in the metric space $\mathbb{C}$. An image of two borders is given in figure 2.8.

Definition 2.12. An open set $D \subset \mathbb{C}$ is *disconnected* if it can be written as the union of two disjoint open sets.

Definition 2.13. An open set $D \subset \mathbb{C}$ is *connected* if it is not disconnected.

Connectivity, being the main topic of this thesis, can be defined in many equivalent ways in topology. In a metric space (like $\mathbb{C}$) we provide a helpful theorem from Dr.

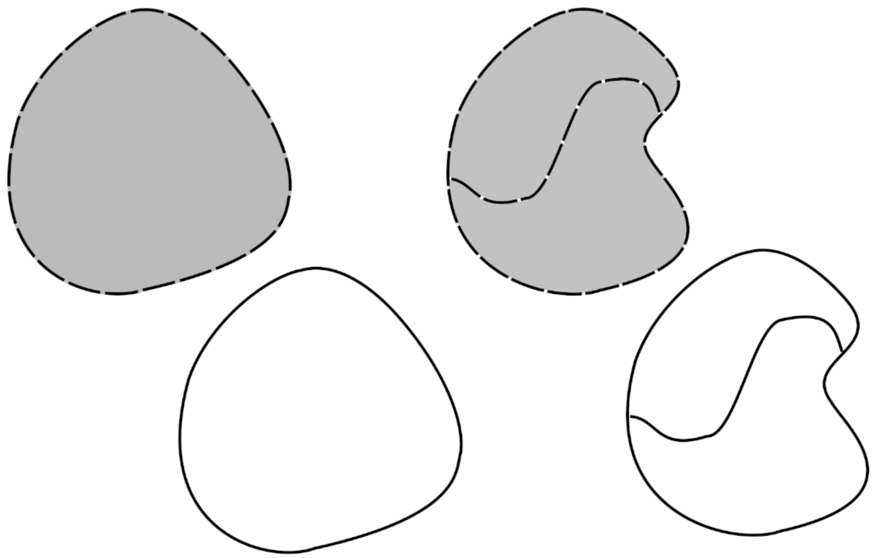


Figure 2.8: Similar open sets and their borders.

John Conway's *Functions of One Complex Variable* [3] as an example which is often insightful in determining whether an open set is connected. It does, however, require the construction of polygonal paths (called simply polygons in Conway's work), which are defined below.

Definition 2.14. For $z_1, z_2 \in \mathbb{C}$, the straight line $[z_1, z_2]$ is called a *segment* and is constructed with the parameterization:

$$[z_1, z_2] = \{tz_2 + (1-t)z_1 : 0 \leq t \leq 1\}$$

Definition 2.15. For any $a, b \in \mathbb{C}$, a *polygonal path* P from a to b is a given by:

$$P = \bigcup_{k=1}^{n} [z_k, z_{k+1}]$$

where $z_1 = a$, $z_n = b$.

The left two images of figure 2.9 demonstrate polygonal paths. The right two examples show non-polygonal paths. The first is disjoint, so it cannot be written

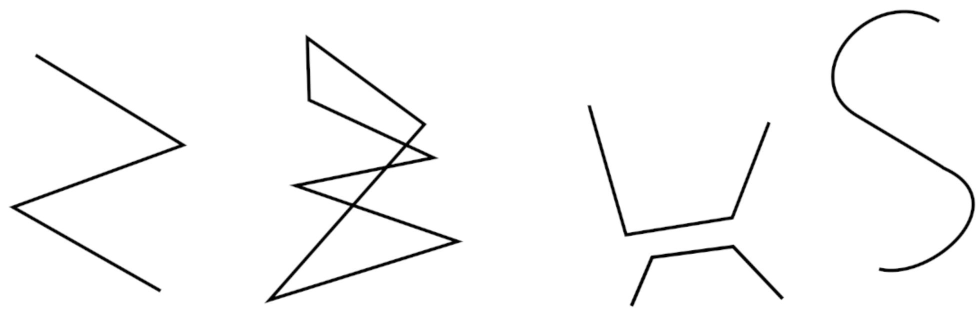


Figure 2.9: Left: polygonal path examples. Right: not polygonal paths.

as a polygonal path (but can be written as two paths). The furthest right could be smooth, demonstrating a discrepancy among analysts. Although not a polygonal path, a near-infinite number of polygonal paths can approximate this path.

With the definition of polygon in mind, we can now create a theorem for connectivity in $\mathbb{C}$. It is worth noting that this particular theorem struggles to generalize to topological spaces that are not metric spaces. Without a sense of distance, a polygonal path becomes more difficult to understand, and there are other methods for determining connectivity.

Theorem 2.16. An open set $D \subset \mathbb{C}$ is connected if for any two points $a, b \in D$ there is a polygonal path from a to b contained in D.

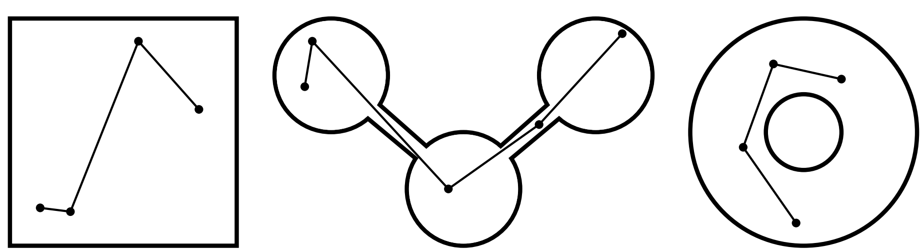


Figure 2.10: Various connected regions and their respective example polygonal paths.

Three connected domains and some path examples are shown in figure 2.10. It is worth reminding the reader that a path must exist for any two points in the region.

We are only showing one such path for demonstration.

Definition 2.17. For a set C and function f, the *components* of C are the maximal connected subsets of the preimage of C under f.

The image/preimage of sets are staples in dynamics and complex analysis alike. Look to figure 2.14 in the following section for what components look like in $\mathbb{C}$.

2.1.2 CLOSURE OF $\mathbb{C}$

Often in mathematics, infinity presents itself as a limit. It is the unattainable representation of that which is beyond every finite value. In dynamics, we need infinity to present normally (like other numbers) so that we can take its actions on the plane into account. To achieve this, we simply call infinity a new point: ∞. This point resides somewhere off the plane, but finding it is difficult. Theoretically, traveling in *any direction* from the origin long enough sends you 'towards' this new point ∞. Then we need every single direction of the plane away from the origin to be mapped to the same point. Fortunately, this mapping already exists and is called stereographic projection.

To achieve this mapping, often called π, we first reside in three-dimensional space $\mathbb{R}^3$. To avoid confusion, we delineate a coordinate in 3-space as vector (a, b, c), and the complex number $z = x + iy$. With the complex plane serving as the plane at $c = 0$, a complex number's real part is thus a and its imaginary part is b. We then nest a unit sphere in the complex plane, with its equator intersecting the unit circle. We plan to map every point in the plane to this sphere. Now pick any point in $z \in \mathbb{C}$, and draw a line through z to the sphere's north pole, $(0, 0, 1) \in \mathbb{R}^3$. Wherever this line intersects the sphere is z^*, the image of z under π (shown in figure 2.11). Most importantly, we have a place to put ∞: right at the top, thus setting $(0, 0, 1) = \infty$. The reader should take a minute to note that moving in any direction away from the

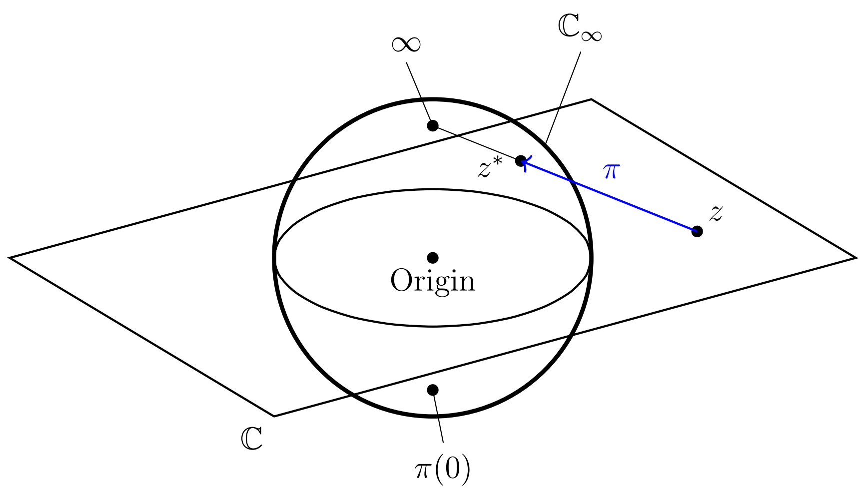


Figure 2.11: The projection map $\pi : \mathbb{C} \to \mathbb{C}_\infty$

origin results in you approaching ∞, but no point in $\mathbb{C}$ maps to ∞. Under the map π, we see that $\pi(0) = (0, 0, -1)$ (the sphere's southern pole). We are now ready for the next definition.

Definition 2.18. The *Riemann sphere* is denoted $\mathbb{C}_\infty$ and is given by: $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ for ∞ the limiting point of $\mathbb{C}$.

Adding ∞ into the complex plane can be seen as plugging a hole remaining in the space since divergent sequences (those approaching ∞) could potentially "converge to ∞". In this sense the point ∞ is viewed as a limit point of $\mathbb{C}$, and including it would complete $\mathbb{C}$. For this reason, $\mathbb{C}_\infty$ is often called the closure of $\mathbb{C}$, and denoted $\overline{\mathbb{C}}$ or $\hat{\mathbb{C}}$.

Definition 2.19. The *stereographic projection* from $\mathbb{C}$ to $\mathbb{C}_\infty$ can be written as $\pi : \mathbb{C} \to \mathbb{C}_\infty$ such that $\pi(z) = t \cdot (2x, 2y, |z|^2 - 1)$ for $z = x + iy$ and $t = \frac{1}{|z|^2+1}$.

Avoiding complete topological justification, we note that the reliability of this new space is guaranteed since $\mathbb{C}_\infty$ and $\mathbb{C}$ are equivalent metric spaces under an isomorphism $\pi : \mathbb{C} \to \mathbb{C}_\infty$. Here, π behaves as a stereographic projection from $\mathbb{C}$ to $\mathbb{C}_\infty$, with $\mathbb{C}_\infty$ appearing as a sphere. Note that ∞ is seated neatly atop the projection at the north pole and 0 hangs from its south pole. The intersection of the plane and sphere is the unit circle (serving as the equator of $\mathbb{C}_\infty$), and all points along the sphere map in a bijective fashion to $\mathbb{C}$.

It is possible to induce many metrics on $\mathbb{C}_\infty$ and from these we choose the chordal metric, due to its versatility in dealing with our newfound point ∞.

Definition 2.20. The *chordal metric* on $\mathbb{C}_\infty$ is defined as $\sigma(z, w) = |\pi(z) - \pi(w)|$. Note that in this case, the absolute value bar denotes the Euclidean length of the now-$\mathbb{R}^3$ vectors z and w after having been mapped through π.

Due to the bijectivity of the map π, we find that $\mathbb{C}_\infty$ and $\mathbb{C}$ are equivalent under π as metrics. In short, we are now able to switch between the use of $\mathbb{C}$ and $\mathbb{C}_\infty$ in matters where it is convenient or useful (such as needing to alleviate ∞ of its significance).

2.2 COMPLEX NUMBERS SYSTEM

A deep understanding of complex-valued functions is paramount to progressing through the capstone proof of this thesis. This section aims to demonstrate the properties of complex-valued functions to give an intuition for the later portions of the thesis.

Any function that maps $\mathbb{C}$ into itself is called a *complex-valued function*. These functions (often called "maps") differ from real-valued functions only in their domain and codomain. From this point forward, unless otherwise specified, all results and discussion surrounding functions are assumed to reside in the complex plane.

Definition 2.21. A function $f : \mathbb{C} \to \mathbb{C}$ is a *complex-valued function*.

Since $\mathbb{C}$ requires two dimensions to visualize, it is difficult to view functions in the normal sense. This is to say we can not use typical x and y axes. We'll require one plane for our domain and another for the range. It is also common to show arrows

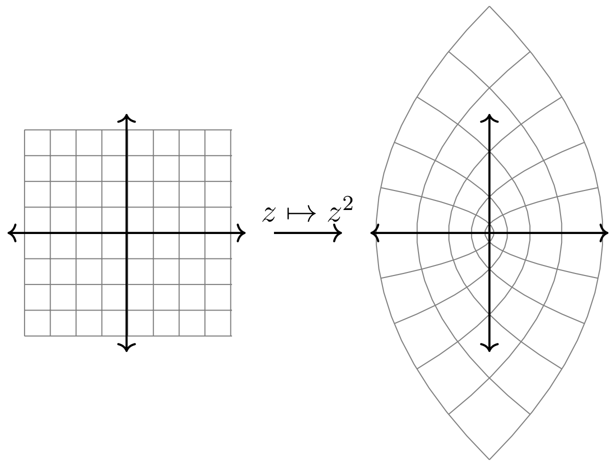


Figure 2.12: Image of the shown grid through z^2.

that demonstrate the movement occurring in the plane. The first idea is exemplified in figure 2.12.

On the left, we see a particular section of the complex plane, and on the right, we see those same lines after transformation $f(z) = z^2$. Later we demonstrate that this particular map has the property of duplicating all regions (mapping two points to the same place), so there are fewer squares visible in the right image than in the left one. Specifically, we might say this map is not surjective: there exists $z, w \in \mathbb{C}$, with $z \neq w$, such that $f(z) = f(w)$. For example $f(1) = f(-1)$, since $(-1)^2 = 1$.

Example 2.22. Suppose $f(z) = z^2$. Then $1 \mapsto 1$, $i \mapsto -1$, and so forth.

While mapping certain points through the function is fun, it may be easier to realize the behavior of a particular function by mapping multiple points at a time. Fortunately, we can map as many points as we want at once by considering the images of sets.

Definition 2.23. For any set $C \subset \mathbb{C}$, the *image* of C through f is denoted $f(C)$ and given by:

$$f(C) = \{w \in \mathbb{C} : f(c) = w,\ c \in C\}$$

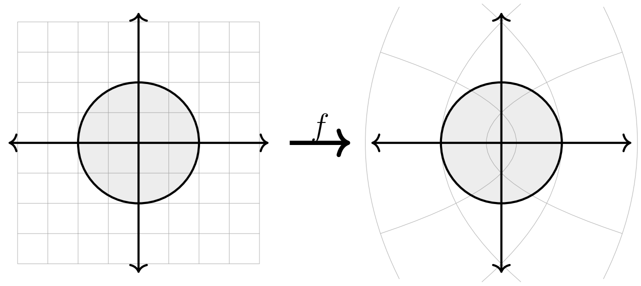


Figure 2.13: Image of the closed disk through $f(z) = z^2$.

Thanks to the properties of $\mathbb{C}$, this can be intuitively thought of as the function moving regions of $\mathbb{C}$ around. An example of doing this mathematically is given next.

Example 2.24. Suppose $f(z) = z^2$ and $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ (the closed unit disk). It is easier in this case to think of f in its polar form. Then suppose $z = re^{i\theta}$ for r the magnitude of z and θ the argument of z. Thus $f(z) = z^2 = \left(re^{i\theta}\right)^2 = r^2e^{i(2\theta)}$. Assuming $z \in \overline{\mathbb{D}}$, we know $|r| \leq 1$ by construction. Thus $|r^2| \leq 1$, showing $r^2e^{i(2\theta)} \in \overline{\mathbb{D}}$ (since any argument θ lands in the unit circle provided $|r| \leq 1$). This proves $f(\overline{\mathbb{D}}) = \overline{\mathbb{D}}$. We also note that $f(1) = 1$, and $f(0) = 0$. These are special cases for the particular map $z \mapsto z^2$. They are the only points that do not move, remaining fixed. Additionally, 0 is the only number in $\mathbb{C}$ having only one preimage (defined next in definition 2.25.)

Shown in figure 2.13 are the depicted domain and image of f respectively. We note that each point in the image of f corresponds to two values, since $f(z) = z^2$. That is, f is a 2-to-1 mapping. As shown, points outside the unit circle are blown up and wrapped counter-clockwise around the origin, while those inside the unit circle sink in. This is because the magnitude is squared, so magnitudes r with $|r| > 1$ bet bigger, while those r with $|r| < 1$ get smaller. The spiraling is the result of the argument of z being added to itself.

One can also take a preimage of a set, which is the equivalent of “undoing” an action caused by a function.

Definition 2.25. For any set $C \subset \mathbb{C}$, the *preimage* of C through f is denoted $f^{-1}(C)$ and given by:

$$f^{-1}(C) = \{z \in \mathbb{C} : f(z) = c, c \in C\}$$

Note the key differences between this definition and its predecessor. In this case, we observe what f maps to C, whereas we previously saw where f sent C. We again observe the example of $z \mapsto z^2$.

Example 2.26. Let $f(z) = z^2$. Then $f^{-1}(z) = \sqrt{z} = z^{1/2}$. Consider the set $C = \{z \in \mathbb{C} : |z - 1 - i| < 1\}$. Thus C is the disk of radius one, centered at $1 + i$. When attempting to find $f^{-1}(C)$ we immediately notice an issue. Each value of C has two possible values that could have been mapped to it. For example, consider $1+i \in C$. In polar form, $1+i = \sqrt{2}e^{\pi i/4}$. Thus $f^{-1}(\sqrt{2}e^{\pi i/4}) = 2^{1/4}e^{\pi i/8+2k\pi}$ for $k \in \mathbb{Z}$ since there are many angles whose doubling is equivalent to $\pi/4$. For this reason, we typically limit the angles to their principal values, thereby removing the "$+2k\pi$" term. For now, we'll assume that there are only 2 such values. Shown in figure 2.14

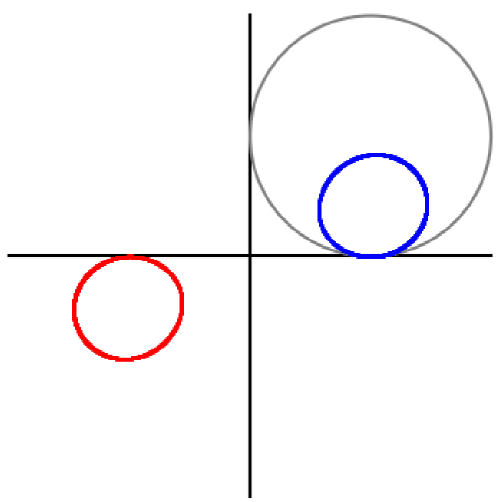


Figure 2.14: 1st Preimage

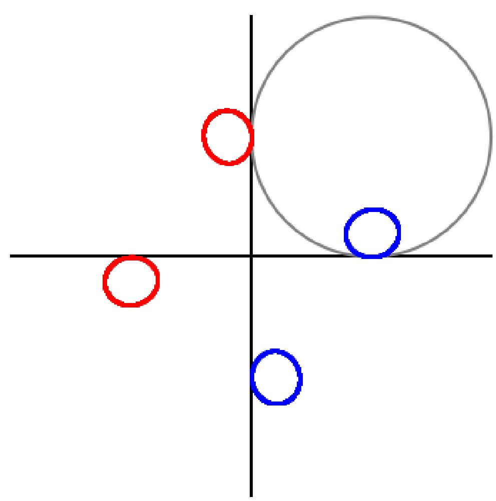


Figure 2.15: 2nd Preimage

are both regions in question. Each of these colored circles maps to the original circle (shown in gray). As shown in the next section, these two colored regions are called the *components* or *connected components* of C under f. Taking $f^{-1}(f^{-1}(C))$ further reveals the components of $f^{-1}(C)$, shown in figure 2.15. Since each component of C has two more components, we end up with four total possible regions that map to C (after two compositions).

Just as there are useful properties of real-valued functions that serve to illuminate their machinations, there are similar properties of complex-valued functions that can help us dig deeper. Our first such result is continuity, defined in its typical sense.

Definition 2.27. A function $f : \mathbb{C} \to \mathbb{C}$ is *continuous* on a set $D \subset \mathbb{C}$ provided that for all $\varepsilon > 0$, there exists some $\delta > 0$ such that:

$$|z - d| < \delta \implies |f(z) - f(d)| < \varepsilon,$$

for $d \in D$. In short, this definition requires that the closer z gets to d, the closer the image of z gets to the image of d.

Definition 2.28. For an open set $D \subset \mathbb{C}$, the function f is said to be *differentiable* on D whenever

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists for $a \in D$. This is, of course, the *derivative* of f at a.

Here we find the first big distinction between the complex and real numbers. In the real numbers, limits exist only when they are equal from both directions (left and right). However, the complex numbers are two-dimensional, thus requiring all two-dimensional paths that exist to agree. This means the existence of a complex derivative is much stronger than the existence of a real one.

Definition 2.29. A function $f : \mathbb{C} \to \mathbb{C}$ is *holomorphic* on a set $D \subset \mathbb{C}$ if f is differentiable in some neighborhood of each $d \in D$.

Definition 2.30. A function $f : \mathbb{C} \to \mathbb{C}_\infty$ is *meromorphic* on a set $D \subset \mathbb{C}$ if either f or $1/f$ is holomorphic on D.

The meromorphic definition may seem overshadowed by its predecessor, but its existence is used interchangeably with holomorphic, especially in dealings with the Riemann sphere. Since the Riemann sphere includes this new point at infinity, we define holomorphic/meromorphic for ∞ next.

Definition 2.31. A function $f(z)$ is holomorphic (or meromorphic) at ∞ if $f(1/z)$ is holomorphic (or meromorphic) at 0.

Example 2.32. Consider $f(z) = z^2$. Note that z^2 is differentiable in a neighborhood of 0 and thus we say $f(z)$ is holomorphic at zero. Is $f(z)$ holomorphic at ∞? By definition, this limit is not finite and cannot exist. However, $1/f(z) = z^{-2}$ is holomorphic at ∞, since $1/f\left(\frac{1}{z}\right) = z^2$ is holomorphic at 0. In this case, as is often done in texts, we say f is holomorphic at ∞ since f is meromorphic at ∞. This example demonstrates the power of the Riemann sphere to remove the uniqueness from infinity, allowing it to be analyzed as any point by simply considering the reciprocal mapping.

We could also similarly have continuity and convergent sequences near ∞. We need only flip over to $1/f$ for points approaching ∞ to tend to 0, making divergent behavior appear to converge to ∞. It is worth noting that neighborhoods of ∞ are typically defined this way in texts.

Definition 2.33. A function $f : \mathbb{C} \to \mathbb{C}$ is *conformal* at z_0 if and only if the derivative of f exists at z_0 and is nonzero.

Conformal maps have the unique property of locally preserving angles. That is, the angle between intersecting lines is unchanged through the transformation. However, conformality does not ensure that parallel lines remain parallel (as seen in figure 2.12).

Definition 2.34. A function $f : \mathbb{C} \to \mathbb{C}$ is *analytic* at a point z_0 if $f'(z_0)$ exists and is continuous at z_0.

Theorem 2.35. *If a function $f : \mathbb{C} \to \mathbb{C}$ is analytic at a point z_0, then:*

$$f(z_0) = \sum_{n=0}^{\infty} a_n (z_0 - a)^n$$

with a nonzero radius of convergence.

This previous result follows from Taylor's Theorem and demonstrates any analytic function to be infinitely differentiable at z_0. This is often called the *polynomial expansion* of f at z_0. Under the umbrella of analytic functions, we can study many other strange types of functions by appealing to their polynomial expansions.

3 DYNAMICS

3.1 INTRODUCTION TO DYNAMICS

Many real-life processes deal with applying a certain action repeatedly or over time. If this action has a function representation, it can be repeatedly iterated, and its long-term behavior categorized. For example, if applying a particular sunscreen blocks 50% of damaging UV rays, one may re-apply to block more. Assuming this process blocks 50% of the remaining rays after each application, we can think of the UV rays blocked as a function of the previous application. That is, for f_n the amount of UV-rays blocked on the n^{th} application, $f_n = 0.5 f_{n-1}$. Then we could suppose that $f_0 = 1$, since after zero applications, you still receive 100% of the UV-rays. We now witness f_n unravel from its starting point, composing each iteration with the previous to see into the future applications of f.

For instance, $f_0 = 1 \implies f_1 = 0.5 \cdot f_0 = 0.5 \cdot 1 = 0.5 \implies f_2 = 0.5 \cdot f_1 = 0.25 \ldots$. To truly encapsulate the long-term behavior of such a sequence, one might form a graph of its outcomes.

We might say that the behavior of this particular function is very predictable. After a time, the sun's rays lessen in impact until they have a negligible impact (if any). In short, this process exhibits very stable behavior whenever the initial value is 1. We can also deal with other initial values, representing stronger or weaker versions of the sun's rays, and view their long-term "stability" as well (barring any environmental/astrological impacts of seriously altering the sun's brightness).

Admittedly, this example is quite simple and demonstrates only a small percentage of the eye-catching behaviors possible for maps of this type. There are many more real-world properties that are studied under the lens of their long-term behavior, with their properties and results being among the most popular mathematics results of recent. This field, prevalent in mathematics, physics, finance, and astronomy (among many others) is often called the study of dynamics. Among such examples include the swinging of a pendulum, population growth/decline, dribbling a basketball, and water flowing over rocks. While the study of such phenomena is difficult, their applications and beauty are found strewn across all fields of mathematics.

Remark 3.1. It is worth noting that the previous sunscreen example resides in $\mathbb{R}$ and it is the goal of this chapter and the next to translate this process to $\mathbb{C}$ and isolate a particular set of interesting behaviors.

3.2 ITERATING IN $\mathbb{C}$

We first give rise to a notational standard for iterating functions. For a function $f(z)$ we denote $f^1 = f(z)$, $f^2(z) = (f \circ f)(z)$, and $f^n(z) = (f \circ f^{n-1})(z)$. This is simply to define a notation f^n, meaning f composed with itself n times. This is the n^{th} step in an iterative sequence.

Definition 3.2. A *fixed point* z of a function f is any z in $\mathbb{C}$ (or sometimes $\mathbb{C}_\infty$) such that $f(z) = z$. That is, $z \mapsto z$.

Example 3.3. For $f : \mathbb{C} \to \mathbb{C}$, let $f(z) = z^2$. Then $f^2(z) = (z^2)^2 = z^4$. Inductively, $f^n(z) = z^{2^n}$. It is usually in the interest of mathematicians to partition space into regions that behave similarly. In this case, we note that values of $z = re^{i\theta}$ with $r > 1$ exhibit diverging behavior. Simply put, $f^n(re^{i\theta}) = \left(re^{i\theta}\right)^{2^n} = r^{2^n}e^{2^n i\theta}$. Then we note that as $n \to \infty$, $r^{2^n} \to \infty$ and so the region outside the unit disk maps further and further outside, approaching ∞. Note that on the Riemann sphere, we can simply write $f^n(\infty) = \infty$. Within the unit disk, $r < 1$, values map further inside, sinking in towards 0. Additionally, $f^n(0) = 0$. Lastly, to cover all of $\mathbb{C}_\infty$, we should also consider values where $r = 1$. Something interesting happens here. Notice that

$f^n(e^{i\theta}) = e^{2^n i\theta}$, which is $r = 1$. This says that the unit circle maps to itself, but $\arg z$ is altered. From complex analysis and our knowledge of roots of unity, we know that all arguments which are rational multiples of π eventually have $f^n(z) = 1$ for some n, and $f^n(1) = 1$. All those z with arguments that are irrational multiples of π never "settle down"; these iterates endlessly careen around the unit circle as n increases.

Thus we have found three distinct regions: two open sets and a closed set. The open sets are $\{z \in \mathbb{C} : |z| > 1\}$ and $\{z \in \mathbb{C} : |z| < 1\}$. These are the sets containing ∞ and 0 respectively. These set's elements contain neighborhoods that likewise tend toward their respective fixed points. However, the closed set bordered between them exhibits a much different behavior – a point's neighbors need not follow its example, need not tend to one, and need not even stay near. These two behaviors are the purpose of the discussion and are detailed further in later chapters.

Now that we have some idea of complex analysis, the topology of the complex plane and Riemann sphere, and a field of math called dynamics which is of interest to us, it is time to take what we know and form the basis of the theory of complex dynamics so we can begin discussing the Mandelbrot set. The usual way to do this is to try and understand the dynamics of a particular family of complex-valued functions, called rational maps. This is the subject of the next chapter. The following theorems and definitions come from Alan F. Beardon's *Iteration of Rational Functions* [2], where he provides a more in-depth discussion of general rational maps. Our focus lies primarily on degree two maps.

3.3 RATIONAL MAPS

Here we begin to explore the dynamics of a certain classification of maps called *rational maps.* Rational maps are constructed from two polynomials: one in the numerator, and one in the denominator. Since analytic maps have their Taylor series expansions as polynomials, understanding the dynamics of rational maps is the first step to understanding more general analytic maps.

Definition 3.4. A *rational map* $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is denoted $R(z)$ and is given by $R(z) = P(z)/Q(z)$, where P and Q are complex polynomials. Rational maps are defined even when $Q(z) = 0$, and in this case, we say $R(z) = \infty$, thanks to the Riemann sphere.

Rational maps are the basis for most complex dynamics and were the first maps discussed during the field's conception. It is worth noting that the polynomial maps of interest in this thesis are typically considered rational maps with $Q(z) = 1$, so the findings of rational maps generally apply to these as well.

We have previously defined $f^0(z) = z$ for any f. The n^{th} *image* of f is denoted $f^n(z)$ and is f composed with itself n times. Similarly, the n^{th} *preimage* of f is f^{-1} composed with itself n times.

We are very interested in composing f with itself, and as a result, the shorthand notation is very lenient in its depiction throughout most academic writing. For our purposes, we say $f^3(z) = f \circ f \circ f(z)$, or $f^{-3}(z) = f^{-1} \circ f^{-1} \circ f^{-1}(z)$. These are also sometimes called the forward or backward images of f respectively, and their use is so frequent we may even omit the typical composition notation and write $fff(z)$ to denote $f^3(z)$. This use is extended to other functions, and juxtaposed functions are assumed to be composed with one another. For example, $R(Q(f(z)))$ is denoted $RQfz$. We also call $f^n(z)$ the n^{th} forward orbit of z, and $f^{-n}(z)$ its n^{th} backward orbit.

Definition 3.5. A fixed point for a function f is sometimes called ζ and is given by $f(\zeta) = \zeta$.

A fixed point for any map is simply a point that maps to itself; one that does not move under the function iteration.

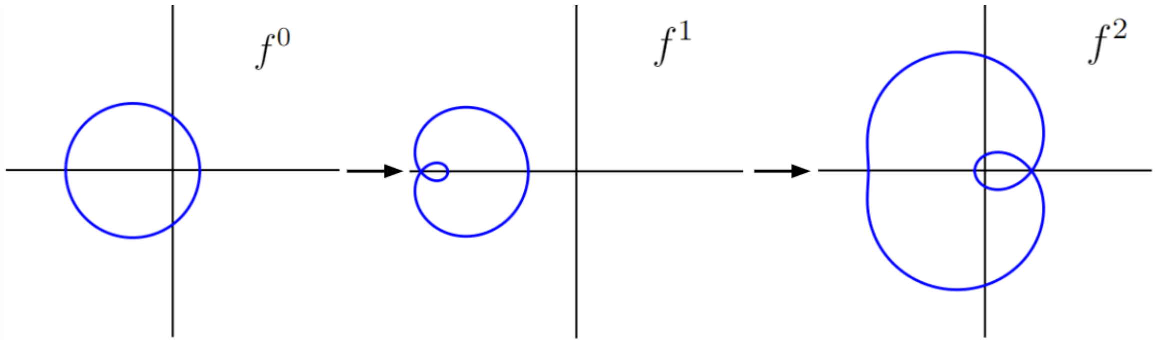


Figure 3.1: Forward image of a circle under $z \mapsto z^2 - 1$.

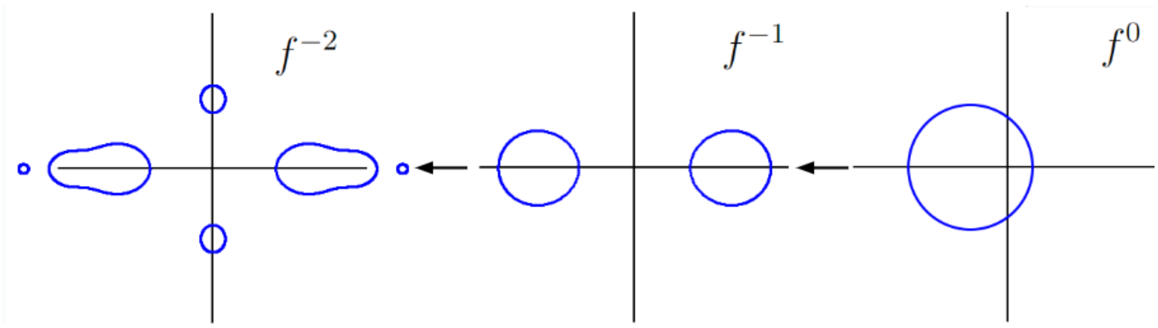


Figure 3.2: Backwards image of a circle under $z \mapsto z^2 - 1$

Example 3.6. Consider the function $f(z) = z^2 + i$. To find the fixed points for this map, we can solve the equation $f(z) = z$. It follows that:

$$\begin{aligned} f(z) &= z \\ z^2 + i &= z \\ z^2 - z + i &= 0 \end{aligned}$$

Thus $z = \frac{1 \pm \sqrt{1-4i}}{2}$ by the quadratic formula. These are illustrated below.

Later in the next chapter, we demonstrate a particular fascination with functions of the form $f(z) = z^2 + c$, for $c \in \mathbb{C}$. This form is sometimes called the *depleted quadratic*, as it lacks a first-degree term. The fixed points of a map with this form can be found by solving the familiar-looking equation $z^2 + c = z$. By the quadratic formula, we find $z = \frac{1 \pm \sqrt{1-4c}}{2}$.

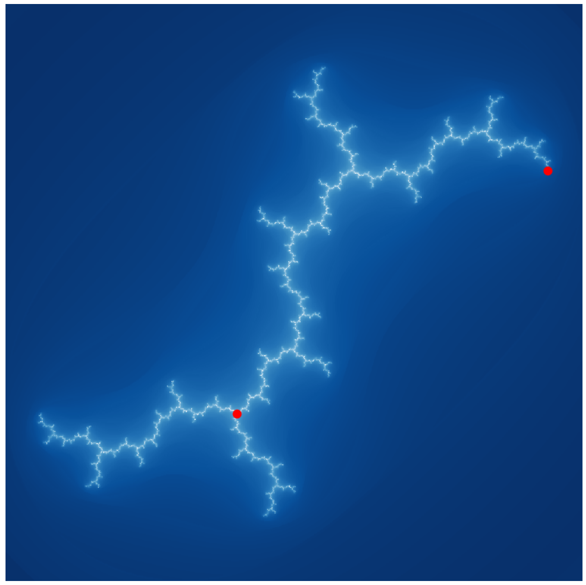


Figure 3.3: A dendrite with map $f(z) = z^2 + i$. Quickly-diverging points are bluer while slowly-diverging points are whiter.

Remark 3.7. We note that on the Riemann sphere, $f(\infty) = \infty$ for any value of c, thus making ∞ a fixed point for the depleted quadratic when dealing with $\mathbb{C}_\infty$.

Definition 3.8. A *cycle* of length n for a value z under$f$ is a fixed point of f^n.

An cycle of length n is a collection of n-many points that each have a cycle of n. For example, the forward orbit of 0 may have a cycle of 4 for a particular map. Then $f^4(0) = 0$. Thus the forward orbit of zero would oscillate as $\{0, f(0), f^2(0), f^3(0), 0, f(0), \ldots\}$.

In the case of the map $f(z) = z^2 + c$, we can solve for the orbits of length two by solving the equation $f^2(c) = z$, as such:

$$\begin{aligned} f^2(z) &= z \\ (z^2 + c)^2 + c &= z \\ z^4 + 2cz^2 - z + c^2 + c &= 0 \end{aligned}$$

This leaves us with a quartic equation to solve. To do this, we note that a fixed point in f also has a cycle of length 2. To show this, let ζ be a fixed point of f. Thus $f^2(\zeta) = f(f(\zeta)) = f(\zeta) = \zeta$. Therefore ζ has a cycle of length 2. Most notably, this means that $z - \frac{1\pm\sqrt{1-4c}}{2}$ are factors of the given quartic, and as such $z^2 - z + c$ evenly divides the quartic polynomial. Then we can perform polynomial long division.

$$\begin{array}{rr} & z^2 + z + (c+1) \\ z^2 - z + c & \overline{)z^4 + 2cz^2 - z + c^2 + c} \\ & \underline{-(z^4 - z^3 + cz^2)} \\ & z^3 + cz^2 - z + c^2 + c \\ & \underline{-(z^3 - z^2 + cz)} \\ & (c+1)z^2 + (-1-c)z + c^2 + c \\ & \underline{-\big((c+1)z^2 - (c+1)z + c(c+1)\big)} \\ & 0. \end{array}$$

Thus, $\big(z^4 + 2cz^2 - z + c^2 + c\big)/\big(z^2 - z + c\big) = z^2 + z + c + 1$. By the quadratic formula, we then conclude:

$$z = \frac{-1 \pm \sqrt{1 - 4(c+1)}}{2} = \frac{1}{2}\left(-1 \pm i\sqrt{3+4c}\right). \tag{3.1}$$

These are the orbits of length two, which are *not* length 1 (fixed). We can now apply this knowledge to show the orbits of length 2 on the dendrite, by letting $c = i$, yielding: $z = \frac{1}{2}\left(-1 \pm i\sqrt{3+4i}\right)$ (shown in figure 3.3).

In the case of the depleted quadratic map, there are always two orbits (the positive and negative). Since there are only two possible points, and each must have a cycle of length two, we can conclude that these points must map to each other. This is a fascinating result but is not true in general nor for other types of maps. We solidify this idea as a theorem.

Theorem 3.9. *Let $f_c(z) = z^2 + c$. The cycles of length 2 of f_c map to each other.*

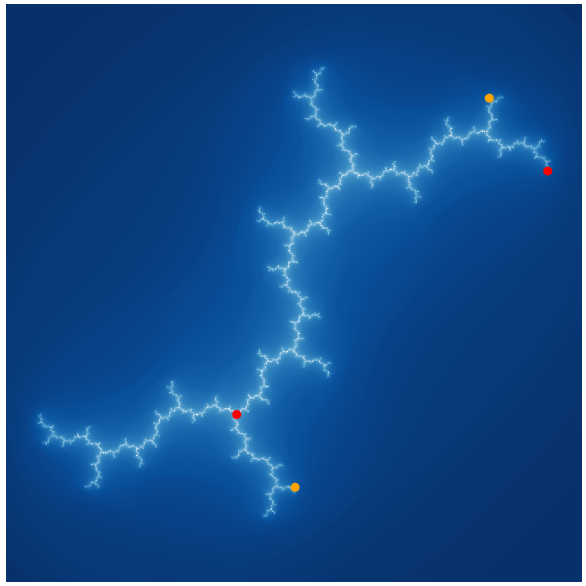


Figure 3.4: A dendrite, with fixed points in red and cycles of length two in orange.

Proof. By a previous result, we note that the orbits of length 2 which are not fixed can be written as $\zeta_1 = \frac{1}{2}\big(-1 + i\sqrt{3+4c}\big)$ and $\zeta_2 = \frac{1}{2}\big(-1 - i\sqrt{3+4c}\big)$. We can now demonstrate that $f(\zeta_1) = \zeta_2$ and $f(\zeta_2) = \zeta_1$.

$$\begin{aligned}
f(\zeta_1) &= (\zeta_1)^2 + c \\
&= \left(\frac{1}{2}\big(-1 + i\sqrt{3+4c}\big)\right)^2 + c \\
&= \frac{1}{4}\big(1 - 2i\sqrt{3+4c} + i^2(3+4c) + c \\
&= \frac{1}{4} - \frac{1}{2}i\sqrt{3+4c} - \frac{3}{4} - c + c \\
&= \frac{1}{2} - \frac{1}{2}i\sqrt{3+4c} \\
&= \frac{1}{2}\big(1 - i\sqrt{3+4c}\big) = \zeta_2.
\end{aligned}$$

Likewise we perform the case of $f(\zeta_2)$:

$$\begin{aligned}
f(\zeta_2) &= (\zeta_2)^2 + c \\
&= \left(\frac{1}{2}\big(-1 - i\sqrt{3+4c}\big)\right)^2 + c \\
&= \frac{1}{4}\big(1 + 2i\sqrt{3+4c} + i^2(3+4c) + c \\
&= \frac{1}{4} + \frac{1}{2}i\sqrt{3+4c} - \frac{3}{4} - c + c \\
&= \frac{1}{2} + \frac{1}{2}i\sqrt{3+4c} \\
&= \frac{1}{2}\big(1 + i\sqrt{3+4c}\big) = \zeta_1.
\end{aligned}$$

Then we have shown $f(\zeta_1) = \zeta_2$ and $f(\zeta_2) = \zeta_1$, and we conclude the orbits of length 2 of f_c map to each other. $\square$

We can also classify cycles/fixed points based on the derivative evaluated at the given point. Analytically we know the derivative to measure the amount of stretch/contraction in a neighborhood of the discussed point. Thus we can classify these points as such:

Definition 3.10. An n-cycle point ζ of the map f^n is:

1. *Repelling* if $|(f^n)'(\zeta)| > 1$,
2. *Indifferent* if $|(f^n)'(\zeta)| = 1$,
3. *Attracting* if $|(f^n)'(\zeta)| < 1$,
4. *Super-attracting* if $|(f^n)'(\zeta)| = 0$.

This definition opens the door to discussing when and where certain cycles occur with a classified behavior, such as finding all the points with an attracting three-cycle. This is touched on further in Chapter 4.

Occasionally we observe a ‘fixed behavior’ for an entire set. To avoid confusion, sets with these properties are called invariant sets.

Definition 3.11. A set C is *forward invariant* in f provided that $f(C) = C$. Similarly, a set is *backward invariant* if $f^{-1}(C) = C$ and *completely invariant* if it is both forward and backward invariant.

Note that this does not necessarily imply elements of C are fixed points. It is still possible for elements of C to move around as long as they stay in C.

Completely invariant sets are instrumental but we will not be using them until the following chapter, where we can form a sense of continuity between function iterates. For now, in keeping with the dendrite theme, the map $z \mapsto z^2 + i$ (figure 3.3) has two invariant sets: the white set and the blue set.

4 JULIA AND FATOU SETS

The Julia and Fatou sets get their name from their math-famous counterparts who studied them. Pierre Fatou and Gaston Julia were French mathematicians around the time after World War I, and just before the precipice of the modern computing era. While dynamics as a field had yet to be conceived, these men are attributed with establishing the theory in the case of holomorphic dynamics. The following theorems and results are continuations from the previous section. They are all found in Beardon's *Iteration of Rational Functions* [2], mostly from Chapters 5 and 6.

4.1 DEFINING THE FATOU AND JULIA SETS

In short, these sets serve to demonstrate the two possible behaviors examined in holomorphic dynamics: regions of stability and instability. Before defining these sets we must first borrow from the theory of equicontinuity.

Definition 4.1. A *family* of functions $\{f_n\}$ is simply a collection of functions of some type. We are most interested in the family of the iterates of f.

Example 4.2. The function $f(z) = z^2$ has a family of iterates:

$$\{f_n\} = \{\dots, f^{-2}, f^{-1}, f^0, f, f^2, \dots\} = \{\dots, z^{1/4}, z^{1/2}, z, z^2, z^4, \dots\}.$$

Definition 4.3. A family of functions $\{f_n\}$ is *equicontinuous* at a point $z_0 \in D \subset \mathbb{C}$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $f \in \{f_n\}$ and all $z \in D$:

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon.$$

For a set $D \subset \mathbb{C}$, the family $\{f_n\}$ is equicontinuous on D if the family $\{f_n\}$ is equicontinuous at each $z_0 \in D$. This in turn implies that for various subsets D_ν on which $\{f_n\}$ is equicontinuous, $\bigcup D_\nu$ is also an equicontinuous region for $\{f_n\}$.

The presence of equicontinuity on a given set D for a family of iterates $\{f_n\}$ means neighborhoods of points in D behave similarly through iteration. If equicontinuity fails, then some local behaviors are not preserved under iteration. This leads us to our first definition of the Fatou set.

Definition 4.4. The *Fatou set* of a function g is denoted $F(g)$ or simply F, and is the maximal open subset of $\mathbb{C}_\infty$ on which the family of iterates of g, $\{g_n\}$, is equicontinuous.

Just from the definition, we first form that F is an open subset of $\mathbb{C}_\infty$, dealing with the entire family of iterates of some function (g in this case). It is also the maximal version of itself, meaning there is one and only one F for a particular family of iterates. We are now ready to define the Julia set.

Definition 4.5. The *Julia set* of a function g is denoted $J(g)$ or simply J and is the complement of F_g in $\mathbb{C}_\infty$. Alternatively, this is the region of $\mathbb{C}_\infty$ on which the family of iterates of g, $\{g_n\}$, is not equicontinuous.

It is now visible to the observer the reasoning for declaring F and J to be stable and unstable outside of simple intuition. These regions of $\mathbb{C}_\infty$ for a given function isolate the preservation of closeness across all the iterates of a particular function. In its truest form, the regions of $\mathbb{C}_\infty$ which belong to J fail to preserve continuity across either the whole region or all of its iterates. The result is that some kind of 'tearing' is happening to the region locally over time.

Similarly, equicontinuity guarantees that regions contained in F stay 'knit-together' as the iterates progress, always guaranteed to have some sense of continuity between any points in f and iterate of the function. In dynamics, J and F are like black and white; complete, utter, and total opposites.

4.2 PROPERTIES OF THE FATOU SET

The Fatou set, while very interesting, gets most of its attention as being the counterpart to the much more chaotic and difficult-to-fathom Julia set. This text is no exception, and we begin to understand the Fatou set as a means to understand J. However, before diving into Fatou sets, we first require some prerequisite ideas from topology. These results help solidify stability in terms of the metric space $\mathbb{C}_\infty$.

Definition 4.6. A set D is *simply connected* if any closed curve can be continuously deformed to a point without leaving D.

In metric spaces, simply connected sets have no holes. If ever a hole occurs in a given set, then the loop around that hole is suddenly no longer able to be contracted to a point. This means their sets can be connected and not simply connected (see the left image of figure 4.2).

Remark 4.7. Our current definition of simply connected allows for disconnected open sets with no holes to be simply connected since no closed curves joining the two spaces need to be considered, as they are not contained in the space.

Theorem 4.8. *The closure of a connected set is connected.*

A connected set in the closure of $\mathbb{C}$ is the collection of all of its accumulation points. Since this set is already connected, it is not jeopardized by the addition of its accumulation points. This is shown below:

It is worth noting the converse of this statement is not necessarily true, and the reader is encouraged to break the converse with a counterexample.

Theorem 4.9. *A domain D is simply connected if and only if its complement is connected.*

This result seems unintuitive at first but can be quickly understood in the context of metric spaces. If the domain D is simply connected, then there are no holes. Thus,

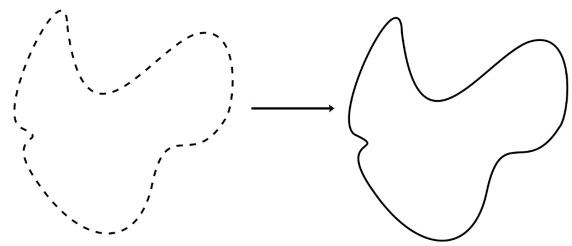


Figure 4.1: The closure of a connected set is connected.

the complement of such as shape cannot have disjoint pieces. This result is most helpfully demonstrated in an example.

Shown in figure 4.2 is a projection of a coffee mug (which can represent an open region in $\mathbb{C}$). Taking its complement in $\mathbb{C}$ reveals two disjoint pieces – the hole in the handle becomes an island, separate from the rest of the set and breaking connectivity.

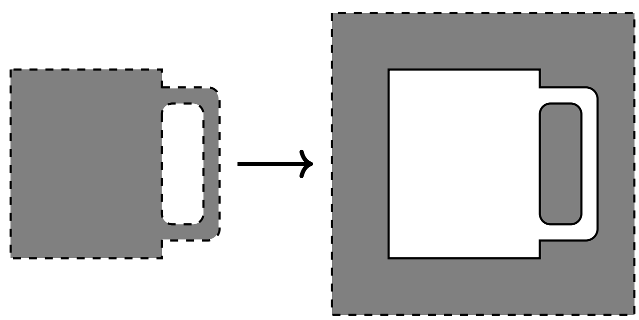


Figure 4.2: A set is not simply connected if its complement is disconnected.

The next image (figure 4.3) is that same coffee mug but with its handle removed. This region has no handle, and thus no problematic holes to be turned into islands. The result is a shape that is most certainly connected.

Theorem 4.10. *A domain D is simply connected if and only if its boundary is connected.*

Intuitively, we can imagine that a shape that is simply connected in $\mathbb{C}$ (no holes) would have one border—a closed loop around the outside (figure 4.3). However, a

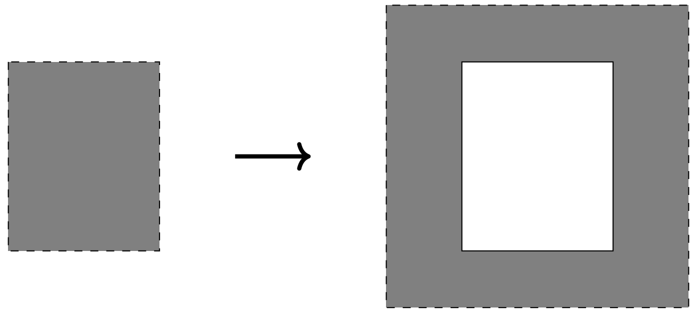


Figure 4.3: A set is simply connected if its complement is connected.

shape with one or more holes would break up its border, since the hole also needs its border (figure 4.2).

4.3 PROPERTIES OF THE JULIA SET

It is worth reminding the reader that the Fatou and Julia sets differ absolutely; they are opposites of one another, and their behavior is such. In general, the Fatou set is wide and stable, while the Julia set is thin and unstable. The Fatou set cradles its fixed points in sinks so that nearby points spiral towards them. The Julia set holds its fixed points on mountain peaks, with neighborhoods diverging from it rapidly and in all directions.

We first consider the Julia set of some simpler rational functions before moving on to degree two polynomials. We set the stage with the deceptively simple map $f(z) = z + 1$. In the spirit of the text, we ask: what is $F(f)$ and $J(f)$? Since the two are complements, it is enough to find one of them. To do this, we should first observe the family of iterates of f. It follows from a simple argument that $f^n(z) = z + n$. Thus for any starting point z, the n^{th} iterate lies n units away in the direction of the positive real axis. This is equivalent to shifting the entire plane by 1 in the real direction of each step of composition. This is loosely demonstrated in figure 4.4.

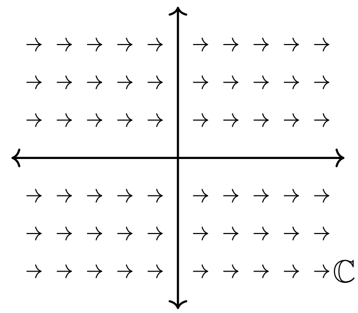


Figure 4.4

On $\mathbb{C}$, $\{f_n\}$ is equicontinuous for every z, since for a given $\varepsilon > 0$ choose $\delta = \varepsilon$ so that $|z - z_0| < \delta \implies |f(z) - f(z_0)| = |z + n - z_0 - n| = |z - z_0| < \delta = \varepsilon$. However, the question of ∞ remains. We first note that ∞ is a fixed point for the map f, since $f(\infty) = \infty$. However, when viewing the Riemann sphere, we see that ∞ behaves like a sort of 'saddle' fixed point, with one side sinking in, and the other drifting away. This behavior prevents equicontinuity at ∞, thereby solidifying $F(f) = \mathbb{C}$ and $J(f) = \infty$.

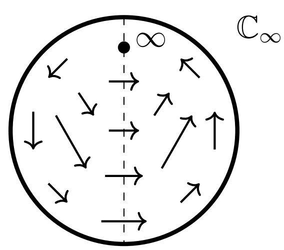


Figure 4.5: Movements of points induced by $z \mapsto z + 1$

Above is the map f on a very rough Riemann sphere that has been tilted a little bit forward to show ∞ at the top. The median line represents the line $\mathrm{RE}(z) = 0$. On the $\mathrm{RE}(z) > 0$ half of the sphere, points converge to ∞. On the $\mathrm{RE}(z) < 0$ side, points diverge from ∞ and remove the ability for $\{f_n\}$ to be equicontinuous, thereby placing $\infty \in J(f)$.

Remark 4.11. While it is possible for F to be empty, these maps are few and far between, with their discoveries previously being a great achievement [2]. Their properties are, however, outside the scope of the thesis.

The thin, wiry nature of J is exemplified in sets called *dendrites*.

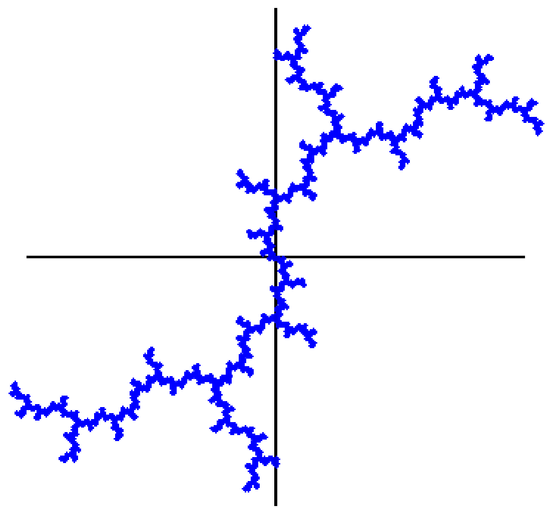


Figure 4.6: Julia set of $z \mapsto z^2 + i$

Intuitively, "thin" and "thick" refer to the size of the interior; thick shapes should have a wide interior, and thin shapes should have an almost empty interior. In the case of J, it is as thin as possible. This is demonstrated by the theorem below.

Theorem 4.12. *If J is not equal to $\mathbb{C}_\infty$, then J has an empty interior. [2]*

This result loosely follows from F consuming the sphere to maintain its equicontinuous nature. Without specifics of the Juila set it is proven by first showing J to be *minimal*, meaning J has no components that map to each other. Since F does consist of these components, it tends to take up more space on the sphere and edges out J.

Some examples are shown in Figure 4.7. Note that the Julia set is the region bordered between solid white and blue (this is the result of a computational approximation of the Julia set).

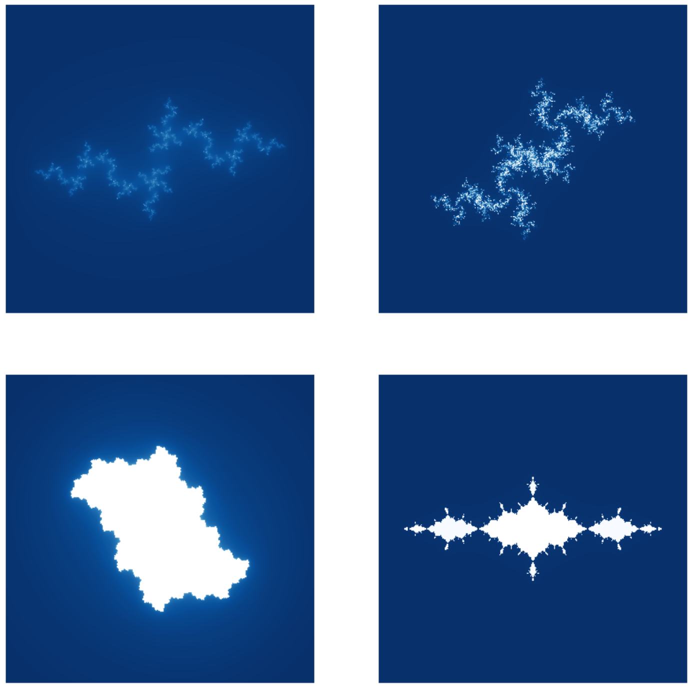


Figure 4.7: Julia sets of $z^2 - 1 + 0.35i$, $z^2 - 0.5i$, $z^2 - 0.226 + 0.7449i$, and $z^2 - 1.2$.

In this section we provide the crux of the thesis; we intend to demonstrate that rational maps of the form $z \mapsto z^2 + c$ for $c \in \mathbb{C}$ exhibit specifically identifiable properties that can allow us to categorize the parameter c in meaningful ways. Specifically, we intend to demonstrate their connectivity and implement a strategy for determining the connectivity of these sets. We then parameterize c by its resulting connected sets and form the Mandelbrot set as a final result before exploring it in the later chapters.

4.3.1 THE BASILICA

For our first example let us consider in full the case when $c = -1$. This produces the map $f(z) = z^2 - 1$. When iterated, this map characterizes regions of the plane by their orbits. We know that depleted quadratic maps have two fixed points (in $\mathbb{C}$) counting multiplicity (barring $c = 0$, who has one fixed point), and ∞ also behaves like a fixed point. These fixed points are given by $\zeta_1 = \frac{1}{2}\big(1+i\sqrt{5}\big)$ and $\zeta_2 = \frac{1}{2}\big(1-i\sqrt{5}\big)$ 3.6. Since $f'(z) = 2z$, we see that $|f'(\zeta_1)| = 2\left|\frac{1}{2}\big(1+i\sqrt{5}\big)\right| = 6$. By similar reasoning we find $|f'(\zeta_2)| = 4$. This classifies both points as repelling and cements them in the Julia set of f, which we denote by $\zeta_1, \zeta_2 \in J(f)$.

The astute may recall we can also find the points with an orbit of length 2 by similar methods. While this is not useful for all maps, in the case of $c = -1$ it is particularly enlightening. To find these orbits, we'll use our established formula as $\zeta_3 = \frac{1}{2}\big(-1+i\sqrt{3-4}\big) = \frac{1}{2}\left(-1+i\sqrt{-1}\right) = \frac{1}{2}\left(-1+i^2\right) = -1$. Similarly $\zeta_4 = 0$. This can be reaffirmed since $f(2) = (-1)^2 - 1 = 0$, and $f(0) = 0^2 - 1 = -1$. Most notably, $|f'(0)| = 0$, proving ζ_3 and ζ_4 to form a sinking orbit of 2. This places ζ_3 and ζ_4 in the Fatou set. Since the Fatou set is defined in terms of equicontinuity, we know that points neighboring -1 form an orbit with neighborhoods of 0. That is, points in the Fatou set near -1 must converge to the orbit between -1 and 0.

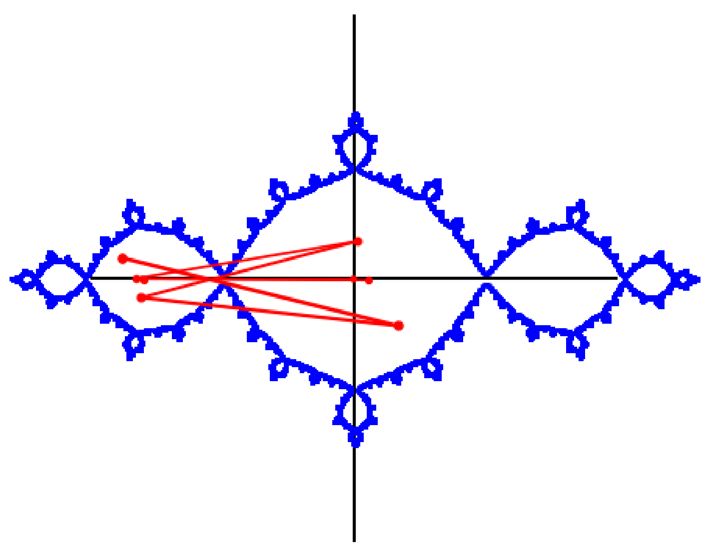


Figure 4.8: An orbit starting at $-1.1 + 0.1i$ converges to the 2-cycle

One can also imagine taking this to the next level, solving increasingly higher-degree polynomials, each with solutions guaranteed to exist by the fundamental theorem of algebra. These points lie in J (since ∞ is fixed). This reasoning concludes that the various bulbs seen in 4.8 must correspond to some orbit length, with many being coupled by their orbits. However, solving these higher-degree polynomials quickly becomes tricky, and often researchers implement a form of Newton's method to find these roots since the derivatives of polynomials are straightforward. Performing this reveals that all bulbs eventually map back to the two-cycle (due to their super-attracting nature). An example is given. Before moving on, we should also take a

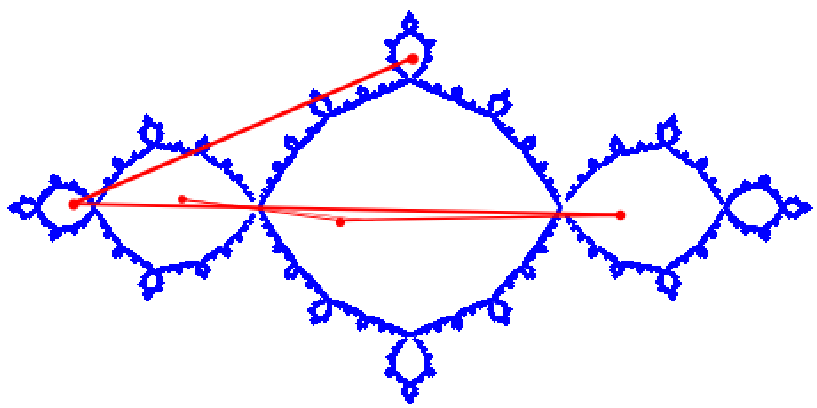


Figure 4.9: A chosen point in one of the northern bulbs converges to the orbit of 2 near zero.

minute to observe F_0 and F_∞. In the case of 0, we find not only that $0 \in F$, but 0 is a member of a super-attracting orbit of length 2. Likewise, ∞ also behaves like an attracting fixed point. Thus the region F_∞ forms the outside of the basilica, and the regions similar to F_0 form the various bulbs. Since these shapes are equicontinuous by definition and completely invariant by a previous result, regions contained in any of these bulbs inevitably expand and fill each bulb under backward iteration. This shows each region inside the basilica to *be* F_0.

The idea of F_0 can help us answer if any maps behave like the given Basilica (are there other Basilicas?). In short, the answer is *yes*! We demonstrate this after the next example, the rabbit.

4.3.2 THE DOUADY RABBIT and VARIANTS

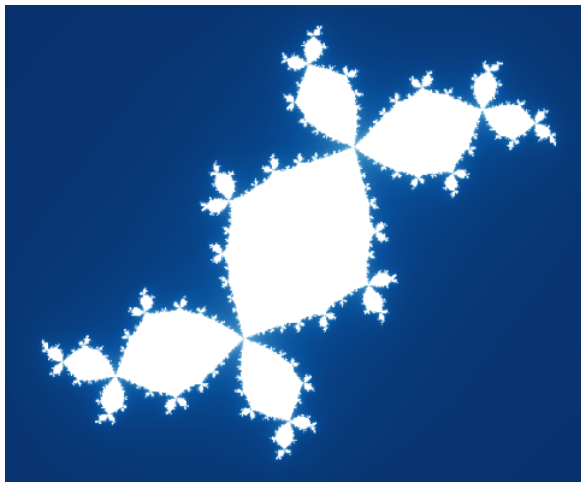


Figure 4.10: Julia set for $c = -0.12256 + 0.74486i$

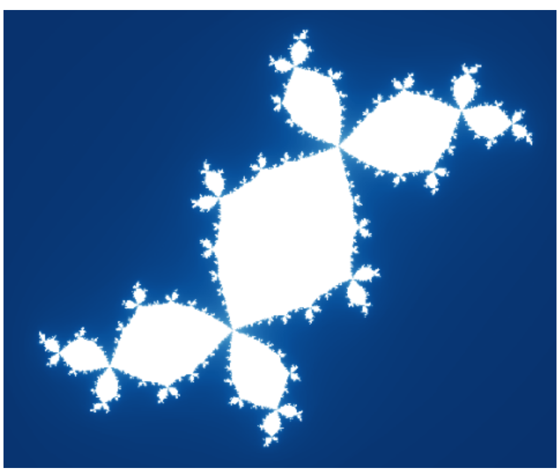


Figure 4.11: Douady Rabbit: $c = -0.12256 + 0.74486i$

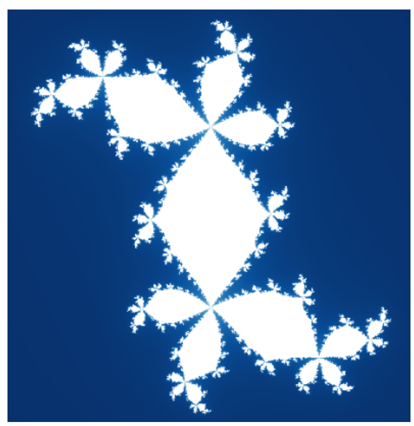


Figure 4.12: Quadruple Rabbit: $c = 0.282271 - 0.530061i$

Similar to the basilica, we note that the formation of bulbs here is due to the length of the cycle at 0. This can be achieved by solving the resulting quadratic from $f^2(0) = 0$ for c. This is the definition for 0 to have a cycle of 2. We first get $c^2+c = 0$, begetting $c = 0$ and $c = -1$ (the basilica).

To find others, one can simply solve higher compositions, such as $f^3(0) = 0$, which yields $(c^2+c)^2+c = 0$. Removing the fact that $c = 0$, we solve the resulting cubic to find that $c_1 \approx -1.75488$, $c_2 \approx -0.12256 + 0.74486i$ and $c_3 = \overline{c_2}$. Note these are the approximate solutions since the exact solutions involve radicals.

Extending this idea further is only natural, but the higher-degree polynomials require computational work. Additionally, each polynomial in terms of c have a degree twice that of the previous, resulting in exponential growth of the number of Julia sets meeting the given criteria, barring repeats due to the repeats such as those discussed in equation 3.6. Below are those for which F_0 is a sinking orbit of length 4. These are sometimes called quadruple rabbits. Both the plane and regular variants exist.

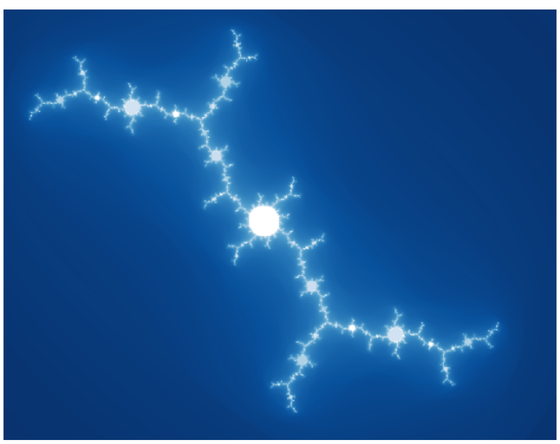


Figure 4.13: Twisted Plane-Like Rabbit: $c = -0.156520 - 1.032247i$

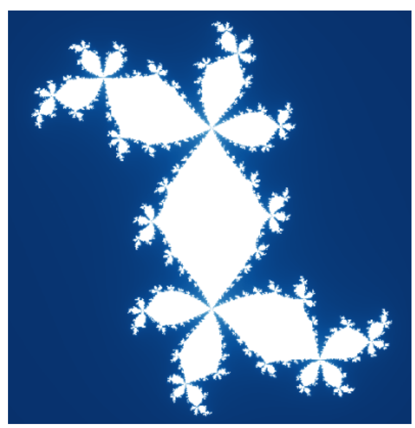


Figure 4.14: Quadruple Rabbit: $c = 0.282271 - 0.530061i$

4.3.3 THE SEAHORSE

The seahorses get their name from the swirly nature they present. These shapes result from high-valued orbits for F_0. One is shown here only to visualize the intricacy possible from Juilia/Fatou sets. Due to this level of detail, clear images of these sets are tough to produce. While these images are clear, they are technically imperfect representations – merely approximations of a shape with infinite intricacy littered throughout.

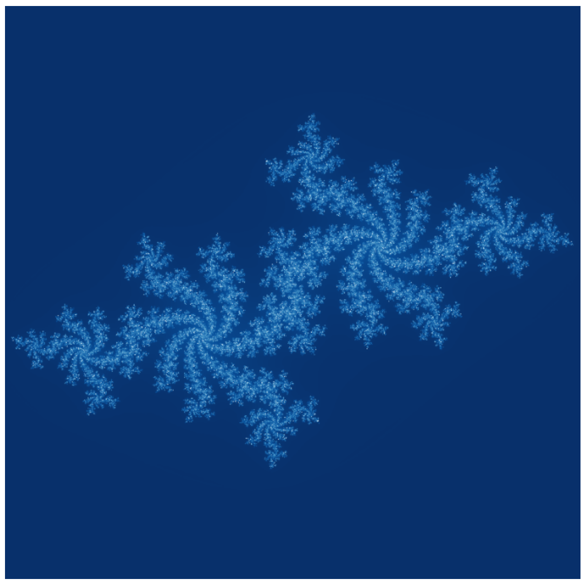


Figure 4.15: A seahorse Julia set for $f(z) = z^2+$

4.3.4 THE DENDRITE

The dendrite is very difficult to find, very difficult to draw, and—most importantly—they represent the edge between maps that have connected Julia sets and those that have disconnected ones. In essence, these dendrites visually show us what it looks like to be on the edge of divergence, poetically demonstrating connectivity as a mathematical consequence of the very shape sets can form. To understand dendrites is to understand Julia sets.

We have previously used the example of $z \mapsto z + i$ as a dendrite. While there are many, they are particularly difficult to find due to their nature of lying between connected and not. We later see that this is because dendrites occur on the border of the Mandelbrot set. For now, we provide an image of $c = i$.

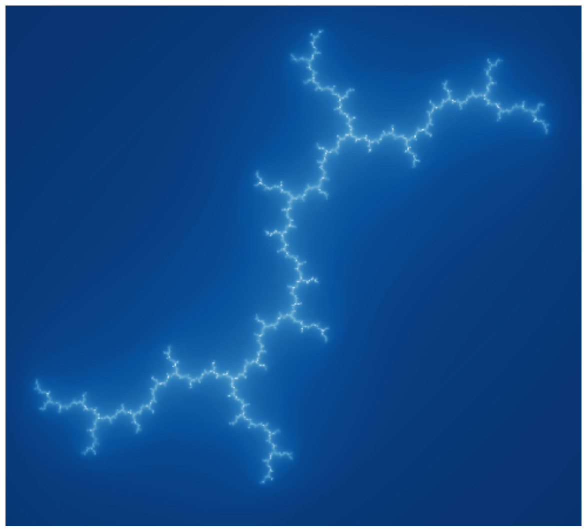


Figure 4.16: Dendrite: $c = i$

So far we have viewed Julia sets that are connected (in this section). To understand this, we begin considering the backward images of F_∞. Of course, if we were to start near ∞, we would never reach the Julia set in backward images. It is thus wise to start near where we believe the border to be. Instead, pick a circle, arc, or line and view its possible preimages. Since F_∞ is equicontinuous and backward invariant, our points *never* map to J, causing the edge of the region to approach the Julia set.

Additionally, since ∞ is a sinking fixed point (or at least behaves like one on $\mathbb{C}_\infty$) we know that iterates tend away from ∞ and toward 0 since we are looking at preimages. Thus the resultant map produces simply connected regions (on $\mathbb{C}_\infty$) for each step in the iteration. Go back far enough, and we meet the border of F_∞, which is J. This is exemplified a in figures 4.17, 4.18, and 4.19.

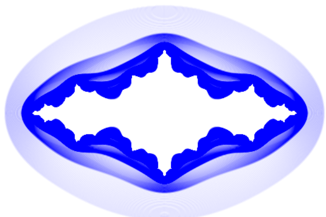


Figure 4.17: Approaching Basilica: $c = -1$

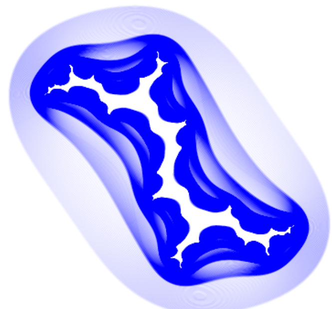


Figure 4.18: Approaching Dendrite: $c = i$

Figure 4.19: Approaching Rabbit: $c = -0.12256 + 0.74486i$

Since each region encroaches on J a little further, we can see that these regions approach J as the backward image $(f_c^{-n}(z))$ approaches ∞. Since each of these regions is simply connected, we state that its complement is connected by theorem 4.9. As such, J is connected.

4.3.5 CANTOR DUST

Julia sets can also be disconnected. Using the previous argument of preimages, we know that maps with disconnected Julia sets have J split repeatedly. For this reason, after splitting infinitely many times, J is not only disconnected but totally disconnected. This means no two points of J are connected. When this happens, we say that a set is constructed of disjoint points. This is often referred to as Cantor dust, due to Cantor's similar examples found in topology. We would say that these sets have *measure zero*—in our case, this means that the set J has no length and no area. This is in stark contrast to connected sets J, which have infinite length.

Note. In figure 4.21, we see the splitting of J happening as the Fatou set at infinity tears through, mapping over regions and removing the possibility for J to exist there. Since F_∞ has no simply-connected preimages (besides the trivial case), we conclude

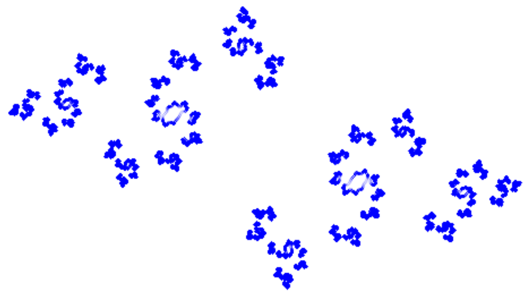


Figure 4.20: Cantor Dust

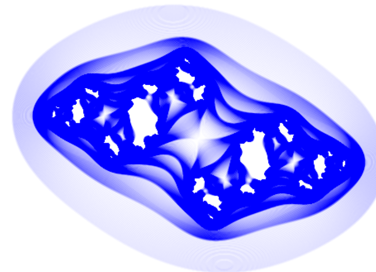


Figure 4.21: Approaching Cantor Dust

that J is not connected.

4.3.6 CRITICAL POINTS

In the previous examples, the points zero and ∞ characterize the Fatou and Julia set for a particular c value. In this subsection we illustrate the reason for this by the use of critical points.

Definition 4.13. A point ζ is a critical point for a function f if $|f'(z)| = 0$.

Note this is the calculus definition of the critical point, and we have previously assigned this behavior to be super-attracting when ζ is a fixed point. It might come as no surprise that for the depleted quadratic $f_c(z) = z^2 + c$, we have $f_c'(z) = 2z$, which has critical points at 0 and ∞ when dealing in $\mathbb{C}_\infty$.

The characteristics of these critical points uniquely determine the dynamics of the map for a given c. For example, supposing you wanted to find Julia sets where zero admits a cycle of one, then solve the polynomial given by $f(0) = 0$ for c. That is $0^2 + c = 0 \implies c = 0$. Then the only map that fixes 0 is z^2. Continuing onward, we may also find those where 0 admits a 2-cycle: $f_c^2(0) = 0 \implies c^2 + c = 0$. Then $c = 0$ and $c = -1$. Here we ignore $c = 0$ since we know it to be a fixed point from the previous result. Additionally, we have seen $c = -1$ to represent the basilica, which was already shown for zero to admit a two-cycle.

Theorem 4.14. *For any n, there exists a $c \in \mathbb{C}$ such that 0 has an n-cycle in $f_c(z) = z^2 + c$.*

The proof of this theorem follows directly from the fundamental theorem of algebra since $f^n(0) = 0$ is a complex-valued polynomial. Trying this same strategy but with a general value z_0 is framed as solving for c in the expression $f_c(z_0) = z_0$, which results in systems similar to that of (3.6) in chapter 3.

Attempting this strategy for ∞ yields $f^n(\infty) = \infty$. Since $f_c^n(z)$ is a polynomial, we know that ∞ is fixed. This means ∞ is a fixed point for all n and for every c of the depleted quadratic. We note that $|(f_c^n)'(\infty)| = \infty$ by similar reasoning. Thus f_c^n meromorphically distinguishes ∞ as a super-attracting n-cycle on the sphere for all n. For this reason, $\infty \in F(f_c)$ for all c.

With this knowledge, we are ready to discuss the Mandelbrot set.

5 THE MANDELBROT SET

So far, we have demonstrated that for any unique c, there is a unique Julia set J_c corresponding to it, under the iteration of the map $z \mapsto z^2 + c$. However, even between the fascinating behaviors exhibited prior, there is one distinct behavior of these Julia sets which we would like to capture: their connectivity. We have observed a few examples demonstrating that only two possibilities occur: either a Julia set is connected (these are the filled Julia sets), or it is completely disconnected (the Cantor-dust Julia sets).

One convenient way of visualizing this would be to take those values of $c \in \mathbb{C}$ who have connected Julia sets and color them (say, black), then leave the values of c who have Cantor Julia sets white. We would have then created a parameter space for c, thereby partitioning $\mathbb{C}$ into two distinct regions. This process produces quite an interesting figure.

Figure 5.1 depicts this partition. The white regions are the values of c which have connected Julia sets, often called the Mandelbrot set. Although to grasp a definition of the Mandelbrot set (often called simply $\mathcal{M}$), we first offer a change in perspective.

5.1 DEFINING $\mathcal{M}$

Unfortunately, defining $\mathcal{M}$ in terms of the connectivity of its Julia sets is difficult and not very helpful to the arguments made later in the thesis. To achieve the desired feats, we first reconsider $\mathcal{M}$ rather as a collection of c values corresponding to bounded sequences.

Our previous discussion has mentioned connectivity and its closeness to F_0. In fact, should 0 be in the Fatou set and remain bounded, then J is connected. This

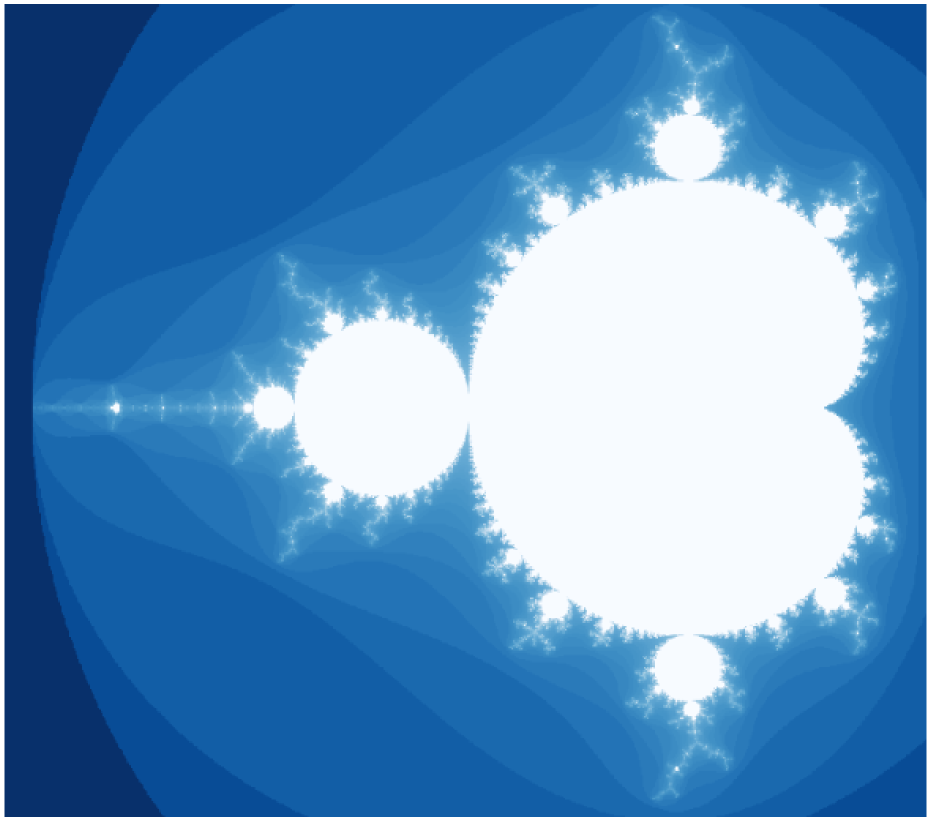


Figure 5.1: The Mandelbrot Set

is because F_0 must form a neighborhood near it to satisfy equicontinuity, thus sandwiching $J = \partial F$ between F_0 and F_∞. If ever F_0 and F_∞ were to touch, then F_0 ceases to exist (0 diverges to ∞) and is absorbed by ∞. This explains the sharp drop-off between connected and disconnected.

However, F_0 can only be separated from F_∞ if $F_0 \cup J$ remains bounded (otherwise those points would be in F_∞. This leads us to the newest definition of $\mathcal{M}$, but to make it easier we first define a recursive polynomial on which to base the definition.

Let $P_c(z) = z^2 + c$, and define the sequence $(c_n) = P_c^n(0)$ as the iterates of 0. We can now define the Mandelbrot set in its most common sense:

Definition 5.1. The Mandelbrot set $\mathcal{M}$ can be defined as:

$$\mathcal{M} = \{c \in \mathbb{C} : (c_n) \text{ is bounded}\}$$

It is worth noting that the next representation of $\mathcal{M}$ is its typical definition in most texts and papers due to its flexibility and simplicity, but there are many such representations of $\mathcal{M}$ useful in other contexts.

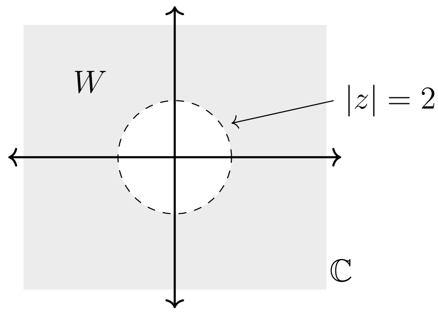
For each bounded sequence (c_n), we say that $c \in \mathcal{M}$. This means that the Julia set corresponding to c has separate Fatou sets for F_∞ and F_0, which distinguishes J as the connected border between them.

5.2 THE CONNECTIVITY OF $\mathcal{M}$

The next logical question to ask is the connectivity of $\mathcal{M}$, but not for the same reasons we have previously. In previous sections, connectivity is almost a gauge of a set's ability to keep itself together; in essence, as a way to remove tears. We have previously seen connectivity as the result of equicontinuity and invariance between maps. In the context of $\mathcal{M}$, however, we plan to use connectivity to answer the question of $\mathcal{M}$: are there Julia sets entirely separate from $\mathcal{M}$? That is, are we missing any? If we were to show that $\mathcal{M}$ is connected, then there would be no Mandelbrot islands, and this means we be able to find every Julia set without ever leaving $\mathcal{M}$. We begin by warming up a useful idea to show if something is connected with a simple proof, as featured in Alan Beardon's *Iteration of Rational Functions*, chapter 10 [2]. We be using this proof to help demonstrate why P_c has the effect of expanding $\mathbb{C}$ for values of $z \in \mathbb{C}$ with $|z| > 2$.

Theorem 5.2. *The Mandlebrot set can be expressed as the following:* $\mathcal{M} = \{c \in \mathbb{C} : |c_n| \leq 2\}$.

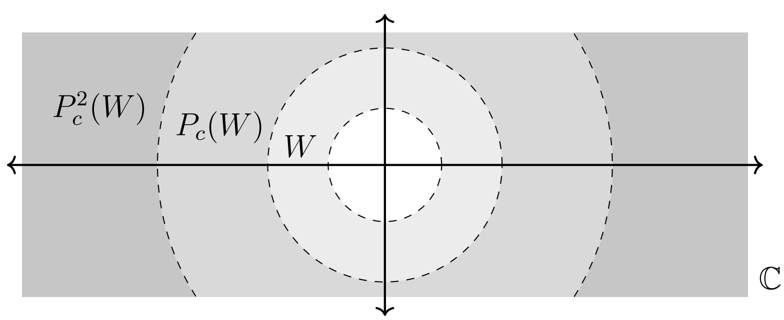
Proof. We start by considering the set: $W = \{z : |z| \geq |c|, |z| > 2\}$. This particular set can vary in construction since $|c|$ is not restricted; however, the construction of z need only guarantee it is strictly greater than 2, with $z \geq |c|$ making the proof a little easier without losing generality.



For each $z \in W$, there exists some $\varepsilon > 0$ such that $|z| \geq 2 + \varepsilon$, seeing as $|z| > 2$ and $|z| \geq |c|$. It follows that:

$$|P_c(z)| = |z^2 + c| \geq |z^2| - |c| \geq |z|^2 - |z| = |z|(|z| - 1) \geq |z|(1 + \varepsilon).$$

Since $1 < 1+\varepsilon$ it follows that $|z|(1+\varepsilon) > |z|$ and consequently $P_c(W) \subset W$. This is a demonstration that P_c maps all z with $|z| > 2$ even further away from the origin. Then, iterating once more, $P_c^2(W) \subset P_c(W)$ by similar reasoning. It follows from a simple inductive argument that $P_c^m(W) \to \infty$ as $m \to \infty$, visually demonstrated below.



We now see that for any sequence (c_n) with an element $|c_k| > 2$, the sequence (c_n) diverges, and as such $c \notin \mathcal{M}$. All sequences (c_n) such that $|c_n| \leq 2$ must remain bounded trivially. Then we can rewrite $\mathcal{M} = \{c \in \mathbb{C} : |c_n| \leq 2\}$. $\square$

We have now shifted from viewing $\mathcal{M}$ as the parameter space of the connectivity of filled Julia sets to viewing $\mathcal{M}$ as the collection of complex sequences bounded by two under the iteration of $P_c(z) = z^2 + c$. Armed with this newfound theorem, we are now prepared to demonstrate the core reasoning behind the connectivity of the Mandelbrot set. The following arguments are made in Dr. Alan Beardon's *Iteration of Rational Functions* [2], and its original conception is found in the paper *Exploring the Mandelbrot Set* by Adrien Douady and John Hubbard [5].

Theorem 5.3. *The set $\mathcal{M}$ is connected.*

Proof. As P_c is iterated, we compose polynomials with polynomials. It then stands to reason that, for a particular entry in the sequence c_k, $c_k = Q_k$ where Q_k is the inductively-defined polynomial $Q_{n+1}(c) = [Q_n(c)]^2 + c$. We now consider the closed disk of radius two, given by $K = \{z : |z| \leq 2\}$. Under Q_n, we know that $Q_{n+1}(\mathbb{C}_\infty \setminus K) \subset Q_n(\mathbb{C}_\infty \setminus K)$ from the previous proof. Thus $Q_{n+1}(K) \supset Q_n(K)$, and we write:

$$\mathcal{M} = \bigcap_{n=1}^{\infty} Q_n^{-1}(K)$$

$$\mathbb{C}_\infty \setminus \mathcal{M} = \bigcup_{n=1}^{\infty} Q_n^{-1}(\mathbb{C}_\infty \setminus K),$$

where the second equality follows from DeMorgan's Law. This shows $\mathcal{M}$ to be compact. Note that for any Q, $Q^{-1}(\mathbb{C}_\infty \setminus K)$ is open (since K is closed). Additionally, $Q^{-1}(\mathbb{C}_\infty \setminus K)$ cannot be expressed as the union of two disjoint open sets and is consequently connected. Lastly, $Q^{-1}(\mathbb{C}_\infty \setminus K)$ must contain ∞. Since each of these three properties is preserved under union, we find $\bigcup_{n=1}^{\infty} Q_n^{-1}(\mathbb{C}_\infty \setminus K)$ to share these traits. Therefore $\mathbb{C}_\infty \setminus \mathcal{M}$ is open and connected.

At this point, we begin constructing a conformal map from $\mathbb{C}_\infty \setminus \mathcal{M}$ to $\{z : |z| > 1\}$. Creating and demonstrating the conformal properties of such a map is the focus of the work of Douady and Hubbard, summarized in the first half of their paper *Exploring* *the Mandelbrot Set* [5]. We only dip our conceptual toes into the argument, which relies upon the theory of functions of two complex variables and Green's functions.

We first get φ_c, an isomorphism guaranteed by the Riemann mapping theorem. While φ_c varies for each c, each is of the form $\varphi_c : \mathbb{C} - K_f \to \mathbb{C} - \overline{\mathbb{D}}$, for K_f the connected components of f. In essence, we would show that Green's function $g(z) = \log|\varphi_c(c)|$ is positive in the neighborhood of ∞, then conjugate the polynomial $P_c(z) = z^2 + c$ to a map $z \mapsto z^d$. Since φ_c is the unique analytic function such that $\varphi_c(z) \to z$ as $z \to \infty$, and $\varphi_c(P_c(z)) = \varphi_c(c)^2$, we then construct its analytic continuation into $\mathbb{C}_\infty \times \mathbb{C}$.

To do this, one first demonstrates the map $g : (z, c) \mapsto g_c(z)$ is continuous and is represented as $g(z, c) = g_c(z) = 2^{-n} g_c(P_c^n(z))$. Thus, for a fixed value of n, we can imagine finding the roots (the 2^n-th roots) of $P_c^n(z)$, denoted $\Phi_n(z, c)$. By its construction:

$$\Phi_n(P_c(z), c) = [\Phi_{n+1}(z, c)]^2$$

$$\text{when } (z, c) \in \{(z, c) \in \mathbb{C}_\infty \times \mathbb{C} : c \in \mathbb{C} \setminus \mathcal{M}, z \in F_c, g_c(z) > g_c(0)\}.$$

In short, it can be shown that the sequence (Φ_n) converges uniformly to a function called simply Φ, which is analytic and behaves like φ_c for values of c near ∞. This is why Φ is called an analytic continuation.

Finally the results demonstrate $c \mapsto \varphi_c(c) = \Phi(c, c)$ for every $c \in \mathbb{C}_\infty \setminus \mathcal{M}$. Seeing as $g_c(z) > 0$ (by the properties of Green's function), then $|\varphi_c(c)| > 1$ and as such φ_c maps $\mathbb{C}_\infty \setminus \mathcal{M}$ into $\{|z| > 1\}$. Since this map is analytic, the local connectivity of $\{|z| > 1\}$ is preserved onto $\mathbb{C}_\infty \setminus \mathcal{M}$, and as such the region is also simply connected. We now recall from an earlier section that a set is simply connected if and only if its complement is connected. This shows $\mathcal{M}$ to be a connected set. $\square$

5.3 ADDITIONAL MANDELBROT SET INSIGHTS

5.3.1 DISCOVERED RESULTS

We have now shown the Mandelbrot set to undeniably be a connected set—to think it all boiled down to finding a map that takes $\mathbb{C} - \mathcal{M}$ to the outside the unit disk. We can now safely draw some additional lemmas of interest and explain their relevance before dipping into the things still unknown about the Mandelbrot set.

Fact 5.4. $\mathcal{M}$ can also be shown to be connected with a more topological argument. It is unclear who was first to do this, but there is a nice argument made in Kahn [6] which is included in the bibliography. It also borrows from the work of Douady and Hubbard, making it a very natural read should you feel comfortable with the previous proof.

Fact 5.5. $\mathcal{M}$ is simply connected [5]. Because of the map Φ we can also conclude $\mathcal{M}$ to be simply connected (that is, $\mathcal{M}$ has no holes). This result may seem dull, but holes on similar maps form whenever the function ends up wrapping back onto itself. These loops can create space inside the set and jeopardize its simple connectivity. These considerations are important to the Mandelbrot set because of its many spirals and links; the simple-connectivity means that any peninsula observed in any region of $\mathcal{M}$ must never loop completely back onto itself.

Fact 5.6. $\mathcal{M}$ is path connected. This is a topological result extending from the previous fact. The theorem mentioned in Chapter 3 of this work is from Conway [3] and also demonstrates this.

Fact 5.7. These facts together produce perhaps the most interesting result of all: all connected Julia sets are continuous deformations of one another. This idea is because each Julia set is related to its respective value of $c \in \mathbb{C}$. Then if we pick any two $c_1, c_2 \in \mathcal{M}$, there must exist a polygonal path between those points contained in $\mathcal{M}$, and thus the Julia sets at both values can be continuously deformed to one another by varying c along the path. (This is the reason that videos of flipping through Julia sets are so satisfying!) Of course, even the Cantor-dust Julia sets lying outside of $\mathcal{M}$ can be continuously deformed to any other $c \in \mathbb{C}$, but having this path in $\mathcal{M}$ shows the way these connected J are linked.

5.3.2 CONJECTURES AND QUESTIONS

The leading conjectures for the Mandelbrot set are some of the most interesting questions in dynamics. Solutions to these problems could open realms to new mathematical principles and ideas that apply to every field where the Mandelbrot set can be found. First among these is the notorious MLC: the 'Mandelbrot Locally-Connected' conjecture. We are currently unequipped to topologically define locally connected in the last few paragraphs of the thesis, but being locally connected at a point z essentially means that every open ball containing z also contains a ball that is connected and contains z. For the Mandelbrot set, this causes problems on the border, where thin, wiry regions may not admit to a small enough connected neighborhood being possible. It was mentioned by Douady and Hubbard [5] that the solution to the MLC is near, though it has been 15 years since and no solution has been found. The local connectivity is the main point of the article which we draw our capstone proof from; Douady and Hubbard require MLC to be true to draw several incredible conclusions about the Mandelbrot set and its border, all available to the reader in [5].

Another currently-unsolved Mandelbrot puzzle is the area of the Mandelbrot set. Researchers have bounded the area of $\mathcal{M}$ ($A_{\mathcal{M}}$) to $1.3744 < A_{\mathcal{M}} < 1.68288$, but is estimated to be around 1.5. The difficulty in these approximations has to do with the very thin regions of $\mathcal{M}$ along its border. If MLC were solved, this could help researchers make better estimates.

The Mandelbrot set is representative of mathematics in the sense that although challenging and difficult to understand, its feats of interest captivate us to solve new problems and answer new questions in hope of understanding something deeper. Without such goals, there is little meaning to continuing in the discovery and creation of mathematics. Mathematics, a lot like the Mandelbrot set, is a great and challenging mystery with captivating beauty.

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The style manual used in this thesis is A Manual For Authors of Mathematical Papers published by the American Mathematical Society.

This thesis was prepared by Eli Shirley using LaTeX.