Foundations of Mathematics and Their Applications in Higher Level Mathematics

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Abstract

This project will explore three foundations of mathematics, and their applications to various areas in higher level mathematics. First, we will learn about mathematicians and logicians who developed various foundations. Then we will examine some key concepts in each foundation. Next, we will examine some key concepts in various higher level mathematics. We will then analyze and exhibit how these foundations apply to these higher level mathematics.

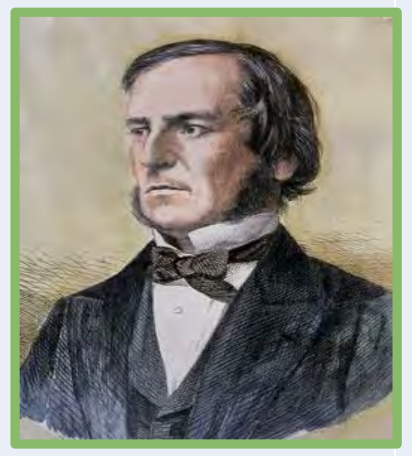
Methodology

As I read the *Introduction to Abstract Mathematics (2ed)* by John Lucas, I discovered that three of the main foundations of higher level mathematics were logic, set theory, and functions. As I continued reading, I was intrigued and fascinated by how these foundations were developed. I was curious about how they apply to higher level mathematics, so I researched the main contributors between these foundations and higher mathematics.

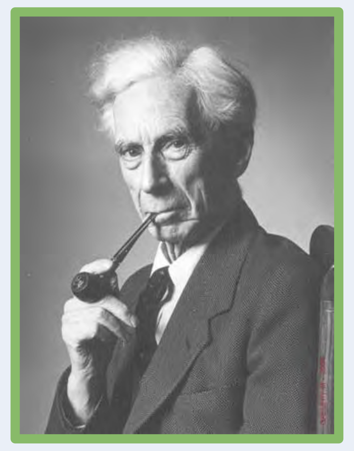
Three Foundations of Mathematics

Logic

George Boole



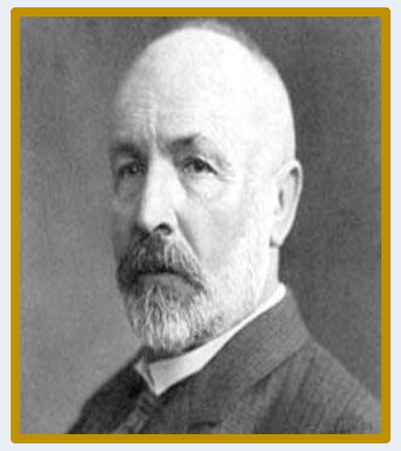
Bertrand Russell



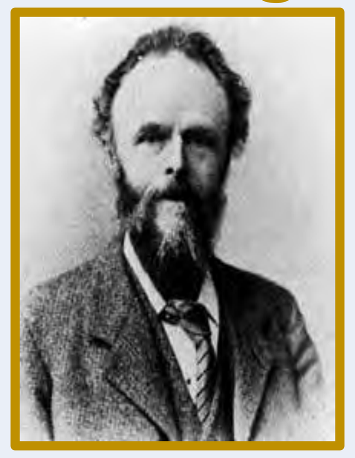
- Tautology
- Axioms and definitions
- Quantifiers (∀, ∃, etc.)
- Conditionals (A ⇒B)
- Deductive/inductive reasoning

Set Theory

Georg Cantor



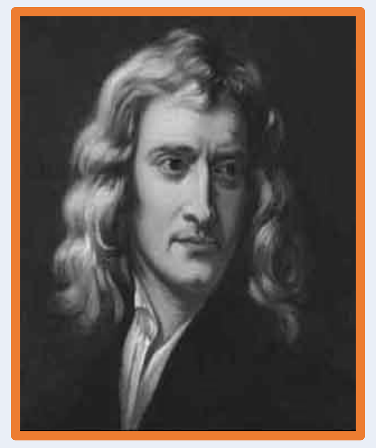
Augustus De Morgan



- Union (∪)
- Intersection (∩)
- Subset (⊆)
- Complement (A^C)

Functions

Isaac Newton



Gottfried Leibniz

- Mapping
- Composition
- One-to-one correspondence

Higher Level Mathematics

Probability

- Probability Function
- Outcomes and Events
- Mutually Exclusive Events
- Complementary Events
- Joint Probability
- Conditional Probability

Topology

- Open sets
- Closed Sets
- Bounded Sets
- Compact Sets
- Countable Sets
- Density

Abstract Algebra

- Equivalence Relations
- Isomorphisms
- Groups
- Rings
- Fields

Results

As a sophomore, I have yet to take most of these courses. As such, I only read three or four chapters in books for each subject, to get a general idea of the main introductory topics. I discovered that there were far too many connections between the foundations and the higher level courses to list here. So, I selected specific topics from higher level mathematics to connect to topics in foundations. Below is a summary of some of the connections I discovered.

Foundations

Higher Level Math	Logic	Set Theory	Functions
Probability	Bayes Theorem: $P(A\|B) = \frac{P(B\|A)}{P(B\|A) + P(B\|A^C)}$ Proof of uses deductive reasoning and Kolmogorov Axioms.	Probability of the union of mutually exclusive events: $P(A \cup B) = P(A) + P(B)$ Joint probability: $P(A \cap B) = P(A)P(B\|A)$	Probability mass function: $p(x) = P(X = x)$ Cumulative distribution function: $F(x) = \sum_{i \le n} p(i)$
Topology	Density of rational numbers: $\forall r_1, r_2 \in \mathbb{R}, \exists q \in \mathbb{Q} \ni r_1 < q < r_2$ Proof uses deductive reasoning, axioms, and logical quantifiers.	Cantor's Nested Interval Theorem: If $\{I_n\}$ is an countable nested sequence of bounded, closed intervals, then $\bigcap I_n \neq \emptyset$.	Countable set: a set that is either finite, or can be put in one-to-one correspondence with the positive integers. Ex: $D = \{2,4,6, \dots\}$ is countable, since $f(x) = \frac{x}{2}$ is one-to-one, and $f(D) = \mathbb{Z}^+$.
Abstract Algebra	Theorem: Equivalence relations invoke partitions. Proof uses deductive reasoning, definition of equivalence relation, and logical quantifiers.	Equivalence class: the set of elements that are related under an equivalence relation Ex: If xRy means $(x - y)$ is divisible by 4, then the equivalence class containing 7 is $[7] = \{\dots, -1, 3, 7, 11, \dots\}$	Isomorphism: a one-to-one, onto mapping that preserves operations Ex: $f(x) = e^x$ is an isomorphism from $\mathbb{R}$ to $\mathbb{R}^+$ that preserves operations: $f(a + b) = f(a)f(b)$

Conclusion

The field of mathematics is defined thorough various foundations, three of which are logic, set theory, and functions. These three foundations give mathematics its language, and apply to higher level mathematics such as abstract algebra, topology, and probability. There are many other foundations (such as number theory) and higher level mathematics (such as analysis) to explore, which would lead to many more connections between them.