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A History of Complex Simple Lie Algebras

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ABSTRACT

In 1869, prompted by his work in differential equations, Sophus Lie wondered about categorizing what he called "closed systems of commutative transformations," while around the same time, Wilhelm Killing's work on non-Euclidean geometry encountered related topics. As mathematicians recognized this as a division of abstract algebra, the area became known as "continuous transformation groups," but we now refer to them as Lie groups.

Patterns and structures emerged from their work, such as describing Lie groups in connection with their associated Lie algebras, which can be categorized in many important ways. In this paper, we focus on Lie algebras over the complex numbers, and how simplicity and the related notion of semisimplicity, as well as root spaces and their representations, reveal that there are, up to isomorphism, surprisingly few simple complex Lie algebras, a result which Killing examined intuitively.

Élie Cartan's influence on the development of the theory of Lie algebras, though chronologically slightly later, was key to making the theory of Lie algebras the influential topic it continues to be today. He brought the rigor Lie preferred to bear on ideas and patterns generated by Killing; among other impacts of his approach, in solving the classification problem of simple complex Lie algebras.

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No mathematician is an island—we emerge from complex systems. Here are a few elements of the system of which I am a part, without whom this thesis might never have been written.

My husband, Thomas Frazier, who puts up with the time this project has subtracted from our schedule and the complications it has added, takes as much home stuff off of my list as he can, and saw the "10PM boundary" come into existence.

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The Woodstock branch library in Portland, Oregon, a place set aside for books and ideas within bike riding distance for some of my most formative years.

CONTENTS

1 INTRODUCTION

In this paper, we will embark on an exploration of Lie algebras, which "ranks among the most important developments in modern mathematics" [16]. Our special interest will be those referred to by the name, which seems at first glance to be contradictory, "simple complex Lie algebras"—simple for their structure, which we will define in detail, and complex as in complex numbers, the algebraic closure of the real numbers. Following in the footsteps of leading early contributors, such as Sophus Lie, Wilhelm Killing, and most of all Élie Cartan, we shall see that there are in fact very few possible structures among simple complex Lie algebras. The importance of Lie algebras and their structures to modern mathematics and physics, about which it is said that "almost every subject in mathematics uses Lie groups" [17] led Nathan Jacobsen to write his book, *Lie Algebras*, published in 1962—less than a hundred years after the first discernible inklings of the topic—which was the first in the field to organize key ideas into a systematic introduction to Lie algebras [16].

Classifying math problems is key to identifying solution strategies, but before we can identify a problem as belonging to a particular category, we must determine a sensible classification method for the problems. Galois theory, for example, revolutionized the study of polynomial equations. By finding connections between field theory and group theory, one major contribution of Galios theory is a system for describing polynomials by their roots, which allows mathematicians to identify whether, and if possible how, polynomial equations can be solved.

Sophus Lie was aware of these developments, for while he was an undergraduate at the University of Christiania, he attended lectures on abstract algebra by Peter Ludvig Meidell Sylow, who found a useful tool for classifying groups. Though Lie claimed not to have understood many of the lectures, it is easy to suspect the idea of using abstract algebra as a classification tool for other problems remained as an influence for his future research. After receiving his undergraduate degree, Lie's career goal was to become a researcher—in what field, he hadn't determined. Before mathematics, he considered biology, chemistry, and physics, and it is plausible that the importance of applications of differential equations in these subjects influenced the formation of his eventual main idea. In 1867–68, he had and refined the idea to investigate what he referred to as "closed systems of infinitesimal transformations," to which we now refer in his honor as "Lie groups." Lie hoped to find a connection between systems of differential equations and their symmetries. This connection would prove to be informative about solutions of differential equations, comparable to how the relationships between fields and groups identified in Galois theory provided tools for classifying polynomial equations [23].

As it turned out, each of these Lie groups is related to a Lie algebra which as desired gives structural information. A *Lie algebra* L is a vector space, with all the familiar properties of vector addition and scalar multiplication, together with a bilinear map called the *bracket* from $L \times L$ to L, where each ordered pair of vectors $(x, y) \mapsto [x, y]$ such that

1. $[x, x] = 0$

2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, known as the Jacobi identity.

When we say the bracket operation is *bilinear*, we mean that it is linear with regard to both elements x and y of the Lie algebra.

Whenever the characteristic of the underlying field is not equal to 2, such as in the complex numbers which we will be using, the first condition is equivalent to requiring that $[x, y] = -[y, x]$. That is, Lie algebras are *anticommutative*.

The Jacobi identity is a particular case of the more general Jacobi Bracket Theorem for partial differential equations [31]. This axiom was not arbitrary: By reordering the Jacobi identity above to C, and considering x, y, and z as corresponding to f, g, annd the derivative respectively, we see a strong resemblance to the familiar product rule $(fg)' = (f')g + f(g')$. In this way, the Jacobi identity encodes derivative-like behavior needed based on Lie's interest in examining continuous transformation groups with the transformations characterized as functions related to systems of differential equations.

Other mathematicians would pick up similar and related ideas—Killing, around the same time as Lie, from a geometrical perspective; Weyl, later, in extending concepts as well making connections to theoretical physics; Dynkin with his diagrams for visualizing structures of Lie algebras; but the greatest contribution to Lie algebra was that of Élie Cartan. In 1894, Élie Cartan's doctoral thesis, "Sur la structure des groupes de transformations finis et continus (The structure of finite continuous groups of transformations)," shed much light on the topic of classifying Lie algebras. As we shall see in detail, he found, by considering the weight space decomposition of simple Lie algebras and the root systems associated with these decompositions, that only certain root systems were possible, and therefore, up to isomorphism, only certain simple Lie algebras [4].

1.1 Examples of Lie Algebras

To understand different kinds of Lie algebras, it will be useful to consider some examples.

Example 1.1 (Abelian Lie Algebras)**.** Let L be any vector space and define $[x, y] = 0$ for all $x, y, \in L$. These Lie algebras are referred to as abelian because, owing to anticommutativity, no other abelian Lie algebras are possible: the usual notion of what it means to be abelian would tell us that in such spaces $[x, y] = [y, x]$, but if we have simultaneously $[x, y] = -[y, x]$ (anticommutativity), as needed by a Lie algebra, it must be the case that $[y, x] = -[y, x]$, which is only feasible if $[y, x] = 0$.

Although at first glance abelian Lie algebras seem too easy to be interesting, they will play important roles. For example, Cartan subalgebras, a necessary structure for understanding the classification of simple Lie algebras, are abelian.

Example 1.2 (General Linear Algebras)**.** More interesting examples of Lie algebras occur when we consider the set of $n \times n$ matrices with complex entries, $\mathfrak{gl}_n(\mathbb{C})$. These are the same elements as in the familiar general linear group, $GL(n, \mathbb{C})$; using German calligraphy to refer to a Lie algebra by otherwise the same letters as a corresponding Lie algebra is a common communicational practice. Within the general linear algebras, we define the bracket product as $[x, y] = xy - yx$ for $x, y \in \mathfrak{gl}_n(\mathbb{C})$, where xy refers to the usual matrix product. The four classic Lie algebras, which will be described in Section 1.4, are *subalgebrassubalgebras* of $\mathfrak{gl}_n(\mathbb{C})$—that is, subsets which are closed under the bracket product $[x, y]$.

As we will make much use of $\mathfrak{gl}_n(\mathbb{C})$ and their subalgebras, it is well to demonstrate that they follow the Lie axioms. The first axiom, requiring that $[x, x] = 0$, is easy to demonstrate, as $[x, x]$ is defined to be $xx - xx$, which is obviously zero. Demonstrating the Jacobi identity requires a bit more work; we begin by developing a formula for $[x, [y, z]]$, one of the three terms of the Jacobi identity, using matrix algebra properties and the definition of $[x, y]$ in this proposed Lie algebra.

$$\begin{aligned}
[x, [y, z]] &= [x, (yz - zy)] \\
&= x(yz - zy) - (yz - zy)x \\
&= xyz - xzy - yzx + zyx.
\end{aligned}$$

Similarly, we find that

$$\begin{aligned}
[y,[z,x]] &= yzx - yxz - zxy + xzy \\
[z,[x,y]] &= zxy - zyx - xyz + yxz.
\end{aligned}$$

Consequently,

$$\begin{aligned}
[x,[y,z]] + [y,[z,x]] + [z,[x,y]] &= xyz - xzy - yzx + zyx + yzx - yxz \\
&\quad - zxy + xzy + zxy - zyx - xyz + yxz \\
&= 0.
\end{aligned}$$

Example 1.3 (One-Dimensional Lie Algebras)**.** The most elementary low-dimensional Lie algebra is the $n = 1$ case of the general linear algebra, $\mathfrak{gl}_1(\mathbb{C})$. By considering any two elements $x, y \in \mathfrak{gl}_1(\mathbb{C})$ and applying the bracket product $[x, y] = xy - yx$, we find that $\mathfrak{gl}_1(\mathbb{C})$ is abelian. In fact, any one-dimensional Lie algebra L will equally prove to be abelian: choosing any element $e \in L$ as a basis, any two elements $\{x, y\}$ can be written in the form $\{ae, be\}$ where a and b are scalars. Then by ordinary vector space properties we have $[ae, be] = ab[e, e]$, but owing to the first axiom of Lie algebras this is equivalent to $ab \cdot 0 = 0$.

Example 1.4 (Two-Dimensional Lie Algebras)**.** When a Lie algebra L is two-dimensional, we have more options as to its structure. It may be the case that L is abelian, in which case we can fully describe it by a basis $\{e_1, e_2\}$ and the relation $[e_1, e_2] = 0$.

On the other hand, if we suppose that a different two-dimensional Lie algebra with basis $\{f_1, f_2\}$ is not abelian, then the relation we use to define it must be in the form $[f_1, f_2] = a_1 f_1 + a_2 f_2$ where a_1 and a_2 are scalars. Then, we may compute the bracket product of any two elements $x = a_1f_1 + a_2f_2$ and $y = b_1f_1 + b_2f_2$ as follows:

$$\begin{aligned}
[x, y] &= [a_1f_1 + a_2f_2, b_1f_1 + b_2e_2] \\
&= [a_1f_1, b_1f_1 + b_2f_2] + [a_2f_2, b_1f_1 + b_2f_2] \\
&= [a_1f_1, b_1f_1] + [a_1f_1, b_2f_2] + [a_2f_2, b_1f_1] + [a_2f_2, b_2f_2] \\
&= a_1b_1[f_1, f_1] + a_1b_2[f_1, f_2] + a_2b_1[f_2, f_1] + a_2b_2[f_2, f_2] \\
&= a_1b_2[f_1, f_2] + a_2b_1[f_2, f_1] \\
&= a_1b_2[f_1, f_2] - a_2b_1[f_1, f_2] \\
&= (a_1b_2 - a_2b_1)[f_1, f_2].
\end{aligned}$$

When we collect the set $[L, L]$ of all bracket products $[x, y]$, where x and y are elements of any Lie algebra L, this set is called that Lie algebra's *derived algebra*. For the non-abelian two-dimensional Lie algebra L, our calculations above show that any element $[x, y]$ of $[L, L]$ is a scalar multiple of the bracket product $[f_1, f_2]$ of our original basis elements. So, $f = [f_1, f_2]$ serves as a one-element basis for $[L, L]$; therefore, the derived algebra $[L, L]$ of any two-dimensional non-abelian Lie algebra L is the one-dimensional Lie algebra.

Example 1.5 (Three-Dimensional Lie Algebras)**.** Among three-dimensional Lie algebras, there are more potential structures. Again, it is informative to consider the Lie algebra's derived algebra, which we can categorize by its dimension and whether it is abelian.

Assuming the Lie algebra is not itself abelian, one possibility is that the derived algebra is one-dimensional, therefore abelian. We can describe this algebra using the basis $\{e_1, e_2, z\}$, with the relations $[e_1, e_2] = z$, $[e_1, z] = 0$, and $[e_2, z] = 0$. This Lie algebra, called the Heisenberg algebra, is used in quantum physics to analyze problems such as those involving quantum harmonic oscillators [13]. It is isomorphic to the algebra of strictly upper triangular 3×3 matrices, with basis

$$e_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Another possibility is that the derived algebra is the one-dimensional non-abelian algebra, in which case the Lie algebra must be the direct sum of the one-dimensional non-abelian Lie algebra and the two-dimensional non-abelian Lie algebra. A three-dimensional Lie algebra may have as its derived algebra either of the two-dimensional algebras above. Or, it may be the case that a three-dimensional Lie algebra's derived algebra is also three-dimensional—that is, that $[L, L] = L$; the Lie algebra's derived algebra is itself, which we shall now see.

1.2 The Special Linear Algebra, $\mathfrak{sl}_2(\mathbb{C})$

One important subalgebra of the general linear algebra $\mathfrak{gl}_2(\mathbb{C})$ is the special linear algebra $\mathfrak{sl}_2(\mathbb{C})$, which is those matrices whose trace (that is, the total of the entries on their diagonals) is equal to zero. Because of this Lie algebra's key role in representation theory, we will often use it to illustrate examples of further concepts.

Any matrix

$$\begin{bmatrix} c & a \\ b & -c \end{bmatrix}$$

in $\mathfrak{sl}_2(\mathbb{C})$ can be written as $ax + by + ch$, using the convenient basis

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This basis can also help us in verifying $\mathfrak{sl}_2(\mathbb{C})$ is a Lie algebra. As mentioned previously, subsets of the general linear algebra $\mathfrak{gl}_2(\mathbb{C})$ such as $\mathfrak{sl}_2(\mathbb{C})$ inherit anticommutativity and the Jacobi identity, so we need only verify closure. Since we have a basis for $\mathfrak{sl}_2(\mathbb{C})$, one easy method to verify closure is to compute bracket products for all pairs of the basis elements, and to observe that the results are elements of $\mathfrak{sl}_2(\mathbb{C})$; as all other elements of the algebra are linear combinations of basis elements, the bracket products of pairs of elements of the algebra will be linear combinations of bracket products of basis elements.

The three bracket products of each basis element and itself are already determined by the first axiom of Lie algebras. The remaining six bracket products can be paired according to anticommutativity, allowing us to describe the full structure with the three relations $[x, y] = h$, $[h, x] = 2x$, and $[h, y] = -2y$, which can be verified using usual matrix multiplication. It is interesting that $[h, x]$ is a scalar multiple of x, and $[h, y]$ is a scalar multiple of y; we will see this again.

Another interesting feature of these three relations is that their results, $\{h, 2x, -2y\}$, are linearly independent of each other, since one is an element of the original basis and the other two are scalar multiples of different basis elements. Therefore, $\{h, 2x, -2y\}$ could also serve as a basis for $\mathfrak{sl}_2(\mathbb{C})$. This serves to prove that $\mathfrak{sl}_2(\mathbb{C})$ is the last type of low-dimensional Lie algebra described above in Example 1.5, where $[L, L] = L$.

1.3 Ideals

As Lie's idea was to classify continuous transformation groups, and many important structural features of a group are defined by the group's subgroups, it is unsurprising that analogous subsets play roles in describing the structures of Lie algebras. Some of these subsets, called *subalgebras*, though smaller are Lie algebras in their own right, inheriting all operational properties from the wider algebra but being also closed under the bracket operation. Not all subalgebras are created equal; some are called *ideals*, defined unsurprisingly as subalgebras $I \subset L$ with the property that, for any $y \in I$ and $x \in L$, $[y, x] \in I$. This property, somewhat stronger than closure, is called *absorption*.

Example 1.6. We have already seen one example of an ideal. Taking $\mathfrak{gl}_2(\mathbb{C})$ as the Lie algebra L, certainly $\mathfrak{sl}_2(\mathbb{C}) \subset \mathfrak{gl}_2(\mathbb{C})$, so to say that the 2×2 special linear algebra is an ideal of the 2×2 general linear algebra requires only verifying that, for any $y \in I$ and $x \in L$, $[y,x] \in I$. As it turns out, an even stronger statement holds: regardless of whether even one of the two is an element of $\mathfrak{sl}_2(\mathbb{C})$, any bracket product of two elements of $\mathfrak{gl}_2(\mathbb{C})$ is an element of $\mathfrak{sl}_2(\mathbb{C})$, due to the following theorem.

Theorem 1.7. *For any two $n\times n$ matrices a and b, the trace of $ab-ba$ is 0.*

Proof. We will use the dot product formula for matrix product entries, so, $(ab)_{ij} = \sum_{k=1}^{n} a_{in}b_{nj}$ and $(ba)_{ij} = \sum_{k=1}^{n} b_{in}a_{nj}$. However, we are only concerned with the entries on the diagonal, where $i=j$; $(ab)_{ii} = \sum_{k=1}^{n} a_{ik}b_{ki}$ and $(ba)_{ii} = \sum_{k=1}^{n} b_{ik}a_{ki} = \sum_{k=1}^{n} a_{ki}b_{ik}$.

To calculate the trace of $[a,b]$, we take the sum of the diagonal entries of $ab-ba$.

$$\begin{aligned}
\mathrm{Tr}(ab-ba) &= \sum_{i=1}^{n}(ab-ba)_{ii} \\
&= \sum_{i=1}^{n}\left((ab)_{ii}-(ba)_{ii}\right) \\
&= \sum_{i=1}^{n}\left(\sum_{k=1}^{n} a_{ik}b_{ki} - \sum_{k=1}^{n} a_{ki}b_{ik}\right) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} - \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ki}b_{ik} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} - \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ki}b_{ik}.
\end{aligned}$$

Noting that the letter assigned to an index variable is unimportant, we can interchange the index variables and rewrite the second term as $-\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki}$, making it apparent that the two terms are identical except for their sign. Therefore, the trace of $[a,b]$ is 0. □

The *trivial ideals* of any Lie algebra L are L itself and $\{0\}$; among Lie algebraists, $\{0\}$ is often written more concisely as 0. When a Lie algebra L has only trivial ideals, and $[L, L] \neq 0$, we refer to that algebra as *simple*. As a consequence, in simple Lie algebras, the center $Z(L)$ is the trivial ideal 0, and since the derived algebra $[L, L]$ must be some ideal due to the impossibility of escaping absorption, $[L, L]$ is the trivial ideal L.

Recalling that $\mathfrak{sl}_2(\mathbb{C})$, like simple Lie algebras, has the property that $[L, L] = L$, we may wonder whether $\mathfrak{sl}_2(\mathbb{C})$ is simple. In fact, we can prove that this is the case fairly directly.

Theorem 1.8. *The special linear algebra on 2×2 matrices, $\mathfrak{sl}_2(\mathbb{C})$, is simple.*

Proof. Let $I \subseteq \mathfrak{sl}_2(\mathbb{C})$ such that there is at least one nonzero element $ax+by+ch \in I$. Now we may examine what other elements must be in I for absorption to hold; we can begin by taking its bracket product with x:

$$\begin{aligned}[x, ax + by + ch] &= a[x, x] + b[x, y] + c[x, h] \\ &= bh - 2cx.\end{aligned}$$

Since $bh - 2cx \in I$, we can compute from it further elements which are required for absorption, such as by applying x again:

$$\begin{aligned}[x, bh - 2cx] &= b[x, h] - 2c[x, x] \\ &= -2bx.\end{aligned}$$

Applying a convenient scalar reveals that $x \in I$.

But then since $[x, y] = h$, we also have $h \in I$. And, applying another convenient scalar, $[-\frac{1}{2}h, y] = -\frac{1}{2}(-2y) = y$. Then, since we have the entire basis $\{x, y, h\} \subset I$, from which we could generate all elements of L as part of the ideal I, it must be the case that $I = \mathfrak{sl}_2(\mathbb{C})$. $\square$

1.4 The Classical Lie Algebras

It was Wilhelm Killing (1847—1923) who conceived of the problem of classifying all simple finite dimensional Lie algebras over the complex numbers. Killing worked worked on this problem for many years, publishing his research in *Mathematische Annalen* [18, 19, 20, 21]. Killing arrived at the conclusion that the only simple Lie algebras were four families of Lie algebras and a small number of Lie algebras that did not fit into any family, although his proofs were incomplete and sometimes wrong, as Cartan mentioned in the introduction to his thesis [4], which completely solved the classification problem.

One of Cartan's main goals in writing his thesis was to rigorously classify all simple Lie algebras over the complex numbers. He reworked the ideas and results of Killing, while adding ideas of his own such as the *Cartan-Killing form.* Many consider Cartan's thesis one of the great works of algebra in the nineteenth century; for example, Poincaré referred to Cartan's work, of which his thesis is a cornerstone, as "among the most important ... of mathematics" [6, 27]. Cartan determined that nearly all simple Lie algebras are isomorphic to one of four families of the classic Lie algebras, A_ℓ, B_ℓ, C_ℓ, or D_ℓ. In addition, there are five exceptional Lie algebras that do not belong to any of the classical Lie algebras: E_6, E_7, E_8, F_4, and G_2 [9, 15].

Élie Cartan (1869–1951) was born and raised in the village of Dolomieu, France, whose population was then about 2300, where his father was the village blacksmith. He was fortunate to be noticed while attending the village primary school and encouraged to continue his education beyond what was typically available to children from working-class families, secondary and university education being a luxury in 19th century France. By noticing and encouraging Cartan, his teacher M. Dupuis and school inspector Antonin Dubost plucked him and his family from obscurity. Cartan would influence his sister Anna to become a mathematics teacher, and his children would all go into lines of work diverging from their roots as rural peasants [25]. His son, Henri Cartan, would become an influential mathematician in his own right and a member of the Bourbaki textbook-writing collective.

But that was all far in the future, as the ten-year-old Élie Cartan began winning scholarships to attend secondary schools and was admitted to the highly selective École Normale Superieure in Paris. There, many of the leading mathematicians of the day were among the faculty he studied under, and his peer group was on the whole very academically inclined. One of his classmates, Arthur Tresse, had been a pupil of Lie, and suggested to Cartan the topic still known as finite continuous transformation groups. Cartan also examined Killing's work on the topic, but found it unsatisfactory, expressing in the introduction to his thesis the need for someone to revisit Killing's results but rigorously [4]. Much of Cartan's doctoral thesis consisted of putting the topic that would be known as Lie algebra on logically firm footing. Disenchantment with Killing's work would have endeared Cartan to Lie, due to the interpersonal and professional problems between Lie and Killing. It may also be the case that Lie's opinion of Killing influenced or reinforced Cartan's; whatever the precise case may have been, Lie had a high opinion of Cartan [12].

Here we will describe each of the families of classical Lie algebras, followed by the exceptional Lie algebras. In some places, we will use terminology in passing which remains to be defined later in our exploration of the topic.

The Special Linear Algebras, A_ℓ. The special linear algebra, A_ℓ, is the set of $(\ell+1)\times(\ell+1)$ matrices with trace 0. For example, the 2×2 special linear algebra, described above, is A_1. In fact, understanding $A_1 = \mathfrak{sl}_2(\mathbb{C})$ and its representations is key to understanding the work of Cartan and Killing.

The Orthogonal Algebras of Odd Dimension, B_ℓ. The orthogonal algebra, B_ℓ, is the set $\mathfrak{o}_{2\ell+1}(\mathbb{C})$ of matrices x of endomorphisms with the property

$f(x(v), w) = -f(v, x(w))$, where v and w are vectors, and f is a transformation whose matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{bmatrix}.$$

These matrices follow the form

$$\begin{bmatrix} 0 & b_1 & b_2 \\ -b_2^T & m & n \\ -b_1^T & p & -m^T \end{bmatrix}$$

where b_1 and b_2 are row vectors with ℓ elements, m, n, p, and q are $\ell \times \ell$ matrices, and both n and p have a property called *skew-symmetry*: taking their transpose in equivalent to taking their additive inverse, so $n^T = -n$ and $p^T = -p$.

When $\ell = 1$, we get B_1, where b_1, b_2, m, n, p, and q are single elements of F, and to make taking their transposes equivalent to taking their additive inverses, $n = p = 0$; so, B_1 is matrices in the form

$$\begin{bmatrix} 0 & b_1 & b_2 \\ -b_2 & b_3 & 0 \\ -b_1 & 0 & -b_3 \end{bmatrix}.$$

The Symplectic Algebras C_ℓ. The set of matrices in the form

$$\begin{bmatrix} m & n \\ p & -m^T \end{bmatrix},$$

where m, n, p are ℓ-by-ℓ matrices, and n and p are symmetric (that is, $n^T = n$ and $p^T = p$), is known as the *symplectic algebra*, C_ℓ.

The Orthogonal Algebras of Even Dimension, D_ℓ. These algebras are very similar to B_ℓ; they are the orthogonal algebra $\mathfrak{o}_{2\ell}(\mathbb{C})$ of endomorphic matrices which follow the formula $f(x(v), w) = -f(v, x(w))$. Matrices in these algebras follow the form

$$\begin{bmatrix} m & -p^T \\ p & n \end{bmatrix},$$

where m, n, and p are $\ell \times \ell$ matrices, and m and n are skew-symmetric.

The Exceptional Lie Algebras. In addition, there are five exceptional Lie algebras that do not belong to any family: E_6, E_7, E_8, F_4, and G_2. The exceptional Lie algebra called E_8 is a 248-dimensional algebra with many applications in particle physics, string theory, and crystallography [22, 30]. The Lie algebra G_2 arises in theoretical physics when one studies mirror symmetry [3]. Lie's rival, Wilhelm Killing, was aware of these as well as the other exceptional Lie algebras, E_6, E_7, and F_4, but rather than grasping their significance, he treated them as a problem to be eliminated [24].

1.5 Solvable and Semisimple Lie Algebras

In group theory, we define solvable groups as those groups whose *derived series*—that is, the sequence of groups formed iteratively by taking the derived group of the previous set—terminates in the trivial group. As this notion is important in Galois theory, finding it echoed in Lie algebra is unsurprising.

In Lie algebra, the *derived series* is the sequence of derived algebras of the previous step in the sequence: $L^{(n)} = [L^{(n-1)}, L^{(n-1)}]$, where $L^{(0)}$ is L itself. In some Lie algebras, by following the derived series sufficiently far, we find a value of n where $L^{(n)} = 0$; such Lie algebras are called *solvable*; such algebras are never simple, because simple Lie algebras have the property $[L, L] = L$ which prevents the derived series from progressing through any smaller ideals toward 0.

Semisimple algebras are defined by different books in two common ways. The most natural way of understanding why such algebras are called semisimple is by defining them as algebras which are direct sums of simple algebras. However, many sources such as [16, 15, 7] define semisimple algebras differently, as algebras whose maximal solvable ideal is 0.

Theorem 1.9 (Alternative definition of semisimple)**.** *The two definitions of semisimple are equivalent.*

Proof. First we must show that semisimple Lie algebras have maximal solvable ideal 0. Let L be a semisimple Lie algebra; then we can say $L = L_1 \oplus L_2 \oplus \cdots \oplus L_n$, where the L_i sets are simple Lie algebras. Let $I \subseteq L$ so that I is a solvable ideal of L; we can decompose I as well, as $I = (I \cap L_1) \oplus (I \cap L_2) \oplus \cdots \oplus (I \cap L_n)$.

Now let us consider the sets in the form $I \cap L_i$. Since the intersection of an ideal and a subalgebra is an ideal in that subalgebra, $I \cap L_i$ is an ideal in L_i. But, L_i is simple; so, either $I \cap L_i = 0$ or $I \cap L_i = L_i$. If, for each i, $I \cap L_i = 0$, then the direct sum of the $I \cap L_i$ algebras would be 0, so let us suppose for contradiction that there exists some i such that $I \cap L_i = L$.

As always, $I \cap L_i \subseteq I$. That is, L_i is a subalgebra of a solvable algebra, so L_i is solvable. However, L_i is defined to be simple, and Lie algebras cannot be both solvable and simple, because a solvable Lie algebra contains an abelian ideal as the penultimate step in its derived series, while a simple Lie algebra's derived algebra, and therefore every step it its derived series, is itself. Our supposition for contradiction must have been wrong: there is no i such that $I \cap L_i = L$, so it must be the case that $I \cap L_i = 0$. The direct sum of any number of 0 sets is 0, as desired.

We must also show that any Lie algebra with maximal solvable ideal 0 is semisimple. We will do so by describing a procedure for determining the decomposition.

If a Lie algebra L with maximal solvable ideal 0 contains no nontrivial ideals, L is simple, therefore semisimple; so, we will focus on the case where a Lie algebra L with maximal solvable ideal 0 contains at least one nontrivial ideal I. We know I is not solvable because the maximal solvable ideal of L is 0. Then we can consider the derived series of I, $I \supseteq I^{(1)} \supseteq I^{(2)} \supseteq ....$

Now let us consider the dimensions of the ideals in that sequence. Certainly a subset cannot have a greater dimension than its superset, so $\dim I \geq \dim I^{(1)} \geq \dim I^{(2)} \geq ...$. Since the dimension is a whole number, if the dimensions of each step of the derived series were strictly decreasing, it would necessarily be the case by the time $i = \dim I$ that $\dim I = 0$, i.e. that I is the trivial ideal 0. However, since I is not solvable, we cannot have $\dim I^{(i)} = 0$ for any i. Instead, it must be the case that there exists $n < \dim I$ such that $I^{(n-1)} = I^{(n)}$, and then for any $m \geq n$, $I^{(m)} = I^{(n)}$.

We will call this stopping point of the derived series J. If J is simple, let $L_1 = J$; that is, J is the first simple Lie algebra in the decomposition of L. However, if J is not simple, choose a nontrivial ideal K of J, and take the derived series of K to its stopping point. That stopping point again either is simple or may be iterated from, just like J and K, until finding a simple ideal which we may call L_1.

Then we can take L' such that $L = L_1 \oplus L'$, which we think of as the "rest of L" in the sense that it contains the structure of L not accounted for in L_1. Starting again with L', we can repeat the process until all needed simple ideals for the decomposition are found, as must happen because in a finite-dimensional space we must eventually run out of ideals. The collected results are a decomposition of L into simple Lie algebras, so L is semisimple. $\square$

Killing's definition of a semisimple Lie algebra, one with no nontrivial abelian ideals, is also equivalent to these two modern definitions [24]. Of the two modern definitions, the second, requiring a maximal solvable ideal 0, is easiest to compare with Killing's definition: an abelian ideal is certainly solvable, and a nontrivial solvable ideal certainly contains an abelian algebra as the penultimate element of its derived series. Perhaps the widespread use of the second definition, despite the first definition's more natural connection to the core idea for which the concept is named, reflects Killing's influence.

1.6 Nilpotent Lie Algebras and Homomorphisms

The derived series, for all its importance, is not the only sequence of ideals which can be generated by iteratively applying the bracket operation, beginning with all possible elements of a Lie algebra. The *descending central series* of a Lie algebra is a sequence of ideals defined by $L^0 = L$ and $L^n = [L, L^{n-1}]$. The similarity of notation between these two sequences can be confusing; as a mnemonic, we can remember that parentheses are used in matched pairs and the sequence term referred to with parentheses on the index, that of the derived series, is the result of taking the bracket product of a matched pair of the previous term.

Like a derived series, a descending central series may eventually have 0 as a term. When this happens—that is, there exists some N so that $L^n = 0$ for all $n \geq N$—the Lie algebra is called *nilpotent*. A sequence of subalgebras of a finite dimensional vector space V where the smallest subalgebra is 0, $0 = V_0 \subset V_1 \subset ... \subset V_n = V$, is called a *flag*.

Homomorphisms, more generally understood as operation-preserving functions, also occur in Lie algebra. Lie algebra homomorphisms are functions between two Lie algebras, preserving the bracket operation. That is, if f is a homomorphism between a Lie algebra domain L and a Lie algebra codomain M, such that, if l_1 and l_2 are elements of L, $m_1 = f(l_1)$, and $m_2 = f(l_2)$, $f([l_1, l_2]) = [m_1, m_2]$. Homomorphisms among Lie algebras can be of special types. Of particular interest are *isomorphisms*, i.e., homomorphisms which establish a one-to-one correspondence between elements of L and of M, and *endomorphisms*; i.e., homomorphisms where the domain L and codomain M are the same Lie algebra.

Like Lie algebras themselves, endomorphisms can be nilpotent. A *nilpotent endo-*

morphism is a homomorphism ϕ where, for some n, $\phi^n = 0$.

Through their mutual friend, the more famous mathematician Felix Klein, Friedrich Engel and Sophus Lie met and began working together in 1884 [26]. Seeing Lie's struggles to make his ideas comprehensible to others in the sixteen years since 1868, Klein suggested that Engel could help. Engel's help was in fact so extensive that "with the collaboration of Friedrich Engel" appears as a subtitle of the volumes of Lie's Theory of Transformation Groups, and the following theorem, for which we can find a proof in [10], bears his name.

Theorem 1.10 (Engel's Theorem)**.** *Let $\mathfrak{gl}(V)$ be the Lie algebras of the endomorphisms of a finite dimensional vector space V and $\mathfrak{g} \subset \mathfrak{gl}(V)$ a subalgebra. Then the following are equivalent:*

1. *Each $x \in \mathfrak{g}$ is a nilpotent endomorphism on V.*

2. *There exists a flag*

$$V = V_0 \supset V_1 \supset \cdots \supset V_n = 0,$$

with codim $V_i = i$ *such that* $\mathfrak{g} \cdot V_i \supset V_{i+1}$*, where* $\mathfrak{g} \cdot V_i$ *indicates that the subalgebra* $\mathfrak{g}$ *is acting on the subspace* V_i*.*

One wonders how, despite such a fruitful collaboration, the working relationship and personal friendship between Engel and Lie broke down in the late 1880s, and Lie's connection with Klein followed a few years later. Klein attributed the breakdown between Lie and colleagues to the thinness of the line between genius and insanity, and certainly mental symptoms of the vitamin B_{12} deficiency from which Lie suffered could have played a role. However, attributing the entirety to such causes seems excessively convenient in absolving Engel and Klein of any responsibility. As it turns out, the rivalry between Lie and Killing provided an additional component for suspicions, regardless of how rational they might have been, to latch onto: beginning in the mid-1880s, Engel also corresponded about transformation groups with Killing, and continued after it was clear that Lie and Killing did not get along. This was enough for Lie to suspect that Killing was, through Engel, stealing ideas for which Lie already felt underappreciated.

Whether Killing did or did not maintain contact with Engel with the goal of extracting Lie's ideas through Engel, and whether such did or did not happen, one thing we can be sure of is that Engel, even after their falling out, had no interest in erasing the accomplishments of Sophus Lie. To the contrary, even after Lie's death, Engel continued editing Lie's work and brought forth several more volumes he referred to as Lie's [26].

2 Representations and Modules

The theory of representations is critical to understanding the classification of Lie algebras. A *representation* of a Lie algebra L on a vector space V is a Lie algebra homomorphism $\rho : L \to \mathfrak{gl}(V)$. This means that $\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$ for $x, y \in L$. The vector space V, together with the representation ρ, is called an L-module. Given a representation $\rho : L \to \mathfrak{gl}(V)$, we say that a subspace W of V is *invariant* if $\rho(x)w \in W$ for all $w \in W$ and $x \in L$. A nonzero representation is said to be *irreducible* if the only invariant subspaces are V itself and the zero space $\{0\}$. As we have mentioned previously, one of the key examples for understanding the classification of simple Lie algebras over the complex numbers is understanding the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$.

2.1 The Adjoint Representation

One very important representation is the *adjoint representation*, $\mathrm{ad}(x)(y) = [x, y]$ for any elements x and y of the Lie algebra; in the same way that $\sin(\theta)$ is commonly written as $\sin\theta$, while we shall in some intricate contexts write $\mathrm{ad}(x)$, it is also common to write or see the adjoint representation written without parentheses as $\mathrm{ad}\, x$. That the adjoint is from L is clear, and as we shall demonstrate a method for writing adx as a matrix in Example 2.1, certainly the adjoint maps into $\mathfrak{gl}(V)$. To convince ourselves that the adjoint is truly a representation, we must also demonstrate that the adjoint is a homomorphism. So, we would like to show that $\mathrm{ad}([x, y]) = [\mathrm{ad}(x), \mathrm{ad}(y)]$. Since we define the adjoint by its effect on another element of the original Lie algebra, we can do so by showing that $\mathrm{ad}([x,y])(z) = [\mathrm{ad}(x), \mathrm{ad}(y)](z)$, as follows:

$$\begin{aligned}
\mathrm{ad}([x,y])(z) &= [[x,y],z] \\
&= [[x,z],y] + [x,[y,z]] \\
&= [x,[y,z]] - [y,[x,z]] \\
&= \mathrm{ad}(x)[y,z] - \mathrm{ad}(y)[x,z] \\
&= \mathrm{ad}(x)\,\mathrm{ad}(y)(z) - \mathrm{ad}(y)\,\mathrm{ad}(x)(z) \\
&= (\mathrm{ad}(x)\,\mathrm{ad}(y) - \mathrm{ad}(y)\,\mathrm{ad}(x))(z) \\
&= [\mathrm{ad}(x), \mathrm{ad}(y)](z).
\end{aligned}$$

While all other steps are basic matrix algebra or our formula definitions of the bracket product and adjoint representation, the less familiar jump in the second line is from the reordered form of the Jacobi identity through which we saw a connection to the product rule:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This hints at the deep connection between the adjoint representation and the Jacobi identity which causes some to describe [14] the Jacobi identity as a property stating that the adjoint representation is truly a representation, as we have now shown.

Since $\mathrm{ad}([x,y])(z) = [\mathrm{ad}(x), \mathrm{ad}(y)](z)$, we can be certain that the adjoint representation is truly a representation.

Example 2.1 (Adjoint Representation of $\mathfrak{sl}_2(\mathbb{C})$)**.** We can determine the adjoint representation of an element by computing its bracket product with each basis element, and it is informative to do so for each element of the basis. Each result from the bracket product multiplication table for x, when written as a linear combination of basis elements, determines, through its coefficients, a column of $\operatorname{ad} x$:

$$[x,x] = 0 = 0x + 0y + 0h$$
$$[x,y] = h = 0x + 0y + 1h$$
$$[x,h] = -2x = -2x + 0y + 0h$$

This yields the result that

$$\operatorname{ad} x = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

By the same method we also can find

$$\operatorname{ad} y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{bmatrix}, \quad \operatorname{ad} h = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Recalling that any element of w of $\mathfrak{sl}_2(\mathbb{C})$ can be written as the linear combination $ax + by + ch$ of these basis elements, since the adjoint representation is a homomorphism, we can summarize the adjoint representation by the formula

$$adw = a \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Among the adjoint matrices of these basis elements, $\operatorname{ad} h$ has some interesting features: as a diagonal matrix, its eigenvalues are the entries on its diagonal, 2, -2, and 0. We previously noticed the nonzero eigenvalues of h in Section 1.2 in the context of generators and relations, as scalar-like behavior of h: $[h,x] = 2x$ and $[h,y] = 2y$.

A contemporary of Sophus Lie, Wilhelm Karl Joseph Killing was born in 1847, into a family where his father was a legal clerk and politician. As his family expectations as to how he would situate himself in life would have been high, but his health as a child was far less than his cleverness, academia was a natural career direction. His secondary school geometry education led him to focus on mathematics, and many of his early career's publications were in geometry. Killing was particularly interested in non-Euclidean geometry which led him to symmetry and ultimately to be one of the founders of the research field of Lie algebras.

Among other contributions, the *Killing form* of a Lie algebra is a matrix which uses the adjoint representation, together with a basis, to encode some important properties of Lie algebras, such as whether the Lie algebra is semisimple. Although we previously demonstrated determining that any non-empty ideal of $\mathfrak{sl}_2(\mathbb{C})$ must be $\mathfrak{sl}_2(\mathbb{C})$ itself, therefore that $\mathfrak{sl}_2(\mathbb{C})$ is simple, these calculations were minimized by the low dimension of $\mathfrak{sl}_2(\mathbb{C})$, and we would want a much more efficient method for dealing with larger Lie algebras.

The entries of the Killing form matrix are calculated according to the formula $\kappa(x, y) = \mathrm{Tr}(\operatorname{ad} x \operatorname{ad} y)$. The Killing form is symmetric, bilinear, and is also associative. That is, $\kappa(x, y) = \kappa(y, x)$ and $\kappa((x, y), z) = \kappa(x, (y, z))$.

Example 2.2 (Killing Form of $\mathfrak{sl}_2(\mathbb{C})$)**.** First we recall from above the adjoint representations of the three elements of the standard basis of $\mathfrak{sl}_2(\mathbb{C})$:

$$\operatorname{ad} x = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \operatorname{ad} y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{bmatrix}, \quad \operatorname{ad} h = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Using these adjoints and the formula above, we calculate the entries, as follows. Each entry on the main diagonal must be computed separately:

$$\begin{aligned}
\kappa(x,x) &= \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} x) \\
&= \operatorname{Tr}\left(\begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) \\
&= \operatorname{Tr}\begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= 0.
\end{aligned}$$

Similarly, $\kappa(y,y) = 0$ and $\kappa(h,h) = 8$.

However, since the Killing form is symmetric—that is, $\kappa(a,b) = \kappa(b,a)$—we compute the remaining entries two at a time:

$$\begin{aligned}
\kappa(x,y) = \kappa(y,x) &= \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y) \\
&= \operatorname{Tr}\left(\begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{bmatrix}\right) \\
&= \operatorname{Tr}\left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right) \\
&= 4.
\end{aligned}$$

By the same process, we find out that $\kappa(x,h) = \kappa(h,x) = 0$ and $\kappa(y,h) = \kappa(h,y) = 0$. So, we find that the Killing form of $\mathfrak{sl}_2(\mathbb{C})$ is

$$\begin{bmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

In contrast to Lie's goal of using symmetry to classify differential equations, Killing's goal of studying geometries in terms of infinitesimal motion led him to investigate Lie algebras. He was, therefore, more interested in the real-valued cases of Lie groups and algebras, though found their complexifications necessary for theoretical and calculational purposes. While Cartan is rightly credited for putting Lie algebra on logically firm footing, Killing's boldness in recognizing patterns and forming conjectures deserves recognition as well. Not all of these conjectures turned out to be accurate; for example, Killing conjectured at one point that the special linear and orthogonal Lie algebras were the only simple Lie algebras, though he would soon discover additional algebras. He also introduced both the Cartan subalgebra and the Cartan matrix of a Lie algebra, and used these tools to examine possibilities among structures of Lie algebras. As a result of this work on a more intuitive level, Killing had found all the exceptional Lie algebras and families of classical Lie algebras, and had convincing if not absolute evidence that the classification was complete.

2.2 Theorems from Lie and Cartan

As we saw in Section 1.5, solvable Lie algebras are those where, given that $L^{(n)} = [L^{(n-1)}, L^{(n-1)}]$, for some sufficiently large value of n, $L^{(n)} = 0$. To better understand solvable Lie algebras, we will need theorems from Lie and Cartan.

Theorem 2.3 (Lie's Theorem)**.** *If L is a solvable Lie subalgebra of $\mathfrak{gl}(V)$, where $\dim V = n < \infty$, then the matrices of L relative to a suitable basis of V are upper triangular.*

Theorem 2.4 (Cartan's Criterion for Solvability)**.** *Let L be a Lie subalgebra of $\mathfrak{gl}(V)$, where $\dim V = n < \infty$. Suppose that $\mathrm{Tr}(xy) = 0$ for all $x \in [L, L]$ and $y \in L$. Then L is solvable.*

Proofs for Theorems 2.3 and 2.4 can be found on p. 16 and p. 20, respectively, in [15]. One notable consequence of this theorem is that, with a suitable choice of basis—in this case meaning a basis in which at least one basis element is in $[L, L]$—the Killing form of a solvable Lie algebra will have a zero row (and column), and, therefore, a determinant equal to zero.

Theorem 2.5 (Cartan's Criterion for Semisimplicity)**.** *Let L be a Lie algebra. Then L is semisimple if and only if its Killing form is nondegenerate; that is, the determinant of the Lie algebra's Killing form is nonzero.*

The original proof for this theorem is the first given in Chapter 4 of [4]. Compared with Cartan's Criterion for Solvability, we see that since semisimple Lie algebras have Killing forms with nonzero determinants, while solvable Lie algebras have Killing forms with determinant zero, any Lie algebra will be either solvable or semisimple but never both.

Example 2.6 (Semisimplicity of $\mathfrak{sl}_2(\mathbb{C})$)**.** We determined in Example 2.2 that the Killing form of $\mathfrak{sl}_2(\mathbb{C})$ is

$$\begin{bmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

Since

$$\det \begin{bmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix} = -128,$$

we can conclude that $\mathfrak{sl}_2(\mathbb{C})$ is semisimple. This fact is precisely what we already expect from knowing that $\mathfrak{sl}_2(\mathbb{C})$ is simple, as per Theorem 1.8, since all simple Lie algebras are semisimple. Although determining directly that $\mathfrak{sl}_2(\mathbb{C})$ is simple was not difficult due to the small number of elements in its basis, the task would grow in difficulty for larger Lie algebras, in which case the process we have illustrated of using the Killing form and Cartan's criterion for semisimplicity would be more efficient.

2.3 Modules and Reducibility of Representations

We refer to a vector space V as an *L-module*, for a Lie algebra L, when there exists a bilinear function $\phi : L \times V \to V$ such that $[x, y].v = x.(y.v) - y.(x.v)$ for all $v \in V$, $x, y \in L$. These L-modules can have *submodules*, subsets which have the absorption property familiar from ideals. Corresponding to simple Lie algebras, L-modules with only the trivial submodules (0 and itself) are called *irreducible*. Also, corresponding to semisimple Lie algebras, *completely reducible* L-modules are those which can be written as direct sums of irreducible L-submodules. Modules and representations are different notations for the same concept, so that whichever is more convenient at the time can be used.

Theorem 2.7. *In the sense that the existence of an L-module implies a representation connecting the module to the Lie algebra, and that the existence of a representation implies a module containing the range of the representation, modules and representations are equivalent.*

Proof. We will show that from any representation, a module can be naturally defined, and likewise from any module, a representation can be naturally defined. First, let us take a representation $\rho : L \to \mathfrak{gl}(V)$, and define an action of L on V by $x.v = \rho(x)(v)$. We want to check whether this action is bilinear as well as whether it follows the other defining formula for L-modules, $[x, y].v = x.(y.v) - y.(x.v)$.

We begin by verifying that this action is bilinear; i.e. that it complies with the two formulas $(ax+by).v = a(x.v)+b(y.v)$ and $x.(av+bw) = a(x.v)+b(x.w)$. First, we use the definition of the action and properties of representations to compute $(ax + by).v$, as follows:

$$\begin{aligned}(ax+by).v &= \rho(ax+by)(v)\\ &= (\rho(ax)+\rho(by))(v)\\ &= \rho(ax)(v)+\rho(by)(v)\\ &= a\rho(x)(v)+b\rho(y)(v)\\ &= a(x.v)+b(y.v),\end{aligned}$$

from which we can see that $(ax + by).v = a(x.v) + b(y.v)$ as needed. Our strategy for checking that this action complies with the second bilinear condition will be very similar, in that we will compute $x.(av + bw) = a(x.v) + b(x.w)$:

$$\begin{aligned}x.(av+bw) &= \rho(x)(av+bw)\\ &= \rho(x)(av)+\rho(x)(bw)\\ &= a\rho(x)(v)+b\rho(x)(w)\\ &= a(x.v)+b(x.w).\end{aligned}$$

We see from this that this action is bilinear.

Now, we need only verify the L-module condition $[x, y].v = x.(y.v) - y.(x.v)$,

$$\begin{aligned}[x,y].v &= \rho([x,y])(v)\\ &= ([\rho(x),\rho(y)])(v)\\ &= (\rho(x)\rho(y)-\rho(y)\rho(x))(v)\\ &= \rho(x)\rho(y)(v)-\rho(y)\rho(x)(v)\\ &= \rho(x)(y.v)-\rho(y)(x.v)\\ &= x.(y.v)-y.(x.v).\end{aligned}$$

Therefore, V is an L-module.

Next, we would like to see that modules similarly define representations. So, let V be an L-module, and define $\rho : L \to \mathfrak{gl}(V)$ by $\rho(x) = x.v$. Since ρ is defined as having the needed domain and codomain, we need only see that ρ is a homomorphism to verify that ρ is a representation. We want to see that $\rho([x,y])(v) = [\rho(x), \rho(y)](v)$:

$$\begin{aligned}
\rho([x,y])(v) &= \rho(xy - yx)(v) \\
&= \rho(xy)(v) - \rho(yx)(v) \\
&= xy.v - yx.v \\
&= x(y.v) - y(x.v) \\
&= x.\rho(y)(v) - y.\rho(x)(v) \\
&= \rho(x)\rho(y)(v) - \rho(y)\rho(x)(v) \\
&= [\rho(x), \rho(y)](v).
\end{aligned}$$

Since each of a homomorphism and a module can define the other, homomorphisms and modules are equivalent. $\square$

Another important tool that we will need is due to Issai Schur (1875–1941), and traditionally known as Schur's Lemma. A proof of Schur's Lemma can be found in [15].

Theorem 2.8 (Schur's Lemma)**.** *Let $\phi : L \to \mathfrak{gl}(V)$ be reducible. Then the only endomorphisms of V commuting with all $\phi(x)$ are the scalars.*

Another important contributor to Lie algebra, and in fact to mathematics more generally with major contributions to fundamentals, differential equations, number theory, theoretical physics, and other topics, was Hermann Weyl. Born in 1885, he emigrated from Nazi Germany in 1933, leaving a chair which Hilbert had held before him at Gottingen to take up a position at Princeton. Lie algebras became his latest research focus late in life—his book on the subject was published in 1952, only three years before his death in 1955—from the direction of his previous work in differential equations and quantum mechanics [1], though it was somewhat before this time that he was the first to refer to the topic previously known as infinitesimal groups by the name we now know, Lie algebras [16].

Theorem 2.9 (Weyl's Theorem)**.** *Let $\phi : L \to \mathfrak{gl}(V)$ be a representation of a semisimple Lie algebra. Then ϕ is completely reducible.*

One consequence of Weyl's Theorem, together with Schur's Lemma, is that any semisimple Lie algebra L will contain, for each of its elements, the semisimple and nilpotent parts of that element's *Jordan decomposition*—by which we refer to writing any matrix as the sum of a simple and a nilpotent matrix. Furthermore, when ρ is a representation of L, where $x = s + n$ is the Jordan decomposition in the Lie algebra, $\rho(x) = \rho(s) + \rho(n)$ is the Jordan decomposition in $\mathfrak{gl}(V)$ [15].

2.4 Representations of $\mathfrak{sl}_2(\mathbb{C})$

In 1914, Cartan determined all irreducible representations of the simple Lie algebras over the complex numbers [5]. We shall see how this works with $\mathfrak{sl}_2(\mathbb{C})$ as an example.

Let V be an L-module, where L is $\mathfrak{sl}_2(\mathbb{C})$. That is, there exists a $\phi : \mathfrak{sl}_2(\mathbb{C}) \times V \to V$ satisfying the defining requirements of an L-module, and therefore a corresponding representation $\rho : \mathfrak{sl}_2(\mathbb{C}) \to V$. Since h is diagonal, $\rho(h)$ will also be diagonal, so $h.v$ will be equal to λv for some value of λ. The set of vectors with the same value for λ is the space $V_\lambda = \{v \in V | h.v = \lambda v\}$, called a *weight space* for the *weight* λ. Since V is finite-dimensional, there can only be finitely many different values of λ, so we can take the greatest. In this case, vectors in V_λ are called *maximal vectors*.

We will determine the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ in this section. That is, we will prove the following theorem.

Theorem 2.10. *Let V be an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ with $\dim V = n+1$.*

Then there exists a nonzero vector $v \in V$ such that $h.v = -nv$, $y.v = 0$, and

$$V = \langle v, x.v, \ldots, x^n.v \rangle$$

over $\mathbb{C}$. Under this basis, the action of h is diagonal with weights $-n, -n+2, \ldots, n$. The action of x has just been described, and the action of y is described in the lemma below.

In other words, Theorem 2.10 tells us that $\mathfrak{sl}_2(\mathbb{C})$ has a unique irreducible representation for every finite dimensional vector space V. We will need several lemmas to prove our theorem.

Lemma 2.11. *If $v \in V_\lambda$, then $x.v \in V_{\lambda+2}$ and $y.v \in V_{\lambda-2}$.*

Proof. We will make use of the fact that V is an L-module. For the purpose at hand, it will be convenient to reorder the L-module condition $[x,y].v = x.(y.v) - y.(x.v)$ as $x.(y.v) = [x,y].v + y.(x.v)$. Then we proceed to calculate:

$$\begin{aligned} h.(x.v) &= [h,x].v + x.h.v \\ &= 2x.v + x.h.v \\ &= 2x.v + x\lambda v \\ &= (\lambda+2)x.v. \end{aligned}$$

Since $h.(x.v) = (\lambda+2)x.v$, $\lambda+2$ is the eigenvalue associated with $x.v$, so $x.v$ is in the eigenspace, or weight space, $V_{\lambda+2}$. The proof for $y.v$ proceeds in quite the same way. □

The actions of L on elements of V are illustrated in Figure 2.1. In this diagram, V is partitioned into eigenspaces.

Figure 2.1: Actions of L on V_n

$$
\begin{tikzcd}
{} \arrow[r] & V_{n-4} \arrow[l, bend right] \arrow[r, "x"] \arrow[loop below, "h"] & V_{n-2} \arrow[l, bend right, "y"'] \arrow[r, "x"] \arrow[loop below, "h"] & V_n \arrow[l, bend right, "y"'] \arrow[loop below, "h"]
\end{tikzcd}
$$

If we choose v from the nontrivial eigenspace which would be furthest left in this diagram, V_{-n}, we satisfy the requirements that $h.v = -nv$ and $y.v = 0$. Furthermore, by repeatedly applying the x action, the set of $n+1$ vectors $\{v, x.v, ...x^n.v\}$ will be from different eigenspaces, thus linearly independent, so will form a basis for V.

We can make further observations about the repeated action of x and y.

Lemma 2.12. *When V_λ is the weight space where, for any v in that space, $y.v = 0$,*

1. $y.x^n.v = -n(\lambda + n - 1)x^{n-1}.v$

2. $y^n.x^n.v = (-1)^n n!\lambda(\lambda + 1)...(\lambda + n - 1).v$

Proof. The first of these two statements is proven using induction, with the easy base case $n = 0$: $y.x^0.v = y.v = 0$ and $-0(\lambda + 0 - 1)x^{0-1}.v = 0$.

Now we consider $y.x^{n+1}.v$; we want to show $y.x^{n+1}.v = -(n+1)(\lambda + n)x^n.v$. To that end, we will use the L-module criterion formula together with our base case. Using the fact that $[x, y].v = x.y.v - y.x.v$ as well as the calculation in the proof of

Lemma 2.11, we have

$$\begin{aligned}
y.x^{n+1}.v &= y.x.x^n.v \\
&= -[x,y].x^n.v + x.y.x^n.v \\
&= -h.x^n.v + x.y.x^n.v \\
&= -(\lambda+2n)x^n.v + -n(\lambda+n-1)x^n.v \\
&= -(n+1)(\lambda+n)x^n.v.
\end{aligned}$$

So, $y.x^n.v = -n(\lambda+n-1)x^{n-1}.v$ as desired.

For the second of these two statements, once again we will use induction on n with the base case $n=1$; clearly, $y.x.v = -h.v = (-1)^1 1!\lambda v$. Assuming $y^n.x^n.v = (-1)^n n!\lambda(\lambda+1)...(\lambda+n-1).v$, we have

$$\begin{aligned}
y^{n+1}.x^{n+1}.v &= y^n.y.x.x^n.v \\
&= y^n.([y,x].x^n.v + x.y.x^n.v) \\
&= y^n.(-h.x^n.v - n(\lambda+n-1)x^n.v) \\
&= y^n.(-(\lambda+2n)x^n.v - n(\lambda+n-1)x^n.v) \\
&= (-(\lambda+2n) - n(\lambda+n-1))y^n.x^n.v \\
&= (-n\lambda - n^2 - \lambda - n)y^n.x^n.v \\
&= -(n+\lambda)(n+1)y^n.x^n.v \\
&= (-1)^{n+1}(n+1)!\lambda(\lambda+1)...(\lambda+n).v.
\end{aligned}$$

□

We see here that not only can we determine into which eigenspace $x^k.v$ any $y^b.x^a.v$ within a given dimensional representation V falls, we can, through use of these scalar formulas and applying the eigenvalues $-n, -n+2, ..., n$ as needed to account for h, determine an exact element of V related to the action of any element of $\mathfrak{sl}_2(\mathbb{C})$. In short, the defining properties of an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ are sufficient to determine the entire behavior of that representation with respect to all elements of $\mathfrak{sl}_2(\mathbb{C})$. Therefore, we may conclude that all irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ of the same dimension are identical up to an isomorphism.

Representation theory pays a good deal of attention to $\mathfrak{sl}_2(\mathbb{C})$ for several important reasons. Among these reasons is that, as the smallest simple complex Lie algebra, $\mathfrak{sl}_2(\mathbb{C})$ makes a more comprehensible example than—for example—the 248-dimensional E_8. It also has applications in physics in its own right, as part of analyzing the Lorentz group [2]. Also, in the next chapter we shall encounter Dynkin diagrams, for which the graph of $\mathfrak{sl}_2(\mathbb{C})$ is a single vertex; for reasons we shall then see, this implies the structure of $\mathfrak{sl}_2(\mathbb{C})$ influences the structure of all other simple complex Lie algebras.

3 Roots and Root Spaces

Cartan and Killing realized that the best way to approach the classification of simple Lie algebras over the complex numbers was to examine their root systems, and eventually their root space decompositions. The requirements of these systems and structures will give rise to our main result, that all Lie algebras are isomorphic to either one of the classical or one of the exceptional Lie algebras.

3.1 Maximal Toral Subalgebras and Root Space Decompositions

We have mentioned Cartan subalgebras in passing, but as we will now use them in detail, it is time to get specific. A *Cartan subalgebra* of a Lie algebra L is a nilpotent subalgebra H such that if $[x, y] \in H$ for all $x \in H$, then $y \in H$. Within the context of finite-dimensional semisimple complex Lie algebras, Cartan subalgebras are made up of diagonal elements; these algebras are called *toral*, so Cartan subalgebras are also known as *maximal toral subalgebras*. Since Cartan subalgebras are made up on diagonal elements, these subalgebras are abelian. Therefore, one way to verify that a proposed Cartan subalgebra is truly maximal is to compute the bracket product of each of its elements with a potential additional element and check whether the result is 0.

We also need to examine the properties of *roots*, which are a type of elements of the dual space H^* of linear functionals from a Cartan subalgebra H to the field over which the Lie algebra was defined, in our case the complex numbers. In particular, each root has a non-empty *root space* defined by the formula

$$L_\alpha = \{x \in L | \text{ for all } h \in H, [h, x] = \alpha(h)x\}.$$

For example, if L contains at least one element x such that $[h, x] = 2x$, we can define a root α so that $\alpha(h) = 2$. This is the case, in particular, in $\mathfrak{sl}_2(\mathbb{C})$, and while $\mathfrak{sl}_2(\mathbb{C})$ is on our minds, we recognize L_α as the familiar one-dimensional weight space generated by x.

Theorem 3.1. *Let L be a semisimple Lie algebra, H a maximal toral subalgebra, and $\Phi \subset H^*$ the set of roots of L (relative to H), and*

$$L = H + \bigoplus_{\alpha \in \Phi} L_\alpha.$$

Let E be Euclidean space, where $\dim E = \ell$ over $\mathbb{R}$. Then

(a) Φ spans E and 0 does not belong to Φ.

(b) If α is a root, then $-\alpha$ is a root. No other scalar multiple of α is a root.

(c) If $\alpha, \beta \in \Phi$, reflecting β across the hyperplane perpendicular to α through the origin, the result of which we can calculate using the formula

$$\beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha,$$

is a root.

(d) If $\alpha, \beta \in \Phi$, then

$$\frac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}.$$

Owing to the importance of the formula

$$2\frac{(\beta, \alpha)}{(\alpha, \alpha)}$$

in these properties, and related calculations, a common shorthand for this formula is $\langle \beta, \alpha \rangle$. In particular, the last of these properties is often written as "$\langle \alpha, \beta \rangle$ is an integer." The integers so calculated are referred to as *Cartan integers*.

A fully detailed proof can be found in Chapter 2 of [15], across several subsections. In particular, the third and fourth subsections of Section 8 prove the properties of such a set of roots, and the fifth subsection summarizes them.

These properties overlap; the third, related to reflection across hyperplanes, is particularly influential, and is related to Weyl groups, which we will discuss further when an important example is at hand. For example, reflecting any root α across the hyperplane defined by itself must give $-\alpha$ as another root, as is required by the second of these properties.

More intricately, the property that $\langle \alpha, \beta \rangle$ is an integer helps show why no scalar multiples of roots, except $k = \pm 1$, are themselves roots. To see why, we use some properties of dot products and trigonometry to find a technique for calculating Cartan integers related to the angle between roots θ, and consider the $\theta = 0$ case.

Since the dot product has the properties that $(\alpha, \beta) = |\alpha||\beta| \cos\theta$ and $(\alpha, \alpha) = |\alpha|^2$, we can calculate $\langle \alpha, \beta \rangle$ as follows:

$$\langle \alpha, \beta \rangle = 2\frac{(\alpha, \beta)}{(\beta, \beta)} = 2\frac{|\alpha||\beta| \cos\theta}{|\beta|^2} = 2\frac{|\alpha| \cos\theta}{|\beta|}.$$

In such a case, we would have $\theta = 0$, so $\cos\theta = 1$. (It suffices to consider the positive case because any negative scalar multiple of α would be a positive scalar multiple of $-\alpha$.) Then, taking the fact that $\langle \alpha, \beta \rangle = 2|\alpha| \cos\theta/|\beta|$ is some integer n between 0 and 4 inclusive, we may restrict the possible values of k by calculating:

$$\begin{aligned} 2\frac{|\alpha| \cos\theta}{|\beta|} &= n \\ 2\frac{|\alpha|}{|\beta|} &= n \\ \frac{|\beta|}{2|\alpha|} &= \frac{1}{n} \\ |\beta| &= \frac{2}{n}|\alpha|, \end{aligned}$$

from which we see by considering values of n that k may only be 2, 1, 2/3, or 1/2, the case where $n = 0$ resulting in k undefined; but when $k = 1$, β is just α, which we have no need to include twice, and as reversing the order of the roots in a pair would allow us to switch between reciprocals, we need only consider one of 1/2 and 2. Therefore, let us consider whether $2\alpha/3$ or 2α is a root.

If $\beta = 2\alpha/3$, we could equivalently say $\alpha = 3\beta/2$, but $3/2$ is not among the allowed values of k. So, $2\alpha/3$ is not a root. Together with another limitation that k cannot be 2, and therefore also not $1/2$ or their opposites, no scalar multiple of a root α other than $-\alpha$ is another root [15].

To gain a better understanding of these properties and how they apply to Lie algebras, it is beneficial to examine them in the context of an example from a familiar Lie algebra. Therefore, in Example 3.2 we will examine a root system of $\mathfrak{sl}_2(\mathbb{C})$

3.2 Root Systems of Lie Algebras

Considering the importance of the properties from Theorem 3.1 in describing Lie algebras, we would like to define a type of structure based on these properties. Relative to a Cartan subalgebra H, we refer to a subset $R \subset H*$ as a *root system* of a Lie algebra L if it has the following properties:

1. The set R is finite, does not contain 0, and spans the Euclidean space E which contains R.

2. For all $\alpha \in R$, the only scalar multiples $k\alpha \in R$ are those where $k = \pm 1$.

3. For all $\alpha \in R$, the reflection s_α across the hyperplane perpendicular to α, defined for calculation purposes by
$$s_\alpha(r) = r - 2\frac{(r,\alpha)}{(\alpha,\alpha)}\alpha,$$
permutes R.

4. For all $\alpha, \beta \in R$,
$$2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$$
is an integer.

Example 3.2 (Root system for $\mathfrak{sl}_2(\mathbb{C})$)**.** As mentioned, a root system is relative to a particular Cartan subalgebra. Since $\mathfrak{sl}_2(\mathbb{C})$ contains only a single Cartan subalgebra, the weight space V_0 generated by h, this is the Cartan subalgebra we will use.

Where $\alpha = 2h$, or we may say in matrix form

$$\alpha = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix},$$

I claim $R = \{\alpha, -\alpha\}$ is a root system for $\mathfrak{sl}_2(\mathbb{C})$, and will show this to be true by checking each property from the above definition.

1. The root system R is finite, does not contain 0, and spans the Euclidean space E which contains R: By inspection, R contains two elements, neither of which is zero. Since $\ell = 1$, we are basically trying to span $\mathbb{R}$; $\{n\alpha\}$ is isomorphic to $\mathbb{R}$ with one possible isomorphism being $f(n\alpha) = 2n$.

2. For all $\alpha \in R$, the only scalar multiples $ka \in R$ are those where $k = \pm 1$: The only two elements of R are precisely a $k = \pm 1$ pair.

3. For all $\alpha \in R$, the reflection $s_\alpha(r)$ permutes R: We have two functions to check, $s_\alpha(r)$ and $s_{-\alpha}(r)$, the first of which proceeds as follows:

$$\begin{aligned} s_\alpha(\alpha) &= \alpha - 2\frac{(\alpha, \alpha)}{(\alpha, \alpha)}\alpha \\ &= \alpha - 2\alpha \\ &= -\alpha \\ s_\alpha(-\alpha) &= -\alpha - 2\frac{(-\alpha, \alpha)}{(\alpha, \alpha)}\alpha \\ &= -\alpha + 2\alpha \\ &= \alpha. \end{aligned}$$

   By inspection, $s_\alpha(r)$ permutes elements of R. We would compute $s_{-\alpha}(r)$ the same way, and determine that this reflection permutes R.

4. For all $\alpha, \beta \in R$, $\langle \alpha, \beta \rangle$ is an integer: Considering the two options α and $-\alpha$, we see that it will be necessary to determine whether each of $\langle \alpha, \alpha \rangle$, $\langle \alpha, -\alpha \rangle$, $\langle -\alpha, \alpha \rangle$, and $\langle -\alpha, -\alpha \rangle$ is an integer. However, recalling the formula

$$\langle \alpha, \beta \rangle = 2\frac{(\alpha, \beta)}{(\beta, \beta)},$$

and observing that we can factor -1 as needed from terms within the formula for the dot product, we have that $(-r, r) = (r, -r) = -1(r, r)$ and $(-r, -r) = -1 \cdot -1(r, r)$; that is, the four cases can differ only in sign, so it suffices to check the simplest, positive, case as follows:

$$\begin{aligned} \langle \alpha, \alpha \rangle &= 2\frac{(\alpha, \alpha)}{(\alpha, \alpha)} \\ &= 2 \cdot 1 = 2. \end{aligned}$$

From this we see that $\langle \alpha, \beta \rangle = \pm 2$ for any pair α, β from our proposed root system.

Having shown that all properties hold, we may conclude that R is in fact a root system for $\mathfrak{sl}_2(\mathbb{C})$.

Root systems can be illustrated and visualized in various ways which help with understanding their properties. For example, a vector diagram for A_1 can depict the two roots α and $-\alpha$ as opposite vectors from a common origin (Figure 3.1). Any root system with a single *simple root*—that is, a positive root which cannot be written as a linear combination of other simple roots—would be depicted in this same way.

Figure 3.1: Root Systems of Rank 1

$$-\alpha \longleftrightarrow \alpha$$

Once we know a root system for a Lie algebra, we can use it to determine that Lie algebra's root space decomposition. First, we define the root space L_α, for each root α, as the set of all elements of the Lie algebra such that, for any element h from the Cartan subalgebra H, $[h, x] = \alpha(h)x$:

$$L_\alpha = \{x \in L | \text{ for all } h \in H, [h, x] = \alpha(h)x\}.$$

Then, the Lie algebra can be decomposed as

$$L = C_L(H) + \bigoplus_{\alpha \in R} L_\alpha,$$

where $C_L(H)$ denotes the centralizer in L of H—that is, the set of all elements x of L such that $[x, h] = [h, x]$. But since H is maximal, $C_L(H) = H$, so the root space decomposition is usually more simply written as

$$L = H + \bigoplus_{\alpha \in R} L_\alpha.$$

Example 3.3 (Root system decomposition of A_1)**.** We have established a root system, $\{\alpha, -\alpha\}$, for A_1. Based on this root system, we would like to examine the root space decomposition of A_1,

$$A_1 = H + L_\alpha + L_{-\alpha}.$$

We already know that H is the Cartan subalgebra generated by h. Using the definition above for the root space L_α, the set of all elements of A_1 such that $[h, x] = \alpha(h)x$ (where x is a variable referring to an A_1 element); but since we chose α so that $\alpha(h) = 2$, L_α is the space where $[h, x] = 2x$. Recognizing that this is the same as the basic relation $[h, x] = 2x$ where x is the usual basis element, we conclude that L_α is the space generated by the basis element x.

Likewise, $L_{-\alpha}$ is defined as the set of all elements of A_1 such that $[h, x] = -\alpha(h)x = -2x$. From this we recognize the relation $[h, y] = -2y$, leading us to conclude that $L_{-\alpha}$ is the space generated by the basis element y. Incidentally, we also may recall that the root spaces L_α and $L_{-\alpha}$ are the same, respectively, as the weight spaces V_2 and V_{-2} from Section 2.4.

3.3 Abstract Root Systems

In or shortly before 1889, Killing realized that the properties of Lie algebra root systems could be further abstracted to a purely geometric object, abstract root systems; he first mentioned them in [20], and this seems to be the first time they were mentioned in any publication. As with so many of Killing's ideas, Cartan confirmed that root systems of Lie algebras and abstract root systems are equivalent, as part of his thesis. This abstraction allows us to set aside all other complicating features of Lie algebras and examine abstract root systems, to understand and completely determine all simple Lie algebras over the complex numbers.

An *abstract root system*, R, is a subset of an ℓ-dimensional Euclidean space E with the following properties:

1. The root system R is finite, $0 \notin R$, and R spans E.
2. For all $\alpha \in R$, the only scalar multiples $k\alpha \in R$ are those where $k = \pm 1$, in both the sense that no other scalar multiplies are permitted and the sense that $-\alpha$ is necessarily a root.
3. For all $\alpha \in R$, the reflection s_α, defined for calculation purposes by
$$s_\alpha(r) = r - 2\langle \alpha, \beta \rangle \alpha,$$
permutes R.
4. For all $\alpha, \beta \in R$, $\langle \alpha, \beta \rangle$ is an integer.

As they are more interesting than the case we previously saw of rank 1, where *rank* refers to the dimension of the Euclidean space E spanned by the set of roots, we shall now determine all root systems of rank 2. Let us consider, for reasons which will shortly become apparent, the product $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle$. By closure of integers under multiplication we know that this product is an integer; and, using the formula we just found, we can be more specific:

$$\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle = 2\frac{|\alpha|\cos\theta}{|\beta|}2\frac{|\beta|\cos\theta}{|\alpha|} = 4\cos^2\theta.$$

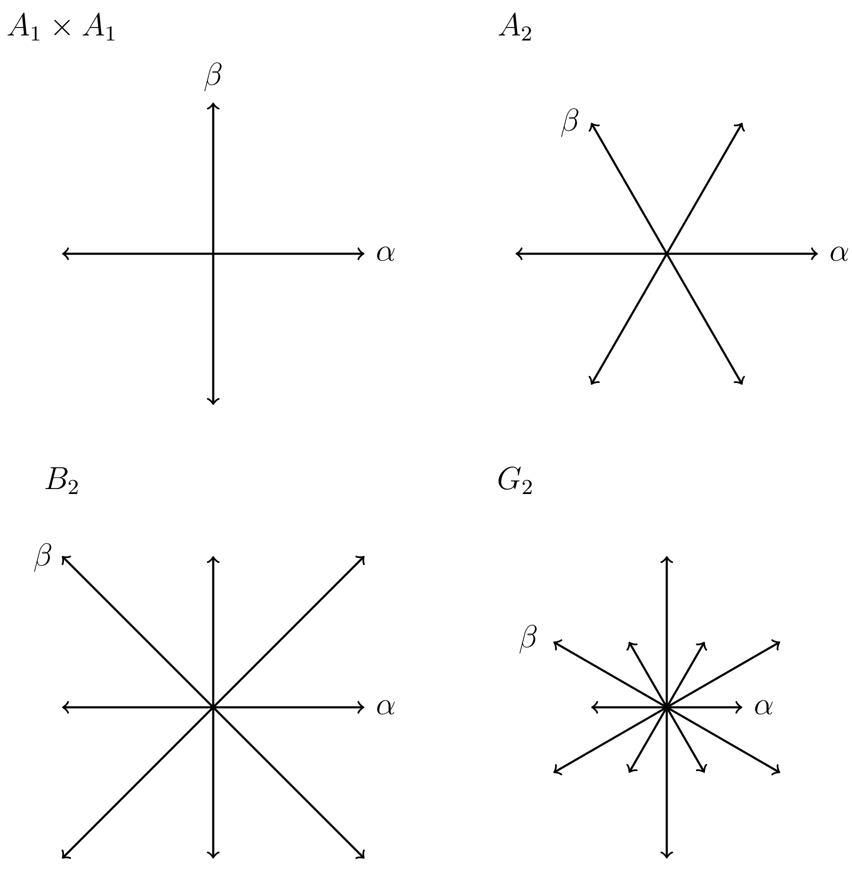
Therefore, we know that $4\cos^2\theta$ is an integer. This is a significant constraint on θ, the angle between roots α and β, since $\cos^2\theta$ is necessarily both an element of the closed interval from 0 to 1 and a rational number whose denominator is 4 or a factor thereof. Furthermore, we need only consider angles between 0 and π inclusive, as we are discussing an actual angle rather than angular motion.

$\cos^2\theta$	$\cos\theta$	θ
0	0	$\pi/2$
$1/4$	$\pm 1/2$	$\pi/3$ or $2\pi/3$
$1/2$	$\pm\sqrt{2}/2$	$\pi/4$ or $3\pi/4$
$3/4$	$\pm\sqrt{3}/2$	$\pi/6$ or $5\pi/6$
1	± 1	0 or π

Table 3.1: Feasible angles between pairs of roots

The situation where the angle between two roots is π describes precisely the requirement that a root's opposite be part of the root system; thus, this angle will appear in all root systems. We can describe the other options by the smallest angle in each: when the smallest angle between roots in a rank 2 system is $\pi/2$ we have $A_1 \times A_1$, while when we have $\pi/3$ and multiples thereof this is A_2, $\pi/4$ and multiples thereof describes B_2, and $\pi/6$ and multiples thereof describes G_2. As we cannot have a root system with $\pi/4$ angles without $\pi/2$, nor $\pi/6$ without both $\pi/2$ and $\pi/3$ nor the converse, nor yet both $\pi/4$ and $\pi/6$ for how that would require the disallowed angle $\pi/12$, we can be sure that all possible root systems of rank 2 are included in Figure 3.2. Furthermore, we have uncovered a powerful way of classifying root systems and their Lie algebras.

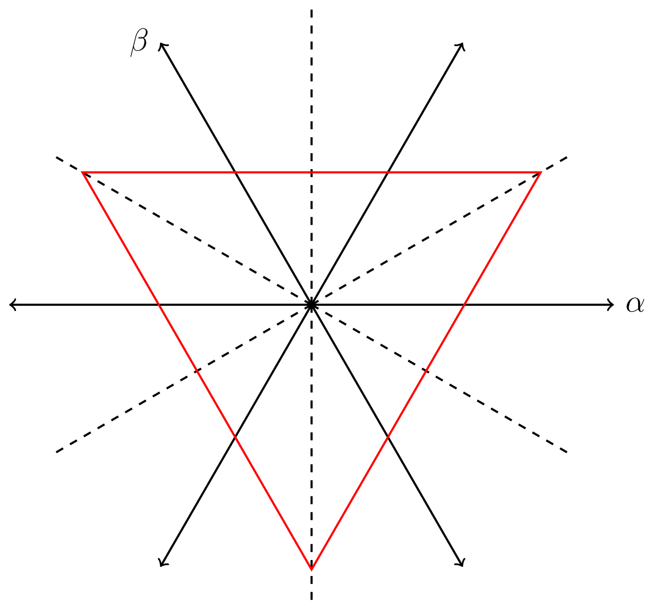
Figure 3.2: Root Systems of Rank 2



On any of these vector diagrams, we can also illustrate hyperplanes, which assist us in examining the Weyl group of the Lie algebra. In rank 2, the hyperplanes are one-dimensional, i.e. lines, through the origin.

Taking the specific example of A_2, with α as a root on the horizontal axis, we have hyperplanes at angles of $\pi/2$ (orthogonal to α), $\pi/6$ (orthogonal to β), and $5\pi/6$ (orthogonal to $\alpha + \beta$). These three hyperplanes split E into six regions, meeting at

Figure 3.3: The Weyl Group of A_2



the origin, each of which includes one root and points with θ strictly within $\pm\pi/6$ of that root; these regions are called *Weyl chambers*. Reflecting across any hyperplane maps the points of one Weyl chamber to the points of another, and therefore the root contained within one Weyl chamber to the root within another. Since we have three reflections, which may be composed in any order, the Weyl group of A_2 is isomorphic to the dihedral group on 3 vertices; that is, to the symmetry group of an equilateral triangle.

3.4 Classification of Root Systems

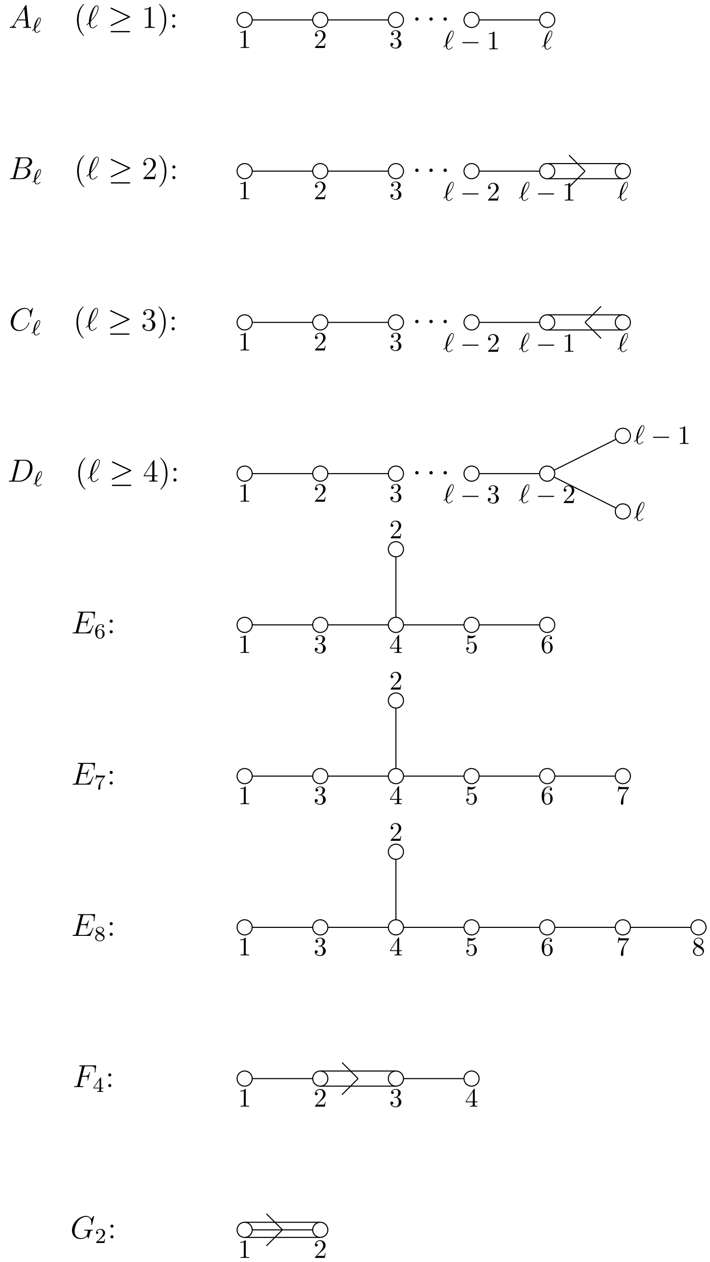
Dynkin diagrams are another important way of illustrating root systems. Named for Evgenii Dynkin, who was also called Eugene since moving from the Soviet Union to the United States, these diagrams illustrate roots as vertices rather than as vectors, and not even all roots; instead, they only illustrate *simple roots*, roots which are positive and not the sum of other positive roots.

Since A_1 has only one simple root, α, the Dynkin diagram representing A_1 is a single vertex. When plotting out any more complicated root system, however, we will have more vertices, and therefore need to decide how and when to connect them. Between any pair of roots, a Dynkin diagram—or the earlier Coxeter diagrams which Dynkin modified—is defined as having $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ edges, but we previously determined that this is $4\cos^2\theta$. So, we will draw a single edge when the angle between roots is $\pi/3$ or $2\pi/3$ as seen in A_2, a double edge when the angle is $\pi/4$ or $3\pi/4$ as seen in B_2, a triple edge when the angle is $\pi/6$ or $5\pi/6$ as seen in G_2, and no edge at all when the angle is $\pi/2$ as seen in $A_1 \times A_1$. We may seem at this point to have overlooked the possibility of drawing four edges, but this would happen when the angle between roots is 0 or π, which we will not draw: the angle π occurs between each root and its opposite, but we are only illustrating simple roots, while the angle 0 would occur between a root and either itself or a positive scalar multiple, but as we have seen, root systems never contain scalar multiples other than ± 1. Since drawing a four-loop on each vertex in order to represent angle 0 between it and itself would not be informative, we never draw four edges between vertices in a Dynkin diagram, nor will we draw any loops.

Dynkin's innovation, transforming Coxeter diagrams to Dynkin diagrams, was to note that the root systems whose diagrams contain a double or triple edge are also those with roots of varying length, and therefore to apply an inequality symbol on the double or triple edges to indicate which is the longer and which the shorter root. This convention makes it feasible to begin with a Dynkin diagram and work backward to the corresponding root system and ultimately to a particular Lie algebra. Therefore, we can use Dynkin diagrams to determine definitively what Lie algebras are possible.

Rather than Dynkin diagrams which would come later, when Cartan proved that

Figure 3.4: Dynkin Diagrams



the simple complex Lie algebras whose Dynkin diagrams are given in Figure 3.4 are, up to isomorphism, the only simple complex Lie algebras, he stated the reasons in terms of *Cartan matrices*, which are matrices whose entries are the Cartan integers. However, his reasons were equivalent to these.

We call a proposed set of n simple roots *admissible* if the roots are linearly independent, $(\alpha, \alpha) = 1$, $(\alpha, \beta) \leq 0$ whenever α and β are distinct, and $4(\alpha, \beta)^2$ is 0, 1, 2, or 3. Furthermore, an admissible Dynkin diagram is the Dynkin diagram of an admissible set of simple roots. We may note that $4(\alpha, \beta)^2$ is a simplified formula for the number of edges between vertices (i.e. the product of symmetrically-located, though possibly distinct, Cartan integers $\langle\alpha, \beta\rangle\langle\beta, \alpha\rangle$) as follows:

$$\begin{aligned}
\langle\alpha, \beta\rangle\langle\beta, \alpha\rangle &= 2\frac{(\alpha, \beta)}{(\beta, \beta)} 2\frac{(\beta, \alpha)}{(\alpha, \alpha)} \\
&= 4\frac{(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)} \\
&= 4(\alpha, \beta)^2.
\end{aligned}$$

Such a set of roots or vertices—we will use whichever terminology is convenient at the time, while understanding that the properties apply in corresponding ways to both the roots and the vertices—has certain properties.

1. As removing some roots could not introduce any inadmissible traits to the set, any subset of an admissible set of roots is itself another admissible set of roots; that is, a root system for a smaller Lie algebra, at least one isomorphic copy of which is contained as a subset of the original Lie algebra.

2. Recalling that the set contains n vertices, the number of directly connected pairs of vertices is at most $n - 1$. The reason for this is somewhat intricate. Since the roots are linearly independent, if we define ϵ as the sum of the roots, we know $\epsilon \neq 0$, so $(\epsilon, \epsilon) > 0$; and, we could calculate (ϵ, ϵ) precisely as

$$\sum_{i=1, j=1}^{n} (\alpha_i, \alpha_j).$$

Of the terms in this summation, n have $i = j$, so $(\alpha_i, \alpha_j) = (\alpha_i, \alpha_i) = 1$; so, the total of these n terms will be n. The remaining terms may be paired (α_i, α_j) with (α_j, α_i). Together these properties of terms allow us to rewrite:

$$0 < n + 2\sum_{i=1}^{n}\sum_{j=1}^{i-1}(\alpha_i, \alpha_j)$$

$$-n < 2\sum_{i=1}^{n}\sum_{j=1}^{i-1}(\alpha_i, \alpha_j)$$

$$n > 2\sum_{i=1}^{n}\sum_{j=1}^{i-1}(\alpha_i, \alpha_j)$$

While we do not know the value of each (α_i, α_j), we know that it is related to the number of edges between α_i and α_j: 0 if they are not connected, and at most $-1/2$ if they are connected. Let us call the number of directly linked pairs of vertices m; then, the total $\sum_{i=1}^{n}\sum_{j=1}^{i-1}(\alpha_i, \alpha_j)$ is at most $-(1/2)m$, giving us

$$n > -2\frac{-1}{2}m = m,$$

that is, the number of vertices must exceed the number of pairs of directly linked vertices.

3. As a consequence of items 1 and 2, we know that an admissible graph contains no cycles, since if an admissible graph did contain a cycle, we could remove the k vertices which are not part of the cycle to make an admissible graph which had n vertices and n directly linked pairs of vertices.

4. Each vertex is an end for no more than three edges. To see why, let us examine the behavior of a vertex α, around which we will refer to each of its k neighbors as β_i for some $i \leq k$. To prevent cycles, we must have that each $(\beta_i, \beta_j) = 0$; that is, all the neighbors of α are orthogonal to each other.

   Now let us select β_0 from within $S = \operatorname{span}\{\alpha, \beta_i\}$ such that β_0 is orthogonal to each previous β_i and $|\beta_0| = 1$; then, $\{\beta_i\}$ where $0 \leq i \leq k$ forms an orthonormal basis for S, which contains α, so α can be described as the total of projection vectors onto β_i. From this fact, we can calculate:

$$\alpha = \sum_{i=0}^{k} (\alpha, \beta_i)\beta_i$$

$$(\alpha, \alpha) = \sum_{i=0}^{k} (\alpha, \beta_i)(\beta_i, \alpha)$$

$$1 = \sum_{i=0}^{k} (\alpha, \beta_i)^2.$$

Since β_0 replaced α as a generator of S, α and β are not orthogonal, so $(\alpha, \beta_0)^2 > 0$, allowing us to further calculate:

$$1 > \sum_{i=1}^{k} (\alpha, \beta_i)^2$$

$$4 > \sum_{i=1}^{k} 4(\alpha, \beta_i)^2.$$

But now each of those terms is the number of edges between α and β_i. As this sum across all neighbors of α is strictly less than four, there can be no more than three edges with each vertex as an end.

An immediate consequence of this fact is that if a pair of vertices are connected to each other by a triple edge, they cannot be connected to anything else. So, the only connected Dynkin diagram with a triple edge is the Dynkin diagram for G_2.

5. Whenever an admissible Dynkin diagram contains a line graph, i.e. a sequence of n vertices α_i connected by single edges, this line can be collapsed to a single vertex called α and calculated as $\alpha = \sum_{i=1}^{n} \alpha_i$. The resulting graph is another admissible Dynkin diagram, which we can verify by checking the properties that $(\alpha, \alpha) = 1$ and $4(\alpha, \beta)^2$ is an allowed value (0, 1, 2, or 3) whenever β is a vertex from off the line.

We previously calculated

$$(\alpha, \alpha) = n + 2 \sum_{i=1}^{n} \sum_{j=1}^{i-1} (\alpha_i, \alpha_j).$$

But since $(\alpha_i, \alpha_j) = -1/2$ whenever $j = i + 1$, which happens $n - 1$ times, and 0 otherwise, we can further simplify and say

$$(\alpha, \alpha) = n + 2 \cdot \frac{-1}{2} \cdot (n - 1) = 1.$$

As for the second property we must check, we will use the fact that $(\alpha, \beta) = \sum_{i=1}^{n} (\alpha_i, \beta)$ since any other root β necessarily has an edge in common with no more than one vertex α_i: otherwise, there would have been a cycle in the original Dynkin diagram, which is not admissible. If none at all, $4(\alpha, \beta)^2 = 0$. If there is an i for which α_i and β share an edge, $(\alpha, \beta) = (\alpha_i, \beta)$, so $4(\alpha, \beta)^2 = (\alpha_i, \beta)^2$. We can summarize these facts by saying that α inherits the edges of the α_i vertices.

Since α inherits the edges of the α_i vertices, we can further conclude that no connected Dynkin diagrams contains two double edges, nor two branch vertices (where a *branch vertex* is a vertex directly connected by single edges with three different vertices), nor one of each. If there were two such features, we could collapse a path between them, creating a vertex with four edges, which is not admissible.

6. To summarize our work up to this point, an admissible graph may have at most one special feature distinguishing itself from a line of singly-connected vertices. It may contain no special features at all, or a branch vertex, or a double edge, or a triple edge. However, with the exception of the triple edge (in which case the triple edge and its two vertices are the entire admissible graph), we have not shown what else may happen in the two or three directions extended from that special feature.

7. To extract a formula to help us further describe line subgraphs of Dynkin diagrams, let us take a line graph with ultimate endpoints α_1 and α_n, define $a = \sum_{i=1}^{n} iv_i$, and calculate (a, a):

$$\begin{aligned}(a,a) &= (\sum_{i=1}^{n} iv_i, \sum_{i=1}^{n} iv_i) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}(i\alpha_i, j\alpha_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} ij(\alpha_i, \alpha_j).\end{aligned}$$

By separately considering terms where $i = j$, so $ij(\alpha_i, \alpha_j) = i^2(\alpha_i, \alpha_i)$, and $i \neq j$, so we can as before pair each with the term with reversed subscripts, we can split the summation to

$$(a,a) = \sum_{i=1}^{n} i^2(\alpha_i, \alpha_i) + 2\sum_{i=1}^{n}\sum_{j=1}^{i} ij(\alpha_i, \alpha_j).$$

But like before, $(\alpha_i, \alpha_j) = -1/2$ when $j = i - 1$ which happens $n - 1$ times, and 0 otherwise, and $(\alpha_i, \alpha_i) = 1$, so

$$\begin{aligned}(a,a) &= \sum_{i=1}^{n} i^2 + 2\sum_{i=1}^{n-1}(i(i+1)\frac{-1}{2}) \\ &= n^2 - \sum_{i=1}^{n-1} i^2 - \sum_{i=1}^{n-1}(i^2 + i) \\ &= n^2 - \sum_{i=1}^{n-1}(i^2 - i^2 - i)) \\ &= n^2 + \sum_{i=1}^{n-1} i \\ &= n^2 + \frac{n(n-1)}{2} \\ &= \frac{n^2 + n}{2}.\end{aligned}$$

8. Next we apply the formula $(a, a) = \frac{n^2+n}{2}$ to an admissible Dynkin diagram which contains a double edge and extends with line subgraphs from each vertex at the end of the double edge. Starting with the longer of the two subgraphs if they are not the same length, if we name the n roots in one line subgraph α_i sequentially from the outer end toward the vertex which is one end of the double edge, then define a as in item 7, and similarly name the m roots in the other line subgraph β_i and correspondingly define b, we have,

$$(a, a) = \frac{n^2 + n}{2}, \qquad (b, b) = \frac{m^2 + m}{2}.$$

We already know $n \geq m$, but we can find out more about the possible values of n and m by considering (a, b); owing to other available information, it is actually more convenient to begin with $(a, b)^2$. Using the facts from the structure of the Dynkin diagram that α_n and β_m are connected by a double edge, i.e. $4(\alpha_n, \beta_m)^2 = 2$ therefore $(\alpha_n, \beta_m)^2 = 1/2$, and $(\alpha_i, \beta_j) = 0$ for all other values of i and/or j, we can calculate:

$$\begin{aligned}(a, b)^2 &= (n\alpha_n, m\beta_m)^2 \\ &= [nm(\alpha_n, \beta_m)]^2 \\ &= \frac{n^2m^2}{2}.\end{aligned}$$

We can also use the Cauchy-Schwarz inequality [8] to investigate $(a, b)^2$. Since a and b are constructed in such a way as to not be scalar multiples of each other,

$$(a, b)^2 < (a, a)(b, b).$$

Combining these formulas, we find

$$\begin{aligned}
\frac{n^2m^2}{2} &< \frac{n(n+1)}{2}\frac{m(m+1)}{2} \\
2n^2m^2 &< n(n+1)m(m+1) \\
2nm &< nm + m + n + 1 \\
0 &< -nm + m + n + 1 \\
0 &> nm - m - n - 1.
\end{aligned}$$

For reasons which will shortly become apparent, we shall now investigate the product of the integers one less than each of n and m:

$$\begin{aligned}
(n-1)(m-1) &= nm - m - n + 1 \\
&= nm - m - n - 1 + 2 \\
&< 0 + 2.
\end{aligned}$$

So we see that the integers one less than each of m and n are a factor pair for a product which is less than 2. If that product is 0, then $m = 1$ and n may be any positive whole number; this describes the diagrams for B_ℓ and C_ℓ. On the other hand, if that product is 1, it must be the case that $n - 1 = m - 1 = 1$, in which case $n = m = 2$, which describes the F_4 diagrams.

That is, the only Lie algebras whose Dynkin diagrams have a double edge are B_ℓ, C_ℓ, and F_4.

9. Finally, let us consider the cases where a Dynkin diagram has one special feature, but rather than a double edge, it is a branch vertex, which we may name δ. Like before, we shall name the other vertices by the line in which they appear, numbered from outside to inside, with α_n, β_m, and γ_p being the neighbors of δ, with the α_i branch being the longest, the β_i branch being next longest, and γ_i the shortest (though none of these branch length comparisons are strict);

that is, $n \geq m \geq p$. Also like before, for each branch we shall have a vector, a and b as before and c correspondingly made from the γ_i roots. Our goal is to determine valid combinations for n, m, and p, the lengths of the three lines extending from the branch vertex.

To form an orthonormal basis for a space containing δ, take $a' = a/|a|$, $b' = b/|b|$, and $c' = c/|c|$, and let d' be some unit vector orthogonal to a, b, and c, but not orthogonal to δ so that we will have $(\delta, d') \neq 0$. Then we can write δ as

$$\delta = (\delta, a')a' + (\delta, b')b' + (\delta, c')c' + (\delta, d')d',$$

from which we may further calculate

$$\begin{aligned}
(\delta, \delta) &= ([(\delta, a')a' + (\delta, b')b' + (\delta, c')c' + (\delta, d')d'], \delta) \\
1 &= (\delta, a')^2 + (\delta, b')^2 + (\delta, c')^2 + (\delta, d')^2 \\
1 - (\delta, d')^2 &= (\delta, a')^2 + (\delta, b')^2 + (\delta, c')^2 \\
1 &> (\delta, a')^2 + (\delta, b')^2 + (\delta, c')^2.
\end{aligned}$$

Now we would like to rewrite each term of the right hand side of this inequality. Beginning with the first term, we have

$$\begin{aligned}
(\delta, a')^2 &= \left(\delta, \frac{1}{|a|}a\right)^2 \\
&= \frac{1}{|a|^2}(\delta, a)^2 \\
&= \frac{1}{(a,a)}(\delta, n\alpha_n)^2. \\
&= \frac{1}{(a,a)}\left(\sum_{i=1}^{n}(\delta, i\alpha_i)\right)^2.
\end{aligned}$$

At this point, we will find it useful to recall the formula for (α, α). We will also take advantage of the fact that $(\delta, \alpha_i) = 1/2$ because these are roots which do not share an edge, as well as the fact that the only nonzero term of $\sum_{i=1}^{n}$ is the

nth term, $(\delta, n\alpha_n)$, to continue simplifying:

$$\begin{aligned}
(\delta, a')^2 &= \frac{2}{n^2+n}(\delta, n\alpha_n)^2 \\
&= \frac{2}{n^2+n}[n(\delta, \alpha_n)]^2 \\
&= \frac{2n^2}{n^2+n}(\frac{1}{2})^2 \\
&= \frac{2n^2}{4n(n+1)} \\
&= \frac{n}{2(n+1)}.
\end{aligned}$$

Applying the same pattern to the second and third terms, we obtain

$$(\delta, b')^2 = \frac{m}{2(m+1)}, \qquad (\delta, c')^2 = \frac{p}{2(p+1)},$$

which we can substitute into the inequality from above, yielding

$$1 > \frac{n}{2(n+1)} + \frac{m}{2(m+1)} + \frac{p}{2(p+1)}.$$

A sequence of basic algebra steps in [28] shows that this is equivalent to

$$\frac{1}{n+1} + \frac{1}{m+1} + \frac{1}{p+1} > 1.$$

Now bearing in mind both this inequality and $n \geq m \geq p$, let us consider possible triplets (n, m, p). First trying a simple case where $n = m = p$, their value is, we have $1/2 + 1/2 + 1/2 = 3/2$, which complies with the inequality; this is D_4. Also, we can notice from this case that $1/(n+1) + 1/2 + 1/2 > 1$ for any value of n; therefore, if $m = p = 1$, n may have any value, giving D_{n+3}. However, if $n = m = p = 2$, we have $1/3 + 1/3 + 1/3 = 1$, which is not strictly greater than 1, so is not an admissible diagram.

One feature we notice from these calculations is that the $1/(p+1)$ contribution to the total is the greatest, due to p being the smallest of the three branch lengths. Therefore we must have that

$$\frac{1}{p+1} > \frac{1}{3},$$

so p must be less than 2; that is, $p = 1$. Then we can say

$$\frac{1}{n+1} + \frac{1}{m+1} > \frac{1}{2},$$

and similarly infer that $m < 3$, so either $m = 1$ or $m = 2$. As we saw before, if $m = p = 1$, n is unbounded and we have D_{n+3}, so let us consider the case when $p = 1$ and $m = 2$:

$$\begin{aligned} \frac{1}{n+1} + \frac{1}{m+1} + \frac{1}{p+1} &> 1 \\ \frac{1}{n+1} + \frac{1}{3} + \frac{1}{2} &> 1 \\ \frac{1}{n+1} &> \frac{1}{6} \\ 6 &> n+1 \\ 5 &> n. \end{aligned}$$

If $(n, m, p) = (4, 2, 1)$, we have E_8; if $(n, m, p) = (3, 2, 1)$ we have E_7; and if $(n, m, p) = (2, 2, 1)$ we have E_6.

Therefore, the Dynkin diagrams illustrated in Figure 3.4 are the only admissible Dynkin diagrams. As a consequence of this, we have our main result:

Theorem 3.4. *The only simple complex Lie algebras are those from the four classical families A_ℓ, B_ℓ, C_ℓ, and D_ℓ and the five exceptional algebras E_6, E_7, E_8, F_4, and G_2.*

Proof. We have already done almost all the work for this result. It only remains to point out that, as we know all the admissible Dynkin diagrams, and a Dynkin diagram determines a simple complex Lie algebra, we also know all the simple complex Lie algebras. □

This unusually tidy set of possibilities is an exceptionally useful feature of Lie algebra. Together with the close connection between Lie algebras and the transformation groups and symmetries which attracted Lie and Killing's attention, this means that mathematicians can be well on their way to understanding a wide variety of phenomena by understanding a comparatively small set of structures.

Though the result is tidy, the story behind the result is a tangled tapestry of human lives: the convergent ideas and insights, but also rivalry, of Sophus Lie and Wilhelm Killing; the ability of Friedrick Engel to communicate with other mathematicians but not to understand the jealousy he provoked; Élie Cartan's fortune, and all of ours, that he had teachers through whose attention he was able to be plucked from obscurity; the escapes of Hermann Weyl and Evgenii Dynkin from hostile countries to the United States where their ideas could reach the main stream of mathematical thought.

By examining and appreciating the motivations and processes of mathematicians before us, we can better understand how mathematics in general and Lie algebra in particular continue to serve as fields of exploration for mathematicians today and in the future.

4 Appendix

4.1 Vector Spaces

Vector spaces over a field F are commonly defined as having the following properties, for any vectors u, v, w from the vector space, and scalars a, b from the field:

1. Associativity of vector addition: $(u + v) + w = u + (v + w)$
2. Commutativity of vector addition: $u + v = v + u$
3. Identity element under vector addition: 0 such that $v + 0 = v$
4. Inverse elements under vector addition: $-v$ such that $v + (-v) = 0$
5. Compatible behavior of scalar multiplication generally: $a(bv) = (ab)v$
6. Compatible behavior of scalar multiplication by the field's multiplicative identity element: $1v = v$
7. Distributivity of scalar multiplication across vector addition: $a(u+v) = au+av$
8. Distributivity of scalar multiplication across field addition: $(a + b)v = ab + av$

Index

BIBLIOGRAPHY

[1] Michael Atiyah, *Hermann Weyl*, Biographical Memoirs, no, 82, National Academies Press, Washington, D.C., 2003.

[2] Luca Bombelli, *Lorentz group*, http://www.phy.olemiss.edu/ luca/Topics/l/lorentz.html.

[3] Andreas P. Braun and Michele Del Zotto, *Mirror symmetry for g2-manifolds: twisted connected sums and dual tops*, Journal of High Energy Physics **2017** (2017), 1–29.

[4] Elie Cartan, *Sur la structure des groupes de transformations finis et continus*, Ph.D. thesis, University of Paris, 1894.

[5] ______, *Oeuvres complétes*, Éditions du Centre National de la Recherche Scientifique (CNRS), Paris, 1984.

[6] Alberto Cogliati, *Continuous groups of transformations: Elie cartan's structural approach*, Ph.D. thesis, Università degli Studi di Milano, 2012.

[7] Karin Erdmann and Mark J. Wildon, *Introduction to Lie algebras*, Springer, New York, NY, 2006.

[8] Arash Farahmand, *6.7 Cauchy-Schwartz Inequality | math 54 notes*, https://math.berkeley.edu/ arash/54/notes/6_7.pdf.

[9] W. Fulton and J. Harris, *Representation theory: A first course*, Springer, New York, NY, 2004.

[10] Brian C. Hall, *Lie groups, Lie algebras, and representations: An elementary introduction*, Springer, New York, NY, 2015.

[11] Thomas Hawkins, *Emergence of the theory of Lie groups*, Springer, New York, NY, 2000.

[12] Sigurdur Helgason, *Sophus Lie, the mathematician*, https://math.mit.edu/ helgason/sophus-lie.pdf.

[13] M. N. Hounkonnou and E. B. Ngompe Nkouankam, *Generalized Heisenberg algebra: application to the harmonic oscillator*, Journal of Physics A: Mathematical and Theoretical **40** (2007), no. 27, 7619.

[14] hoyland (https://math.stackexchange.com/users/14722/hoyland), *Jacobi identity - intuitive explanation*, Mathematics Stack Exchange, URL:https://math.stackexchange.com/q/113785 (version: 2012-02-26).

[15] J.E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer, New York, NY, 1972.

[16] Nathan Jacobson, *Lie algebras*, Dover Publications, Inc, Mineola, NY, 1962.

[17] Lizhen Ji, *Sophus Lie: A real giant in mathematics*, Notices of the ICCM **3** (2015).

[18] Wilhelm Killing, *Die zusammensetzung der stetigen endlichen transformations-gruppen*, Math. Ann **31** (1888), 252–290.

[19] ———, *Die zusammensetzung der stetigen endlichen transformations-gruppen*, Math. Ann **33** (1888), 1–48.

[20] ———, *Die zusammensetzung der stetigen endlichen transformations-gruppen*, Math. Ann **34** (1889), 57–122.

[21] ———, *Die zusammensetzung der stetigen endlichen transformations-gruppen*, Math. Ann **2** (1890), 161–189.

[22] A. Garrett Lisi, *An exceptionally simple theory of everything*, 2007.

[23] J. J. O'Connor and E. F. Robertson, *Sophus Lie*, https://mathshistory.st-andrews.ac.uk/Biographies/Lie/, 2000.

[24] ______, *Wilhelm Karl Joseph Killing*, https://mathshistory.st-andrews.ac.uk/Biographies/Killing/, 2005.

[25] ______, *Elie Cartan*, https://mathshistory.st-andrews.ac.uk/Biographies/Cartan/, 2014.

[26] ______, *Friedrich Engel*, https://mathshistory.st-andrews.ac.uk/Biographies/Engel/, 2014.

[27] Henri Poincaré, *Rapport sur les travaux de M. Cartan*, Acta Mathematica **38** (1914), 137–145.

[28] Brooks Roberts, *Lie algebras: Course notes from 2018-2019*, https://www.webpages.uidaho.edu/ brooksr/liealgebraclass.pdf.

[29] Jean-Pierre Serre, *Algèbres de Lie semi-simples complexes*, W. A. Benjamin, New York, NY, 1966.

[30] David Shiga, *'most beautiful' math structure appears in lab for first time*, https://www.newscientist.com/article/dn18356-most-beautiful-math-structure-appears-in-lab-for-first-time/, 2010.

[31] A. P. Soldatov, *Jacobi brackets*, https://encyclopediaofmath.org/index.php? title=Jacobi_brackets&oldid=12643, 2006.

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