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Restrictions on Topological Symmetry Groups of the 3-Rung Möbius Ladder on the Torus

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by

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ABSTRACT

In this work, we discuss properties of the 3-rung Möbius ladder embedded on the surface of a torus. We present proofs on restrictions of topological symmetry groups of the Möbius ladder with and without the assumption of preserving orientation. Specifically, we show that $\mathbb{Z}_2$ is the only possible non-trivial orientation-preserving topological symmetry groups, and also that $\mathbb{Z}_2$ and D_2 are the only possible nontrivial topological symmetry groups.

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1 Introduction

Stereochemistry is the study of the three-dimensional structure of molecules and topology is the study of properties of spaces that are invariant under any continuous deformation. While both of these areas are interesting to study within their respective fields of chemistry and mathematics, the topological processes of studying spaces have become increasingly useful as new and more complex molecules are synthesized and discovered. Sixty years ago, all known molecular structures had *planar graph* representations: their graphs can be drawn on the plane in such a way that its edges intersect only at their endpoints, that is, the edges of the graph do not cross each other. While topology could be used to determine the stereochemistry of these molecules, they could be more easily determined using simple techniques from geometry [4].

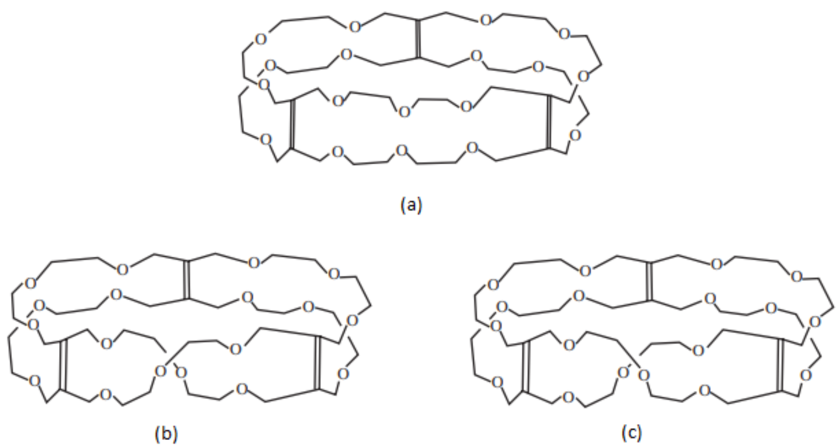


Figure 1.1: Walba, Richards, and Haltiwanger's molecules [5]

In 1982, Walba, Richards, and Haltiwanger synthesized the first molecular 3-rung Möbius ladder as shown in Figure 1.1(b,c) [5]. Notice that the outer loop of this molecule resembles the sides of a Möbius strip. This, in conjunction with the 3-rungs joining the sides of the hollow strip, makes the 3-rung Möbius ladder molecule aptly named.

This molecule is of particular importance as it was one of the first to be considered *topologically complex*, meaning it could not be deformed into a plane and thus did not have a planar graph representation. Walba was able to synthesize the 3-rung Möbius ladder by first creating molecular ladders. The sides of these ladders are composed of a polyether chain of 60 atoms which are all oxygens and carbons while the rungs of the ladder were composed strictly of carbon-carbon double bonds. This meant that the sides and the rungs of the ladder were chemically distinct from each other. The ends of the ladder were then forced to attach, but these ladders did not join to themselves in the same way: they took the form of a cylindrical ladder as in Figure 1.1(a), or a Möbius ladder with a left or right twist as shown in Figure 1.1(b,c). In order to demonstrate that the synthesized molecules had the structure of a 3-rung Möbius ladder, the chemists used NMR (nuclear magnetic resonance) to show that a some of the molecules produced were *chiral*. A molecule is said to be *chemically achiral* if it can be deformed into its mirror image, otherwise, it is said to be *chemically chiral*. Here it is clear that the cylindrical 3-rung ladder is achiral, but it may be less clear that the 3-rung Möbius ladder is chiral [4].

Walba strongly suspected — but was unable to show — that a 3-rung Möbius ladder must be chiral. It was not until 1986 that Jon Simon, a topologist, proved that a 3-rung Möbius ladder cannot be deformed into its mirror image such that sides go to sides and rungs go to rungs [12]. This distinction is important as the rungs and the sides of the molecule in question are chemically distinct. It was further shown by Flapan in 1989 that this result holds true for any Möbius ladder with an odd number of rungs greater than one [3].

Topologists often like to think of these molecules in terms of *graphs*, or a collection of vertices and edges connecting the vertices. Once a molecule has a given mathematical interpretation, it is also necessary to talk about the space in which we are analyzing a given graph. Whenever a graph is embedded in a particular space, it is referred to as an *embedding* of the given graph in that space.

Speaking in a general sense, topology is used in stereochemistry to recognize when one embedding of a graph cannot be deformed into another embedding, that is, to be able to recognize distinct molecules, and to evaluate their topological properties. Molecules that are structurally related to each other are called *isomers*. The question of determining when two molecules are structurally related naturally asks for some kind of topological analysis, particularly among topologically complex graphs. The *topological stereoisomers* of a given molecule are those molecules that have the same abstract graph as the given molecule, but their embedded graphs cannot be deformed into one another [4].

The *topological symmetry group* for an embedding in a topological space is the group of automorphisms of the embedding which are induced by homeomorphisms of the topological space. Determining the topological symmetry group of an embedding of a molecule in a given space can supply chemists with important information about the structure of that molecule.

In past work, the topological symmetry groups of the Möbius ladder have been classified when embedded in the 3-sphere, S^3. This was done with the intent of describing the topological symmetry groups of the Möbius ladder in $\mathbb{R}^3$ as topological symmetry groups are the same for graphs embedded in $\mathbb{R}^3$ as in S^3 [6]. It has also been shown that $\mathbb{Z}_2$ occurs as a topological symmetry group for the 3-rung Möbius ladder embedded in the torus [13]. In our work, we are interested in which homeomorphisms of the torus induce automorphisms of the Möbius ladder. We will present several restrictions for the topological symmetry groups of the Möbius ladder embedded on the torus, T^2.

1.1 Topology

Here we present some topological ideas which will be used in our arguments.

Definition 1.1. A *homeomorphism*, or *topological equivalence*, is a bijective continuous function between topological spaces with a continuous inverse [1].

Definition 1.2. Let X and Y be topological spaces. We say f is an *embedding* of X into Y provided that $f : X \to Y$ is an injective map, $f : X \to f(X)$ is a homeomorphism, and $f(X)$ has the induced topology from Y [1].

Definition 1.3. A *graph* $G = \{V, E\}$ is a set of vertices, V, and a set of edges, $E \subseteq \{\{x, y\} | x, y \in V, x \neq y\}$.

It will also be beneficial to make a distinction between certain edges on given graphs. Recall that the Möbius ladder has distinct chemical compositions for the rungs and the sides. If we were to translate the Möbius ladder directly into a graph, we would lose this information, so we consider the rungs and the sides to be distinct when discussing its graph.

Definition 1.4. Let X be a graph considered as a topological space with the discrete topology and Y a topological space. If an embedding f of X into Y exists, we call $f(X)$ an *embedded graph*.

Definition 1.5. An *automorphism* of a graph is defined as a bijection from the graph to itself, taking vertices to vertices and edges to edges in such a way that adjacent vertices are taken to adjacent vertices [4].

Definition 1.6. A graph embedded in three-dimensional space is *topologically achiral* if it can be deformed into its mirror image. Otherwise it is *topologically chiral* [5].

The above definitions help us to translate the molecular ladder into mathematical terms. Since we are also concerned with the stereochemistry of this molecule, and its graph is topologically complex, we will require techniques from topology to help us.

Definition 1.7. For a particular embedded graph G in a topological space X, the *topological symmetry group* is the group of automorphisms of G induced by homeomorphisms on X. Similarly, a *finite topological symmetry group* is the group of automorphisms of G induced by homeomorphisms of finite order on X [6].

From now on, we will refer to a topological symmetry group of finite order as a TSG.

In our work we will look at the topological symmetry group of a particular embedding of the 3-Rung Möbius ladder in the torus. It should be noted that there are many different ways in which one can embed the 3-rung Möbius ladder in the torus. We will show some of these in section 3.1.1.

Definition 1.8. Let A and B be subsets of $\mathbb{R}^3$, and let $h : A \to B$ and $g : A \to B$ be homeomorphisms. We say h and g are *isotopic* if there exists a continuous function $F : A \times [0, 1] \to B$ such that $F(x, 0) = h(x), F(x, 1) = g(x)$, and for every fixed $t \in [0, 1]$, the function $F(x, t)$ is a homeomorphism [4] .

Definition 1.9. Let $h : T^2 \to T^2$ be a homeomorphism. We say h is *orientation-preserving* if h preserves the orientation of the torus. Otherwise, h is said to be *orientation-reversing* [2] .

In certain cases, we wish to consider all possible homeomorphisms, while in others, we might only consider orientation-preserving homeomorphisms. It should be noted that all homeomorphisms are either orientation-preserving or orientation-reversing. When considering orientation-preserving homeomorphisms, we are actually referencing a different topological symmetry group than when considering all possible homeomorphisms. We will continue to refer to a topological symmetry group without the orientation-preserving restriction as a TSG. When only considering homeomorphisms which are orientation-preserving, we would then refer to a orientation-preserving topological symmetry group which we denote TSG_+.

2 The Torus

The typical definition of a *torus* shown in Figure 2.1 is given by the direct product of two circles, $S^1 \times S^1$. This definition also allows us to easily define the *longitude* as the first circle, l, and the *meridian* as the second, m. Since all the embeddings we will be discussing are on a torus, we will first look at different ways in which we can represent the torus. Being able to look at a torus from different perspectives will be a powerful tool in our future arguments.

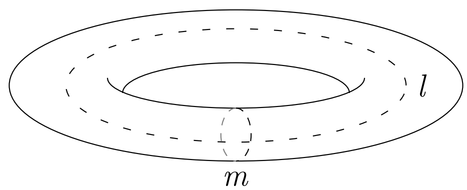


Figure 2.1: The Torus [13]

2.1 Identification Spaces

There are many different ways that one can view the torus. For the purpose of examining our embeddings, we will almost always avoid looking at molecules embedded directly on the surface of the torus itself. This is because it is hard to understand how the surface of the torus changes when going through transformations. Luckily, topology gives us a lovely tool that allows us to visually and intuitively simplify the space that we are looking at: *identification spaces*.

Definition 2.1. Let X be a topological space and let $\mathcal{P}$ be a family of disjoint nonempty subsets of X such that $\bigcup \mathcal{P} = X$, called the *partition* of X. We form a new space, Y, called an *identification space*, as follows: the points of Y are the members of $\mathcal{P}$ and, if $\varphi : X \to Y$ sends each point of X to the subset of $\mathcal{P}$ containing it, the topology on Y is the largest for which φ is continuous. This topology is called the *identification topology* on Y. We think of Y as the space obtained from X by identifying each of the subsets of $\mathcal{P}$ to a single point [1] .

We will consider two primary identification spaces of the torus: the unit square and the unit hexagon. We can construct the torus from the unit square by identifying opposite edges of the square along with identifying all four corner points. This process is shown in Figure 2.2.

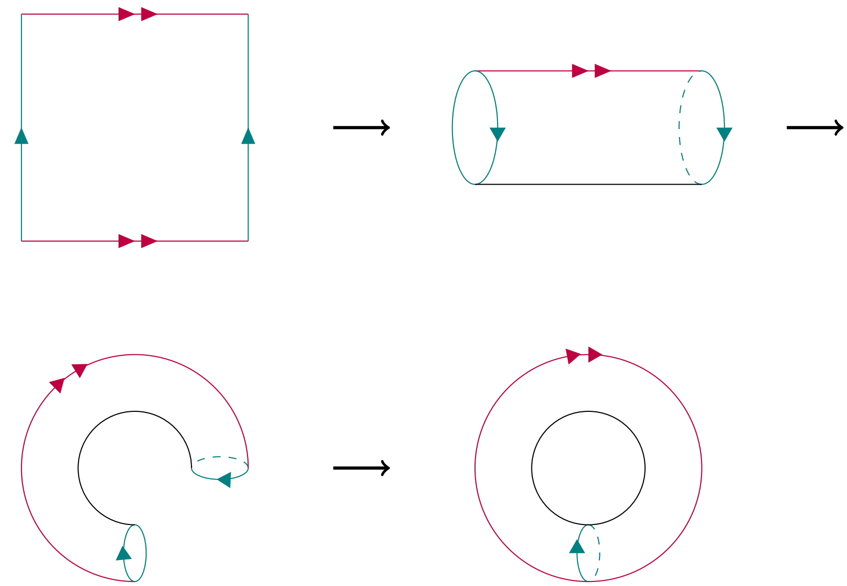


Figure 2.2: Construction of the torus from the unit square

Notice that we can easily follow the longitude and meridian on the unit square, as one pair of edges that we identified will serve as the longitude (red) and the other pair of edges will serve as the meridian (green).

We perform a similar process to construct the torus from a unit hexagon, this time instead identifying three pairs of edges along with the six corner points. It is important to note in this case that not only are the longitude and meridian not on the borders of the identification space, but also that the longitude and meridian can be represented in multiple ways. This is entirely dependent on the twist of the tube as demonstrated in step 2 of Figures 2.3 and 2.4. While the direction of the twist will cause the longitude to appear in a different location on the unit hexagon, it still acts as a longitude on the torus.

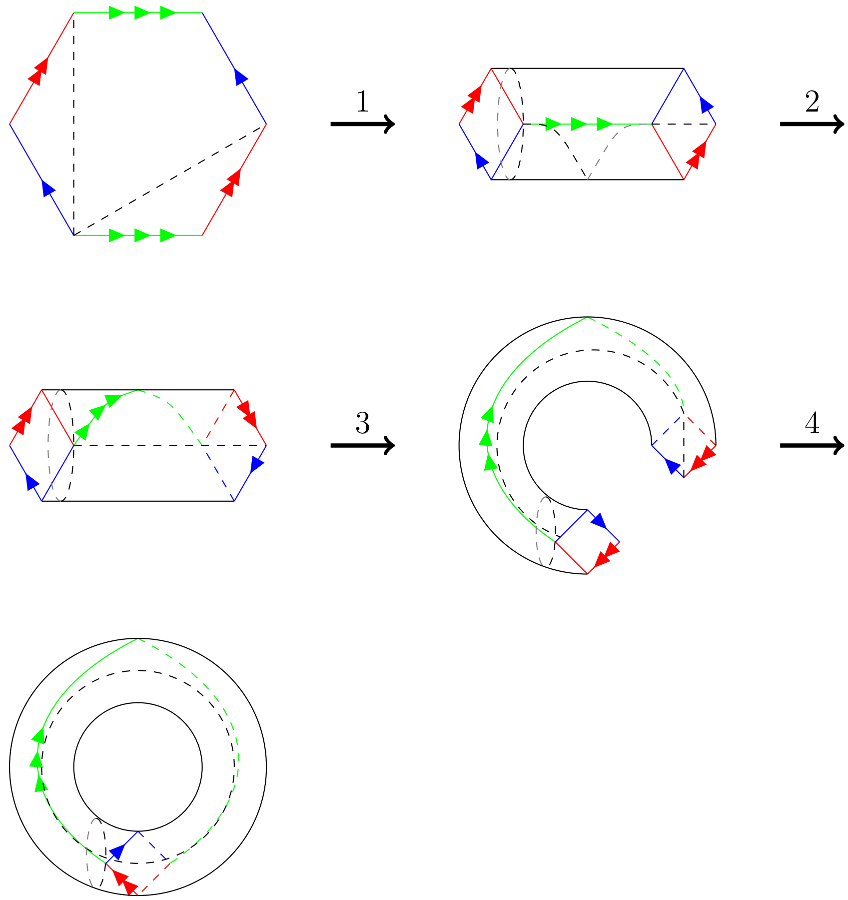


Figure 2.3: Construction of the torus from the unit hexagon (Forward Twist)

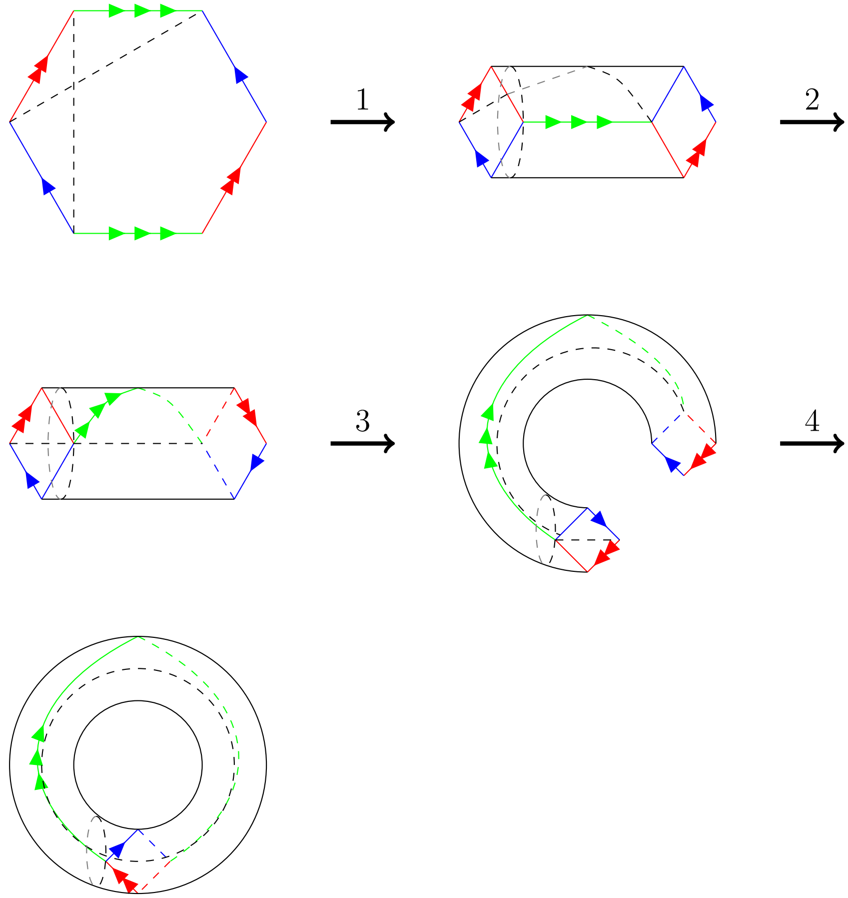


Figure 2.4: Construction of the torus from the unit hexagon (Alternate Longitude)

2.2 Torus Knots

In our work, we would like to appeal to the sides of a Möbius ladder as a torus knot. Recall that the general structure of the Möbius ladder with 3-rungs is shown in Figure 2.5. It will sometimes be useful to refer to the *loop* of the ladder without the rungs. Since the loop is a subset of the embedded graph, if we are able to show that a property is not held between two loops, then it will also not be held between two embeddings of the graph. This can be stated with confidence as we are considering the sides and the rungs of the ladder to be distinct.

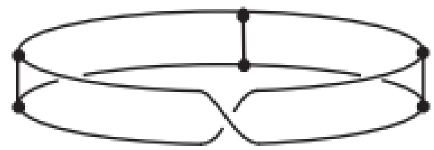


Figure 2.5: 3-rung Möbius Ladder [4]

Definition 2.2. A *loop* in a topological space X is a continuous function $\alpha : [0,1] \to X$ such that $\alpha(0) = \alpha(1)$, and we shall say the loop is *based* at the point $\alpha(0)$.

We define the loop of the 3-rung Möbius ladder on the torus shown in Figure 2.6 as the sides of the ladder based at 1, and follows the vertices of the ladder where the rungs meet the sides continuously. So, our automorphisms must send the loop to itself through continuous deformation.

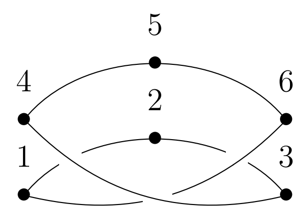


Figure 2.6: The loop of the 3-rung Möbius ladder [13]

Now that we have a loop alone that can be embedded on the torus without the additional rungs, we have a structure which naturally corresponds with torus knots.

Definition 2.3. For relatively prime integers p and q, a (p, q) *torus knot* is a closed loop on the torus which wraps longitudinally p times and meridianally q times [9] .

It should be noted that wrapping in the longitudinal direction p times corresponds to crossing the meridian a net total of p times. Similarly, wrapping in the direction of the meridian q times corresponds to crossing the longitude a net total of q times.

By viewing the loop of any embedded Möbius ladder as a torus knot, we can extend many of the classifications of torus knots to possible embeddings that we may want to consider. One classification result that will be of use later states that p and q must be relatively prime in order for a (p, q) torus knot to be an knot rather than a link. If p and q are not relatively prime then the resulting embedding is actually a link of multiple loops rather than a single loop [9]. Since we wish to describe the singular loop of the 3-rung Möbius ladder, this result applies to the embeddings we are interested in as well. Now that we have these loops defined in terms of functions, we have a natural topological tool for determining whether two different loops are equivalent: *homotopy*.

Definition 2.4. Let X, Y be topological spaces and $f, g : X \to Y$ be continuous functions. Then we say that f is *homotopic* to g if there exists a continuous function $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all points $x \in X$. Furthermore, the function F is called a *homotopy* from f to g [1] .

For the cases we are interested in, we do not need to meet all requirements of an isotopy since homotopy supplies us with the deformation we desire. We also have that the relation of homotopy is an equivalence relation. This gives us another method to compare loops and embeddings in certain cases.

Definition 2.5. The equivalence class of a loop L under the equivalence relation of homotopy is denoted $[L]$ and called the *homotopy class* of L [7].

Definition 2.6. The set of homotopy classes of loops in a topological space X based at some point p forms a group referred to as the *fundamental group* [1].

We use the fundamental group of the torus to describe the homotopy class of the loop. It has already been shown that the fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$ [1]. Homotopy classes can be seen as ordered pairs (p, q) where the first coordinate represents the number of times the loop circles the longitude, and the second coordinate represents the number of times the loop circles the meridian. The sign of the given integer determines the direction that the loop will wrap, with direction being assigned in relation to where the meridian and longitude intersect (much like a coordinate system).

Because of the structure of our embeddings, there is an additional concern when discussing homotopy classes. Consider the loop in Figure 2.7 embedded on the unit square. Recalling that the green line serves as a longitude and the red line serves as a meridian, we attribute the positive directions for each as right and up accordingly. If we consider tracing the loop beginning from the origin, it is clear that the loop circles the longitude twice in the positive direction and circles the meridian once in the positive direction. This indicates the loop would have a homotopy class of $(2, 1)$. However, we could also begin by tracing the loop from the top right of the identification space and travel in the reverse direction. Doing so does not change the embedding in a significant way as the loop of the molecule is non-oriented. When obtaining the homotopy class this way, we see that that we instead get $(-2, -1)$. Even though these two homotopy classes are distinct in having different integers, there is no difference in these homotopy class as we are not worried about a direction in our loop. We must be mindful of this in the future as this indicates homotopy classes of the form (p, q) and $(-p, -q)$ are actually equivalent under the conditions of our problem.

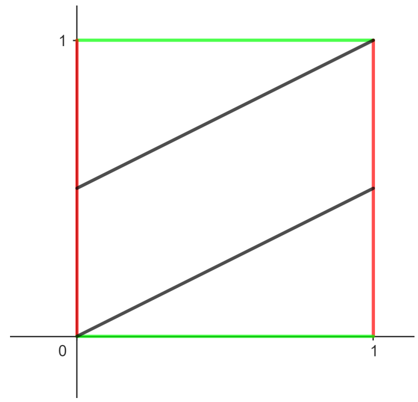


Figure 2.7: Loop with homotopy class $(2, 1)$ or $(-2, -1)$

2.3 Automorphisms of the Graph and Homeomorphisms of the Torus

In our work, we would like to discuss both orientation-preserving and orientation-reversing homeomorphisms. To do this, we must first discuss what topological symmetry groups we are interested in and what they must look like.

Notice that the graph of the 3-rung Möbius ladder has 6 vertices, that is, its vertex set consists of 6 elements. Furthermore, any automorphism of this graph must have a bijection from the vertex set to itself. This indicates that any possible topological symmetry group must be a subgroup of the symmetric group S_6 as we are considering permutations of these 6 possible vertices and S_6 is precisely that.

The nontrivial subgroups of S_6 up to isomorphism are: $\mathbb{Z}_6, \mathbb{Z}_5, \mathbb{Z}_4, \mathbb{Z}_3, \mathbb{Z}_2, D_6,$ $D_5, D_4, D_3, D_2, A_6, A_5, A_4, S_6, S_5, S_4, \mathbb{Z}_5 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4, (\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_4, (\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, D_3 \times D_3, D_4 \times \mathbb{Z}_2, D_3 \times \mathbb{Z}_3, A_4 \times \mathbb{Z}_2, S_4 \times \mathbb{Z}_2, S_3 \wr \mathbb{Z}_2$ [6]. Luckily, we can further refine this list by specifically considering homeomorphisms of the torus, as this is the space for the topological symmetry group in which we are interested.

It has been previously shown that all finite order homeomorphisms of the torus correspond, up to isotopy, to elements of $GL(2, \mathbb{Z})$ [8] . This means that when looking at the embedding of a 3-rung Möbius ladder on a torus, we should consider all possible subgroups of S_6 which are isomorphic to subgroups of $GL(2, \mathbb{Z})$. Then we obtain the following list of possible nontrivial subgroups: $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6, D_2, D_3, D_4, D_6$.

These are the base restrictions we are going to build upon. There is, however, another way in which we can classify these topological symmetry groups: those induced by orientation-preserving homeomorphisms and those induced by orientation-reversing homeomorphisms. As previously stated, orientation-preserving homeomorphisms and orientation-reversing homeomorphisms comprise all possible homeomorphisms. In our case, we gain useful insight looking at orientation-preserving homeomorphisms, so we consider these first.

It has been shown that that a homeomorphism of the torus is orientation-preserving if and only if it corresponds to an element of $SL(2, \mathbb{Z})$ [8]. So, to consider orientation-preserving topological symmetry groups of the 3-rung Möbius ladder on the torus, we can refine our list even further by considering subgroups of S_6 which are isomorphic to groups generated by elements of $SL(2, \mathbb{Z})$ which have finite order. Doing this eliminates the dihedral groups and leaves us with the following possible orientation-preserving topological symmetry groups: $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$.

Matrix	Order	Rotation	Matrix	Order	Rotation
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	1	0	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	2	π
$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$	6	$\frac{\pi}{3}$	$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$	3	$\frac{4\pi}{3}$
$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$	6	$\frac{\pi}{3}$	$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$	3	$\frac{4\pi}{3}$
$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	4	$\frac{\pi}{2}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	4	$\frac{3\pi}{2}$
$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$	3	$\frac{2\pi}{3}$	$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$	6	$\frac{5\pi}{3}$
$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$	3	$\frac{2\pi}{3}$	$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$	6	$\frac{5\pi}{3}$

Table 2.1: The finite order $SL(2,\mathbb{Z})$ elements up to conjugacy

When we would like to consider all possible homeomorphisms, we need to return to considering subgroups of $GL(2,\mathbb{Z})$. Recall that we can look at both orientation-reversing and preserving symmetry groups by considering transformations of the torus by elements of $GL(2,\mathbb{Z})$ [8]. A complete list of nontrivial finite subgroups of $GL(2,\mathbb{Z})$ up to conjugacy is shown below [10].

1. $\mathbb{Z}_2$ afforded by $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

2. $\mathbb{Z}_2$ afforded by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3. $\mathbb{Z}_2 \times \mathbb{Z}_2$ afforded by $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

4. $\mathbb{Z}_2 \times \mathbb{Z}_2$ afforded by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

5. $\mathbb{Z}_3$ afforded by $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$

6. $\mathbb{Z}_4$ afforded by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

7. $\mathbb{Z}_6$ afforded by $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

8. Σ_3 afforded by $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

9. Σ_3 afforded by $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

10. $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$ afforded by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

11. $\mathbb{Z}_6 \rtimes \mathbb{Z}_2$ afforded by $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Recall earlier in this section that we used a slightly different list of subgroups of $GL(2,\mathbb{Z})$ when looking for TSG's. The groups in these lists are still isomorphic to each other, so this list still captures all the possible TSG's. We use this particular list over the other as this list includes a list of elements from $GL(2,\mathbb{Z})$ which generate the given groups. Many of these groups will be quickly eliminated as candidates for TSG's, and thus it is not important to know exactly which groups are isomorphic to each other. The one that will be important in our results is $\mathbb{Z}_2 \times \mathbb{Z}_2$, which is isomorphic to D_2. From now on we will refer to this group only as D_2.

It is often easier to work with the generators of these groups in order to determine if they could possibly generate symmetry groups for an embedding. A table of all generators of finite subgroups of $GL(2,\mathbb{Z})$ is shown in Tables 2.2 and a table of all elements conjugate to the generators is shown in Table 2.3.

Matrix	Order	Eigenvalues	Eigenvectors	Trace
$R_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	2	$-1, -1$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	-2
$R_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	2	$-1, 1$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$	0
$R_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	2	$-1, 1$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	0
$R_4 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	2	$-1, 1$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	0
$R_5 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$	3	$\frac{1}{2}(-1 \pm i\sqrt{3})$	$\begin{bmatrix} \frac{1}{2}(1 \pm i\sqrt{3}) \\ 1 \end{bmatrix}$	-1
$R_6 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	4	$i, -i$	$\begin{bmatrix} i \\ 1 \end{bmatrix} \begin{bmatrix} -i \\ 1 \end{bmatrix}$	0
$R_7 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$	6	$\frac{1}{2}(1 \pm i\sqrt{3})$	$\begin{bmatrix} \frac{1}{2}(1 \pm i\sqrt{3}) \\ 1 \end{bmatrix}$	1
$R_8 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$	2	$-1, 1$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$	0
$R_9 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	4	$i, -i$	$\begin{bmatrix} -i \\ 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix}$	0

Table 2.2: Matrices which generate all finite subgroups of $GL(2, \mathbb{Z})$ up to conjugacy

Matrix	Conjugate Form	Eigenvalues	Eigenvectors	Trace
R_1	$\begin{bmatrix} bc-ad & 0 \\ 0 & bc-ad \end{bmatrix}$	$bc-ad$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	± 2
R_2	$\begin{bmatrix} bd-ac & a^2-b^2 \\ d^2-c^2 & ac-bd \end{bmatrix}$	$\pm(ad-bc)$	$\begin{bmatrix} (a\pm b)/(c\pm d) \\ 1 \end{bmatrix}$	0
R_3	$\begin{bmatrix} bc+ad & -2ab \\ 2cd & -bc-ad \end{bmatrix}$	$\pm(ad-bc)$	$\begin{bmatrix} a/c \\ 1 \end{bmatrix} \begin{bmatrix} b/d \\ 1 \end{bmatrix}$	0
R_4	$\begin{bmatrix} -ad-bc & 2ab \\ -2cd & ad+bc \end{bmatrix}$	$\pm(ad-bc)$	$\begin{bmatrix} b/d \\ 1 \end{bmatrix} \begin{bmatrix} a/c \\ 1 \end{bmatrix}$	0
R_5	$\begin{bmatrix} bd+ca+cb & -a^2-ab-b^2 \\ c^2+cd+d^2 & -ac-ad-bd \end{bmatrix}$	complex	complex	± 1
R_6	$\begin{bmatrix} ac+bd & -a^2-b^2 \\ c^2+d^2 & -ac-bd \end{bmatrix}$	complex	complex	0
R_7	$\begin{bmatrix} ac+ad+bd & -a^2-ab-b^2 \\ c^2+cd+d^2 & -ac-bc-bd \end{bmatrix}$	complex	complex	± 1
R_8	$\begin{bmatrix} ac-bd & b^2-a^2 \\ c^2-d^2 & bd-ac \end{bmatrix}$	$\pm(ad-bc)$	$\begin{bmatrix} (a\mp b)/(c\mp d) \\ 1 \end{bmatrix}$	0
R_9	$\begin{bmatrix} -ac-bd & a^2+b^2 \\ -c^2-d^2 & ac+bd \end{bmatrix}$	complex	complex	0

Table 2.3: Conjugates of Elements of $GL(2,\mathbb{Z})$

3 Results

In this section, we present two different specific embeddings of the 3-rung Möbius ladder in the torus and also present restrictions on TSG's and TSG_+'s for any 3-rung Möbius ladder embedded in the torus.

3.1 Different Embeddings of the Ladder

Moving forward, we will focus on two examples of embeddings of the Möbius ladder. Woods showed that $\mathbb{Z}_2$ occurred as an orientation preserving topological symmetry group for the embedding in Figure 3.1 [13]. Notice that this embedding wraps around the longitude twice and the meridian once, which corresponds to the loop of the embedding having a homotopy class of $(2, 1)$. We will refer to any future embedding with a homotopy class of (p, q) as $f_{(p,q)}$.

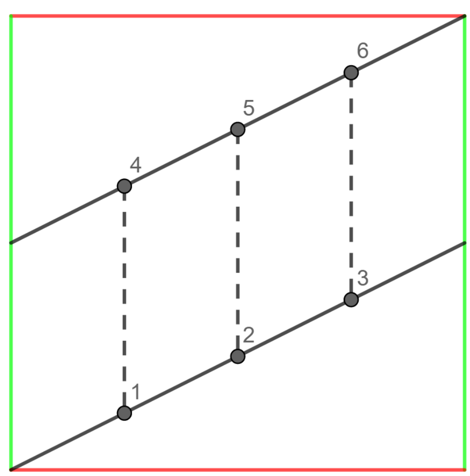


Figure 3.1: $f_{(2,1)}$ Ladder Embedding

We will also consider the following $f_{(1,1)}$ embedding shown in Figure 3.2. Notice that this embedding is distinct from the previous as their respective homotopy classes are not equal.

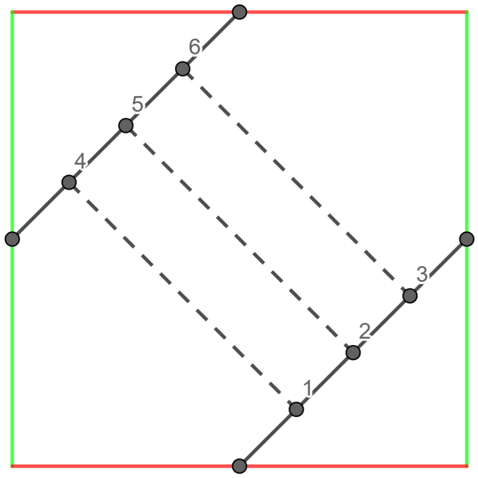


Figure 3.2: $f_{(1,1)}$ Ladder Embedding

In fact, we could create a general embedding of the 3-rung Möbius ladder by first creating a (p, q) torus knot and then adding the 3 rungs joining two adjacent sides of the loop. There is, however, another way in which we could create an embedding, and that is by first creating a loop with a homotopy class of $(0, 0)$. This embedding shown in Figure 3.3, $f_{(0,0)}$, is a trivial embedding of and will be excluded from our future arguments.

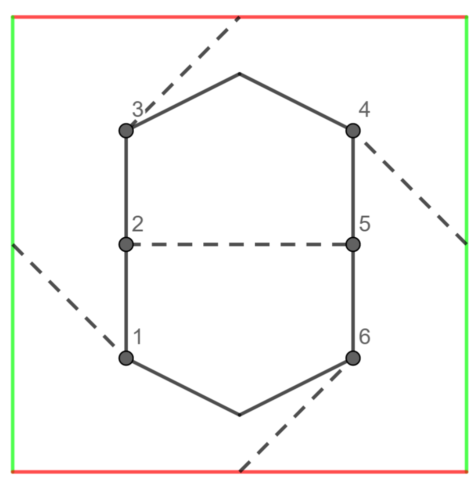


Figure 3.3: $f_{(0,0)}$ Ladder Embedding

One thing we may wish to analyze about these new embeddings are the possible topological symmetry groups which can occur for them. We could perform a similar analysis to the work done by Woods on the $f_{(2,1)}$ embedding, but we will need to take a different approach to show something more general.

3.2 $\mathbb{Z}_2$ Is the Only Possible Candidate for TSG_+

Here we will present both a visual argument and a rigorous argument showing $\mathbb{Z}_3, \mathbb{Z}_4$, and $\mathbb{Z}_6$ cannot occur as orientation-preserving topological symmetry groups. By eliminating these groups from our list of possible groups as outlined in Section 2.3, we will easily be able to show the desired result. We begin with the visual argument.

A Visual Argument

Let $f_{(p,q)}$ be an embedding of the Möbius ladder on the torus, and L be the loop on which $f_{(p,q)}$ lies; that is, L is the sides of the ladder. Recall that L is equivalent to a (p, q) torus knot and it follows that p and q must also be relatively prime. While this is helpful, it excludes 3 other possible types of embeddings: $f_{(p,0)}$,$f_{(0,q)}$, and $f_{(1,1)}$. We will consider the $f_{(1,1)}$ embedding in the formal argument and leave the others as special cases for this argument. Notice then that $p \neq q$ for all possible cases we consider for the visual argument.

Recall that the possible non-trivial orientation-preserving topological symmetry groups for the 3-rung Möbius ladder are $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$, and $\mathbb{Z}_6$. We will proceed by demonstrating why $\mathbb{Z}_4$ cannot occur as an orientation-preserving topological symmetry group.

First, consider applying the homeomorphism h_1 to the unit square, S, which corresponds to the rotation by $\pi/2$. Notice in Figure 3.4 that the red meridian line maps to the line which will become the longitude line when the unit square is identified as the torus. Similarly, the green longitude line will become meridian line. It follows that $h_1(L)$ will now cross the longitude a net total of p times and the meridian q times. Then we have that $[h_1(L)] = (q, p)$.

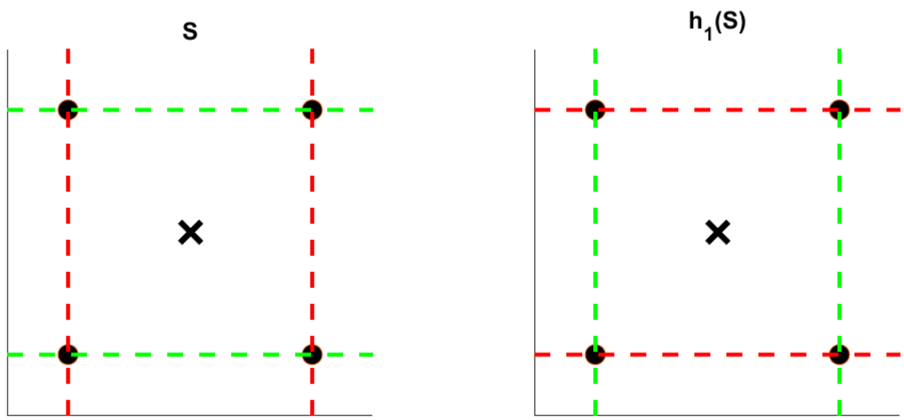


Figure 3.4: Rotation of Square by $\pi/2$

Then we have that $[h_1(L)] \neq [L]$ as $p \neq q$. Thus $h_1(L)$ is not isotopic to L and furthermore the homeomorphism h_1 does not induce an automorphism on $f_{(p,q)}$. A similar argument can be used to show that the homeomorphism h_2, which corresponds to a rotation of the unit square by $3\pi/2$, does not induce an automorphism on $f_{(p,q)}$. Since neither h_1 nor h_2 induce an automorphism of our embedding $f_{(p,q)}$, and h_1 and h_2 are the only elements in $SL_2(\mathbb{Z})$ of order 4 up to conjugacy, it follows that $\mathbb{Z}_4$ is not an orientation-preserving topological symmetry group of any embedding $f_{(p,q)}$.

To eliminate $\mathbb{Z}_3$, consider applying the homeomorphism h_3 which corresponds to the rotation of the hexagon by $2\pi/3$ to the unit hexagon, H, as shown in Figure 3.5. Notice that the red meridian line in H gets mapped to the line which will be identified as a longitude in $h_3(H)$. It follows that if we have a loop which crosses the meridian p times that $h_3(L)$ will cross the longitude p times. But the original loop crosses the longitude q times, and since p and q are distinct, we would have that $[L] \neq [h_3(L)]$. Therefore h_3 does not induce an automorphism on $f_{(p,q)}$.

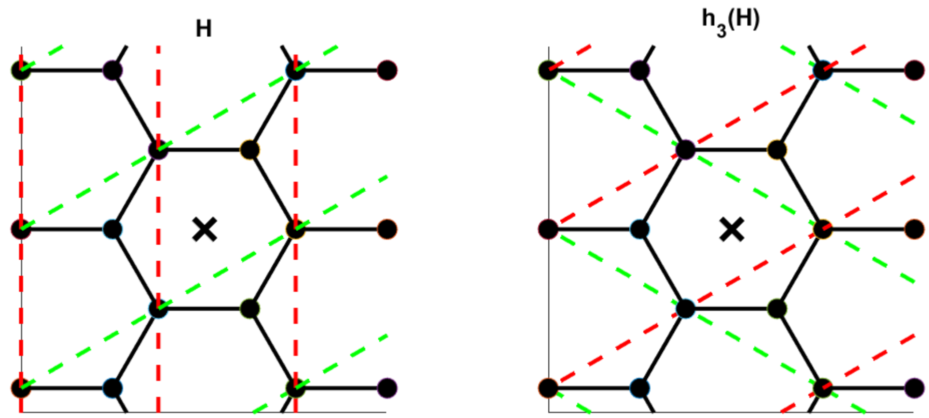


Figure 3.5: Rotation of Hexagon by $2\pi/3$

When we apply the homeomorphism h_4 which corresponds to the rotation of the unit hexagon by $4\pi/3$, we obtain a similar result shown in Figure 3.6. Now the green longitude line in H gets mapped to the line which will be identified as a meridian in $h_4(H)$. It follows that if we have a loop which crosses the longitude q times that $h_4(L)$ will cross the meridian q times. But L crosses the meridian p times. Thus we have that $[L] \neq [h_4(L)]$. Therefore h_4 does not induce an automorphism on $f_{(p,q)}$.

Since neither h_3 nor h_4 induce an automorphism of our embedding $f_{(p,q)}$, and h_3 and h_4 are the only elements of order 3 up to conjugacy, it follows that $\mathbb{Z}_3$ is not an orientation-preserving topological symmetry group of any embedding $f_{(p,q)}$.

Next, we demonstrate $\mathbb{Z}_6$ cannot occur by applying the homeomorphism h_5 which corresponds to the rotation of the hexagon by $\pi/3$ to H as shown in Figure 3.7. Notice that the green longitude line in H gets mapped to the line which will be identified as a meridian in $h_5(H)$. It follows that if we have a loop which crosses the longitude

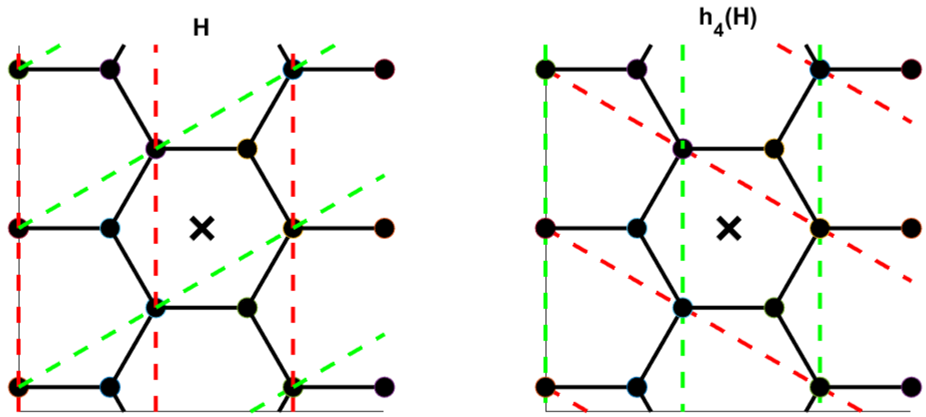


Figure 3.6: Rotation of Hexagon by $4\pi/3$

q times that $h_5(L)$ will cross the meridian q times. But the original loop crosses the meridian p times, and since p and q are distinct, we would have that $[L] \neq [h_5(L)]$. Therefore h_5 does not induce an automorphism on $f_{(p,q)}$.

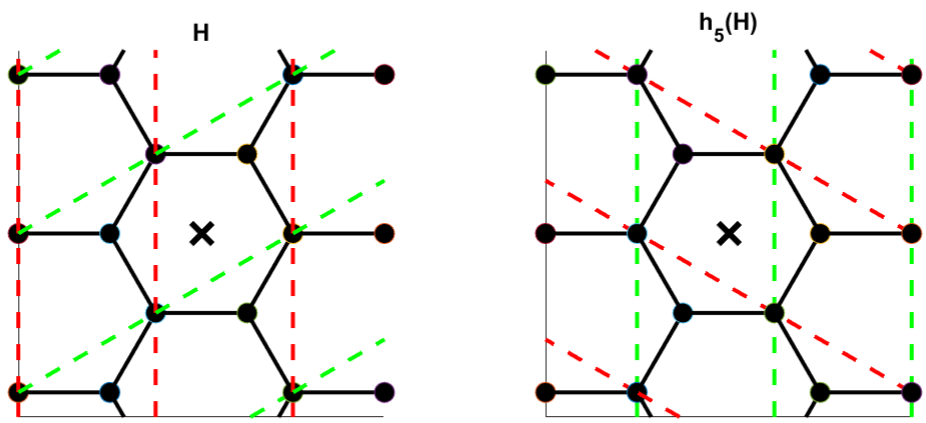


Figure 3.7: Rotation of Hexagon by $\pi/3$

When we apply the homeomorphism h_6 which corresponds to the rotation of the unit hexagon by $5\pi/3$ as shown in 3.8, we obtain a similar result: the red meridian line in H gets mapped to the line which will be identified as a longitude in $h_6(H)$. It follows that if we have a loop which crosses the meridian p times that $h_6(L)$ will cross the longitude p times. But L crosses the longitude q times. Thus we have that $[L] \neq [h_6(L)]$. Therefore h_6 does not induce an automorphism on $f_{(p,q)}$.

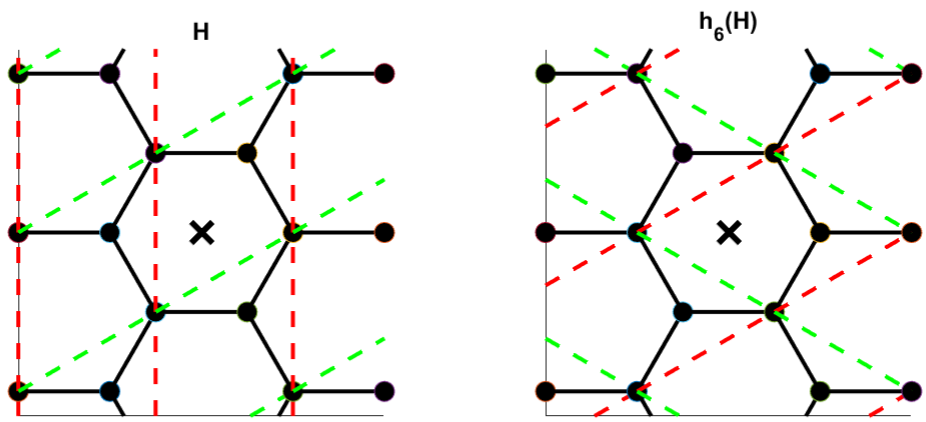


Figure 3.8: Rotation of Hexagon by $5\pi/3$

Since neither h_5 nor h_6 induce an automorphism of our embedding $f_{(p,q)}$, and h_5 and h_6 are the only elements of order 6 up to conjugacy, it follows that $\mathbb{Z}_6$ cannot be an orientation-preserving topological symmetry group of any embedding $f_{(p,q)}$.

Finally, we examine the special cases of $f_{(p,0)}$ and $f_{(0,q)}$. Notice that applying any of the previous homeomorphisms we considered will switch at least one element of the homotopy classes. That is, after a homeomorphism is applied, the number of crossings on the longitude will switch with the number of crossings on the meridian or meridian crossings with longitude crossings. In either case, this excludes $\mathbb{Z}_3, \mathbb{Z}_4$, and $\mathbb{Z}_6$ as possible candidates for orientation-preserving topological symmetry groups as none of the corresponding homeomorphisms preserve homotopy classes.

Therefore it must be the case that for any embedding on the torus, $\mathbb{Z}_3, \mathbb{Z}_4$, and $\mathbb{Z}_6$ cannot occur as orientation-preserving topological symmetry groups.

While this provides an intuitive argument for why these orientation-preserving symmetry groups cannot occur for these embeddings of the Möbius ladder on the torus, this result can be shown more concretely and more generally by utilizing techniques from linear algebra.

Result Via Linear Algebra

Here we present a proof that $\mathbb{Z}_2$ is the only possible non-trivial orientation-preserving topological symmetry group for a Möbius ladder embedded on the torus.

As a precursor, consider Figure 3.9 which shows the plane tiled with unit squares which have the loop of a $f_{(2,1)}$ embedded within. For our purposes, it suits us to consider the tiling only with a line drawn to $(2,1)$ as shown in Figure 3.10. Notice that this line drawn from the origin of the tiling to the point $(2,1)$ represents a vector. Recall that orientation-preserving homeomorphisms of the torus must correspond to elements of $SL(2,\mathbb{Z})$.

Notice that in order for the loop of the embedding to be sent to itself, $(2,1)$ must be sent to some scalar multiple of itself, as anything else would change the number of times the loop crossed the longitude and meridian and thus the homotopy class of the loop itself. Notice that this indicates $(2,1)$ must be an eigenvector of any proposed transformation in order for that transformation to be an automorphism. A similar argument can be extended to any $f_{(p,q)}$ embedding. We will use this to help us prove Theorems 3.1 and 3.2.

Theorem 3.1. *The finite cyclic group $\mathbb{Z}_2$ is the only possible finite non-trivial orientation-preserving topological symmetry group of the 3-rung Möbius ladder with sides and rungs distinct, embedded on the torus.*

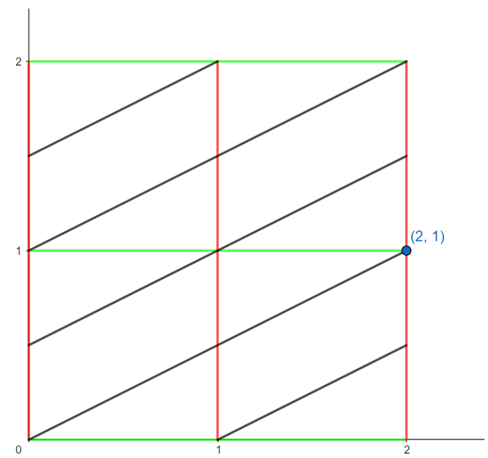


Figure 3.9: Tiling of a $(2, 1)$ Loop

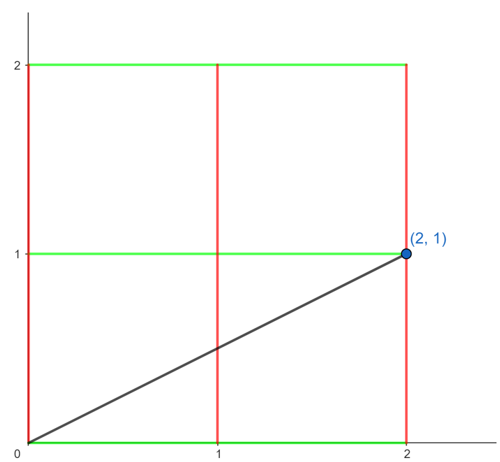


Figure 3.10: $(2, 1)$ vector

Proof: For an embedding $f_{(p,q)}$, $\binom{p}{q}$ must be an eigenvector of any possible orientation-preserving homeomorphism, h. That is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \lambda \begin{pmatrix} p \\ q \end{pmatrix} \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \ , \ p, q \in \mathbb{Z},$$

If this were not the case then $[f] \neq [h(f)]$, and so h would not induce an automorphism on f. Notice $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right) = \left(\begin{smallmatrix} ap+bq \\ cp+dq \end{smallmatrix}\right)$ is a matrix with integer entries which is equal to $\binom{\lambda p}{\lambda q}$. Thus, in order for this equality to hold in must be the case that λ is a rational value.

We can look at the specific form of this eigenvalue by taking the characteristic polynomial and solving for λ. When doing this we obtain

$$\lambda = \frac{\mathrm{tr} \pm \sqrt{\mathrm{tr}^2 - 4}}{2},$$

where $\mathrm{tr} = a + d$ is the trace of the our matrix.

Note that the trace must be an integer as $a, d \in \mathbb{Z}$. Further, notice that in order for this eigenvalue to be rational,$\mathrm{tr}^2 - 4$ must be a perfect square. That is, $\mathrm{tr}^2 - 4 = y^2$ for some $y \in \mathbb{Z}$. Using the fact that our trace must be an integer the solutions to this equation can be obtained by brute force. Doing so for this equation yields $\mathrm{tr} = \pm 2$ and $y = 0$.

So, in order for h to induce an automorphism on our embedding, it must be the case that the trace of the matrix to which h corresponds must be ± 2. Then it follows that any matrix with a trace not equal to ± 2 cannot correspond to a homeomorphism which would induce an automorphism on $f_{(p,q)}$. By comparing this to our table of elements of finite order of $SL(2, \mathbb{Z})$ (Table 2.1) we can see that the only elements with trace ± 2 are those that have an order of 1 or 2. This indicates that any other higher-order matrix cannot be a homeomorphism for the embedding.

Therefore it must be the case that for any embedding of the 3-rung Möbius ladder on the torus, $\mathbb{Z}_3, \mathbb{Z}_4$, and $\mathbb{Z}_6$ cannot occur as orientation-preserving topological symmetry groups. Eliminating these from our list of possible candidates from earlier leaves us with $\mathbb{Z}_2$ as the only possible non-trivial orientation-preserving topological symmetry group. $\square$

3.3 $\mathbb{Z}_2$ and D_2 Are the Only Possible Candidates for TSG

Instead of restricting ourselves to the cases where we are only considering orientation-preserving symmetry groups, we can instead consider both orientation-preserving and reversing symmetry groups. To do this, we will consider all possible finite subgroups of $GL(2, \mathbb{Z})$ which are categorized by the list in Section 2.3.

First we will look at the $f_{(2,1)}$ embedding. With a particular embedding in mind, it is easiest to consider how the possible generators transform the given embedding to determine if the generator induces an automorphism or not. This can be done by comparing the homotopy classes of the embedding before and after it has been transformed by a generator as was done in the visual argument in Section 3.2.1.

So, we can then compare the homotopy class of $f_{(2,1)}$, $[f_{(2,1)}]$, to the homotopy classes of the various transformed embeddings. Performing these comparisons, as shown in Table 3.1, demonstrates that there is only one transformation for which the homotopy class of the loop is the same as the homotopy class of the loop under that transformation: $[R_1(f_{(2,1)})] = (-2, -1) = (2, 1) = [f_{(2,1)}]$. For each other transformation, this is false and therefore none of the other transformations induce an automorphism on $f_{(2,1)}$. This immediately removes all subgroups from consideration aside from $\mathbb{Z}_2$, for which R_1 is a generator.

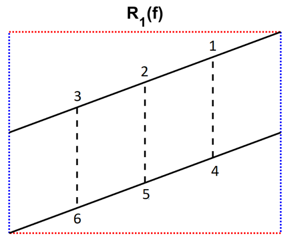
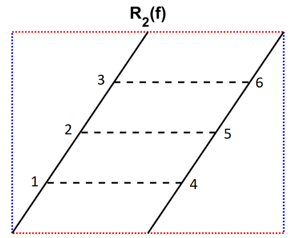
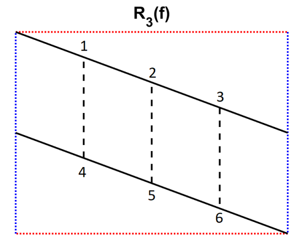
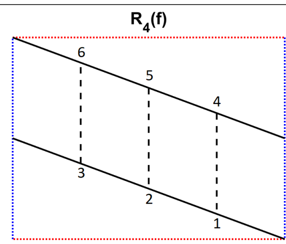
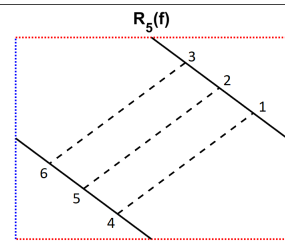
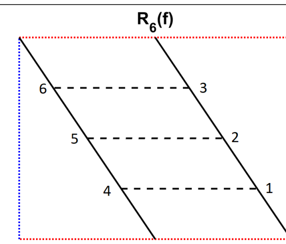
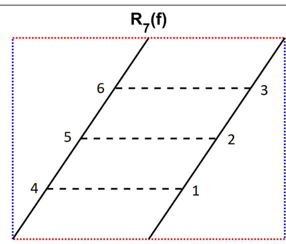
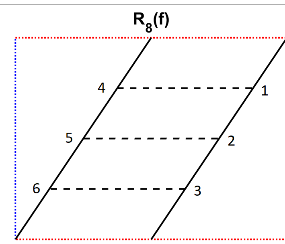
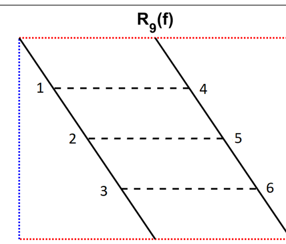
 $[R_1(f)] = (-2, -1)$	 $[R_2(f)] = (1, 2)$	 $[R_3(f)] = (2, -1)$
 $[R_4(f)] = (-2, 1)$	 $[R_5(f)] = (-1, 1)$	 $[R_6(f)] = (-1, 2)$
 $[R_7(f)] = (1, 2)$	 $[R_8(f)] = (-1, -2)$	 $[R_9(f)] = (1, -2)$

Table 3.1: Transformations of $f = f_{(2,1)}$

We will now consider how each of the generators transform the $f_{(1,1)}$ embedding, g. Notice that there are many cases where the homotopy class does not immediately remove the possibility of a particular transformation being an automorphism. We can see that $[g] = [R_1(g)] = [R_2(g)] = [R_8(g)]$, so we have 3 candidates which do induce an automorphism on g and thus 3 possible topological symmetry groups.

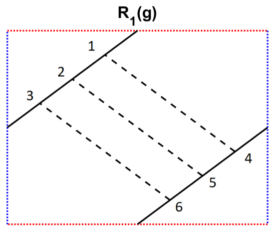
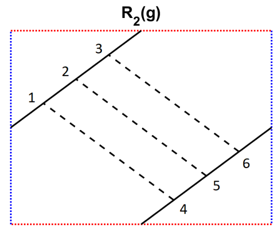
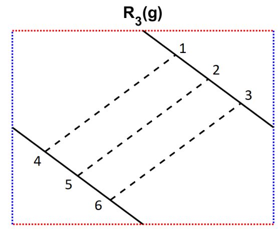
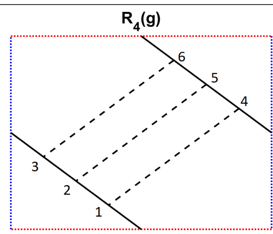
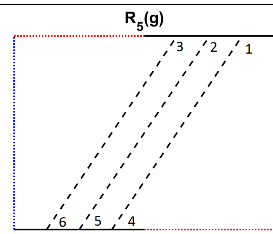
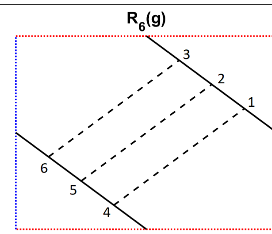
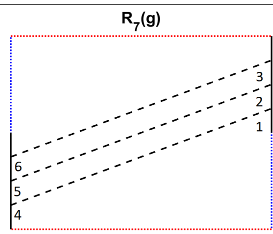
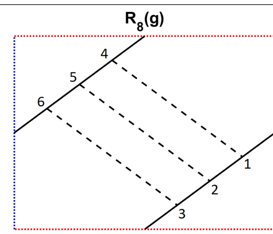
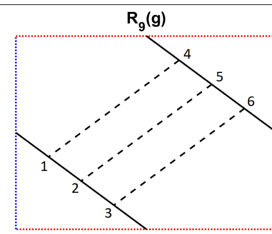
 $[R_1(g)] = (-1,-1)$	 $[R_2(g)] = (1,1)$	 $[R_3(g)] = (1,-1)$
 $[R_4(g)] = (-1,1)$	 $[R_5(g)] = (-1,0)$	 $[R_6(g)] = (-1,1)$
 $[R_7(g)] = (0,1)$	 $[R_8(g)] = (-1,-1)$	 $[R_9(g)] = (1,-1)$

Table 3.2: Transformations of $g = f_{(1,1)}$

We can use a similar technique from the proof of the previous theorem to generate the following restrictions on the possible topological symmetry groups.

Theorem 3.2. *The finite cyclic groups* $\mathbb{Z}_2$ *and* D_2 *are the only possible finite non-trivial topological symmetry groups of the 3-rung Möbius ladder with sides and rungs distinct, embedded in the torus.*

Proof: Recall that there are 9 distinct finite subgroups of $GL(2,\mathbb{Z})$ up to conjugacy. As in the proof of Theorem 3.1, it is still the case that if we have an embedding

$f_{(p,q)}$, then $\binom{p}{q}$ must be an eigenvector with some rational eigenvalues for a homeomorphism in order for it to induce an automorphism on the embedding. Then it must be the case that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \lambda \begin{pmatrix} p \\ q \end{pmatrix} \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z}) \ , \ p,q \in \mathbb{Z}.$$

Notice $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right) = \left(\begin{smallmatrix} ap+bq \\ cp+dq \end{smallmatrix}\right)$ is a matrix with integer entries which is equal to $\left(\begin{smallmatrix} \lambda p \\ \lambda q \end{smallmatrix}\right)$. Thus, in order for this equality to hold in must be the case that λ is a rational value. However, if we look at the Table 2.2 we find that R_5, R_6, R_7 and R_9 all have complex eigenvalues and eigenvectors. It is also shown that any element conjugate to these generators will also have complex eigenvalues and eigenvectors as seen in the Table 2.3. Then it must be the case that none of these generators could induce an automorphism on our embedding and thus any group they generate cannot be a topological symmetry group. Eliminating these subgroups of $GL(2,\mathbb{Z})$ leaves us with $\mathbb{Z}_2$ and D_2 as our only possible topological symmetry groups. $\square$

4 Future Work

In this work we have placed restrictions on the topological symmetry groups for the 3-rung Möbius ladder embedded on the torus, but we have not confirmed that the possible groups occur. We know in previous work from Woods that $\mathbb{Z}_2$ occurs as an orientation-preserving topological symmetry group for the $f_{(2,1)}$ embedding, but we have not yet shown it is possible for D_2. In the future we hope to either show that D_2 does occur as topological symmetry group for some embedding, or that it cannot occur and thus further restrict the possible topological symmetry groups.

Other possible routes include investigating the structure of topological symmetry groups of different embeddings on the torus or even continuing to look at the 3-rung Möbius ladder on different topological spaces.

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