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# TOPOLOGICAL PROPERTIES OF A 3-RUNG MÖBIUS LADDER

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# TOPOLOGICAL PROPERTIES OF A 3-RUNG MÖBIUS LADDER

by

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## ABSTRACT

In this work, we discuss the properties of the 3-rung Möbius ladder on the torus. We also prove $\mathbb{Z}_2$ is an orientation preserving topological symmetry group of the 3-rung Möbius ladder with sides and rungs distinct, embedded in the torus.

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# 1 INTRODUCTION

The field of topological stereochemistry was developed from the research of topologist Jon Simon in an attempt to answer the questions of chemist David Walba. In 1982, Walba, Richards, and Haltiwanger became the first chemists to synthesize a molecule that they believed to have the structure of a 3-rung Möbius ladder [5]. The *Möbius ladder* is similar to a Möbius strip, but the strip is replaced with a ladder that contains three rungs. Figure 1.1 below illustrates the structure of the 3-rung Möbius ladder and its mirror image. By the *mirror image* of an object we mean the object as seen in a mirror [5].

Figure 1.1: A 3-rung Möbius ladder and its mirror image [5]

The molecule Walba and his researchers created was constructed from carbon and oxygen atoms. The sides of the ladder were made from a polyether chain consisting of carbon and oxygen atoms and the rungs were formed by carbon-carbon double bonds with no oxygen atoms. So, the sides and rungs of the ladder are chemically different. To prove that the molecule had the structure of a 3-rung Möbius ladder, the researchers used nuclear magnetic resonance to gather evidence that the molecule was chemically chiral. A molecule is said to be *chemically achiral* if it can be deformed into its mirror image. Otherwise, it is said to be *chemically chiral* [5].

As seen in Figure 1.2(a), the molecular cylinder is its own mirror image and so it is chemically achiral. The evidence of chirality led the researchers to believe they had

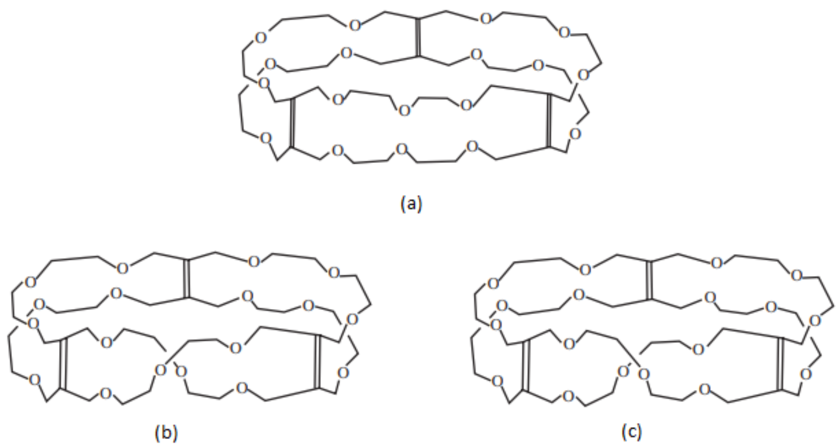


Figure 1.2: Different structures of Walba, Richards, and Haltiwanger's molecule [5]

successfully synthesized the first molecular Möbius ladder. The researchers created many ladders and then forced the ends to join together. They assumed that some of the ladders would have joined in a half-left twist and some in a half-right twist as seen in Figure 1.2(b, c).

Amino acids in living organisms are typically chiral. This means that organisms may have different reactions to different enantiomers. A chiral molecule and its mirror image together are called *enantiomers* [4]. We see this effect in medicine for humans where one enantiomer may have the desired effect, and the other is often ineffective. However, in the 1960s a chiral drug, Thalidomide, was given to pregnant women as a *racemic mixture* meaning it had a $50 : 50$ mixture of the two enantiomers [4]. The left-handed enantiomer helped cure morning sickness and the right-handed enantiomer caused terrible birth defects. Chemists are able to separate the enantiomers to make medicines that eliminate these side effects, but this is a costly process. So, it is of importance to determine whether or not a molecule is chemically chiral [4].

Walba conjectured that even with complete flexibility the 3-rung Möbius ladder could not be deformed into its mirror image so that rungs would go to rungs and sides would go to sides, but he was unable to prove it. When the sides and rungs of the

3-rung Möbius ladder are not chemically different it is possible to deform the half-left twist into its mirror image, the half-right twist, and vice versa [11]. However, when the sides and rungs are chemically different as they are in this molecule, the 3-rung Möbius ladder cannot be deformed into its mirror image. In 1985, Simon completed the proof that the embedded graph of a Möbius ladder with 3 rungs cannot be deformed into its mirror image such that rungs go to rungs and sides go to sides [11].

Chemists have long been interested in finding molecules that are structurally related to each other. These molecules are called *isomers* and the type of interest to topologists is the topological stereoisomer. The *topological stereoisomers* of a given molecule are those molecules that have the same abstract graph as the given molecule, but as embedded graphs one cannot be deformed into the other [4]. Topologists have become interested in two branches of stereochemistry. The first is recognizing when one embedding of a graph cannot be deformed into another embedding and the second is evaluating the properties that are presented by the deformation.

## 1.1 Topology

Now, we transition into the topological ideas that we use in this work. Here we present more formal definitions of the mathematical terms we have introduced above.

**Definition 1.1.** A *homeomorphism*, or *topological equivalence*, is a one-to-one and onto continuous function between topological spaces with a continuous inverse [1].

**Definition 1.2.** Let $A$ and $B$ be subsets of a set $M$, which is a subset of $\mathbb{R}^n$. We say that $A$ is *ambient isotopic* to $B$ in $M$ if there is a continuous function $F : M \times [0,1] \to M$ such that for each fixed $t \in [0,1]$ the function $F(x,t)$ is a homeomorphism, $F(x,0) = x$ for all $x \in M$, and $F(A \times \{1\}) = B$. The function $F$ is said to be an *ambient isotopy* [4].

Since this definition corresponds with our intuitive conception of a *deformation* we use the words ambient isotopy and deformation interchangeably.

**Definition 1.3.** Let $X$ and $Y$ be topological spaces. If $f : X \to Y$ is a one-to-one map, $f : X \to f(X)$ is a homeomorphism, and $f(X)$ has the induced topology from $Y$, we call $f$ an *embedding* of $X$ into $Y$ [1].

**Definition 1.4.** A *graph* is a finite collection of vertices together with disjoint edges connecting pairs of vertices, with the requirement that there is at most one edge between any pair of vertices and every edge has two distinct vertices [4].

**Definition 1.5.** Let $X$ be a graph considered as a topological space with the discrete topology and $Y$ a topological space. If an embedding $f$ of $X$ into $Y$ exists, we call $f(X)$ an *embedded graph.*

**Definition 1.6.** An *abstract graph* is a graph that is considered independent of any particular embedding in three-dimensional space [4].

**Definition 1.7.** A graph embedded in three-dimensional space is *topologically achiral* if it can be deformed into its mirror image. Otherwise it is *topologically chiral* [5].

With his proof, Simon showed that the molecular Möbius ladder with 3 rungs that are chemically different from the sides is topologically chiral and that the 3-rung Möbius ladder and its mirror image are topological stereoisomers. This result led topologists to question if it was the abstract structure of the graph or the particular embedding that made the molecule chiral. Flapan answered this question in 1989 with her proof that no embedding of a Möbius ladder with an odd number of rungs greater than one can be deformed into its mirror image in such a way that rungs go to rungs and sides go to sides [3].

**Definition 1.8.** An *automorphism* of a graph is defined as a bijection from the graph to itself, taking vertices to vertices in such a way that adjacent vertices are taken to adjacent vertices [4].

One way to determine if a molecule is chemically chiral is to examine all of its topological stereoisomers. Given a particular embedded graph of the molecule, performing an automorphism that is not induced by a deformation of the embedded graph in space creates a topological stereoisomer. There are only a finite number of automorphisms of a given graph, so enumerating all topological stereoisomers should be possible. Whether or not a particular automorphism can be induced by a deformation depends on the embedding of the graph. So, determining the topological symmetry groups of a molecule is the first step.

**Definition 1.9.** For a particular embedded graph $G$ in a topological space $X$, a *topological symmetry group* is the group of automorphisms of $G$ induced by homeomorphisms on $X$ [6].

**Definition 1.10.** Let $A$ and $B$ be subsets of $\mathbb{R}^3$, and let $h : A \to B$ and $g : A \to B$ be homeomorphisms. We say $h$ and $g$ are *isotopic* if there exists a continuous function $F : A \times [0,1] \to B$ such that $F(x,0) = h(x), F(x,1) = g(x)$, and for every fixed $t \in [0,1]$, the function $F(x,t)$ is a homeomorphism [4].

**Definition 1.11.** Let $h : \mathbb{R}^3 \to \mathbb{R}^3$ be a homeomorphism. If $h$ is isotopic to the identity map, then we say $h$ is *orientation preserving.* If $h$ is isotopic to a reflection map, then we say $h$ is *orientation reversing* [4].

An important result in topology is that every homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ is isotopic to either the identity map or to a reflection map, but not to both [4]. So, orientation reversing homeomorphisms map an embedded graph to its mirror image or a deformation of its mirror image, while orientation preserving homeomorphisms map an embedded graph to itself or a deformation of itself in space [4]. We are interested only in orientation preserving topological symmetry groups so we only need to consider orientation preserving homeomorphisms.

The 3-rung Möbius ladder has six vertices, or six places where the rungs meet the sides of the ladder. For this reason the topological symmetry groups must be subgroups of the symmetric group $S_6$. Flapan and Lawrence discovered all of the orientation preserving topological symmetry groups of a 3-rung Möbius ladder graph in the 3-sphere, $S^3$. As in knot theory, they chose to embed graphs in the 3-sphere instead of $\mathbb{R}^3$ noting that the topological symmetry groups are the same whether the graph is embedded in $\mathbb{R}^3$ or $S^3$ [6].

The non-trivial subgroups of $S_6$ up to an isomorphism are: $\mathbb{Z}_6, \mathbb{Z}_5, \mathbb{Z}_4, \mathbb{Z}_3, \mathbb{Z}_2, D_6, D_5, D_4, D_3, D_2, A_6, A_5, A_4, S_6, S_5, S_4, \mathbb{Z}_5 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4, (\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_4, (\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, D_3 \times D_3, D_4 \times \mathbb{Z}_2, D_3 \times \mathbb{Z}_3, A_4 \times \mathbb{Z}_2, S_4 \times \mathbb{Z}_2, S_3 \wr \mathbb{Z}_2$ [6]. Flapan and Lawrence were able to find a complete list of the nontrivial groups which occur as orientation-preserving topological symmetry groups for some embedding of the 3-rung Möbius ladder in $S^3$. The groups are as follows: $\mathbb{Z}_6, \mathbb{Z}_3, \mathbb{Z}_2, D_6, D_3, D_2, D_3 \times D_3, \mathbb{Z}_3 \times \mathbb{Z}_3, (\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_2, D_3 \times \mathbb{Z}_3$ [6].

The Möbius ladder can be embedded in $S^3$, but it could also be embedded in other topological spaces such as the surface of a torus, $T^2$. In this work, we seek to classify the orientation preserving topological symmetry groups of some embedding of the 3-rung Möbius ladder in the torus. This question has not been investigated elsewhere. We do not yet have a complete list of the orientation preserving topological symmetry groups of some embedding of the 3-rung Möbius ladder in the torus.

In chapter 3 we present a proof of one group that occurs as an orientation preserving topological symmetry group of the 3-rung Möbius ladder with sides and rungs distinct, embedded in the torus. To complete this proof we present a formal definition of the torus and discuss its topological properties in chapter 2.

# 2 Properties of the Torus

In order to consider embeddings of the molecular 3-rung Möbius ladder in $T^2$ we need to examine its properties. The *torus* as shown in Figure 2.1 is commonly defined as the product of two unit circles, $S^1 \times S^1$. We call the first circle the *longitude, l,* and the second circle the *meridian, m.*

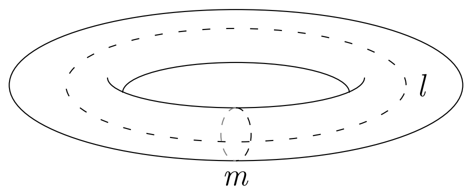


Figure 2.1: The torus

## 2.1 The Torus as an Identification Space

It is useful to define the torus as an identification space when we are interested in the embeddings on its surface. In this section we present two ways to construct the torus as an identification space.

**Definition 2.1.** Let $X$ be a topological space and let $\mathcal{P}$ be a family of disjoint nonempty subsets of $X$ such that $\bigcup \mathcal{P} = X$, called the *partition* of $X$. We form a new space, $Y$ called an *identification space*, as follows: the points of $Y$ are the members of $\mathcal{P}$ and, if $\varphi : X \to Y$ sends each point of $X$ to the subset of $\mathcal{P}$ containing it, the topology on $Y$ is the largest for which $\varphi$ is continuous. This topology is called the *identification topology* on $Y$. We think of $Y$ as the space obtained from $X$ by identifying each of the subsets of $\mathcal{P}$ to a single point [1].

**Definition 2.2.** Take $X$ to be the unit square $[0, 1] \times [0, 1]$ in $\mathbb{R}^2$ with the subspace topology, and partition $X$ into the following subsets:

1. the set $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ of four corner points;
2. sets consisting of pairs of points $(x, 0), (x, 1)$, where $0 < x < 1$;
3. sets consisting of pairs of points $(0, y), (1, y)$, where $0 < y < 1$;
4. sets consisting of a single point $(x, y)$ where $0 < x < 1$ and $0 < y < 1$.

The resulting identification space is the *torus* [1].

Intuitively the torus can be formed by rotating the bottom segment forwards to glue it to the top of the unit square to form a cylinder. Then stretch the cylinder around and glue the ends of the cylinder together so that the four corners all lie on top of each other. This creates the torus from the unit square as shown in Figure 2.2.

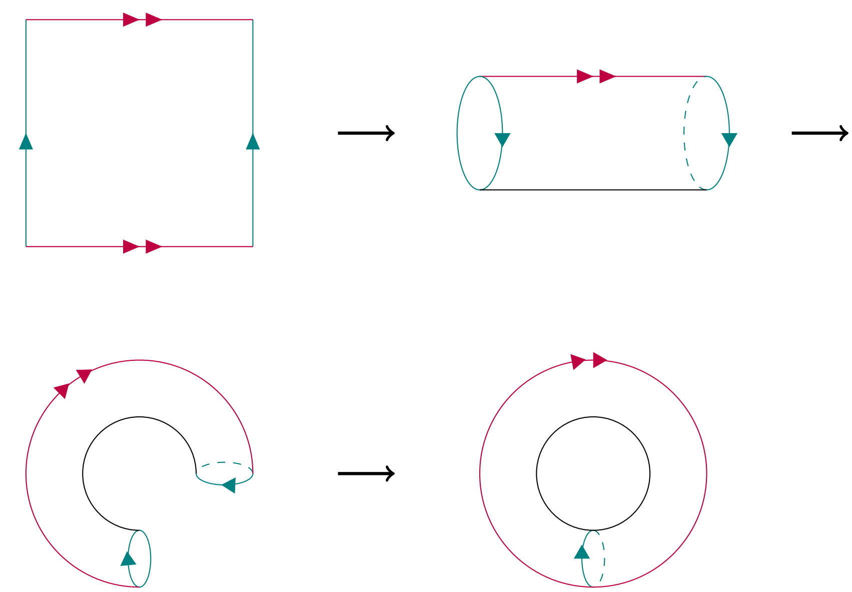

Figure 2.2: Construction of the torus from the unit square

We can also create the torus as an identification space from a regular hexagon. We define a new coordinate system so that the $x$-axis passes through the hexagon at

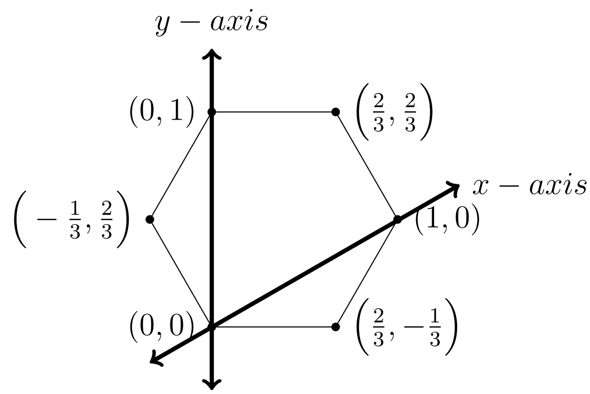


Figure 2.3: The unit hexagon

points (0, 0) and (1, 0) and the $y$-axis passes through the hexagon at points (0, 0) and (0, 1) as shown in Figure 2.3. We call this hexagon a *unit hexagon* in $\mathbb{R}^2$.

As demonstrated in Figure 2.4, the first step is to rotate the top and bottom of the hexagon up out of the page and glue them together. Next, add a 180 degree twist by leaving the left side alone and rotating the bottom right corner up out of the page. Finally, stretch the shape around so the ends are glued together.

**Definition 2.3.** Take $X$ to be the unit hexagon in $\mathbb{R}^2$ with the subspace topology, and partition $X$ into the following subsets:

1. the set $\{(0,0),(0,1),(1,0),(\frac{2}{3},\frac{2}{3}),(-\frac{1}{3},\frac{2}{3}),(\frac{2}{3},-\frac{1}{3})\}$ of six corner points;
2. sets consisting of pairs of points $(x,-\frac{1}{2}x),(x,-\frac{1}{2}x+1)$ where $0<x<\frac{2}{3}$;
3. sets consisting of pairs of points $(x,-2x),(x+1,-2x)$ where $-\frac{1}{3}<x<0$;
4. sets consisting of pairs of points $(x,x+1),(x+1,x)$ where $-\frac{1}{3}<x<0$;
5. sets consisting of a single point $(x,y)$ where $0<x<1$ and $0<y<\frac{2}{3}$;
6. sets consisting of a single point $(x,y)$ where $-\frac{1}{3}<x<0$ and $\frac{2}{3}<y<x+1$;
7. sets consisting of a single point $(x,y)$ where $0<x<\frac{2}{3}$ and $\frac{2}{3}<y<-\frac{1}{2}x+1$;

8. sets consisting of a single point $(x, y)$ where $0 < x < \frac{2}{3}$ and $-\frac{1}{2}x < y < 0$;

9. sets consisting of a single point $(x, y)$ where $\frac{2}{3} < x < 1$ and $x - 1 < y < 0$;

The resulting identification space is the *torus*.

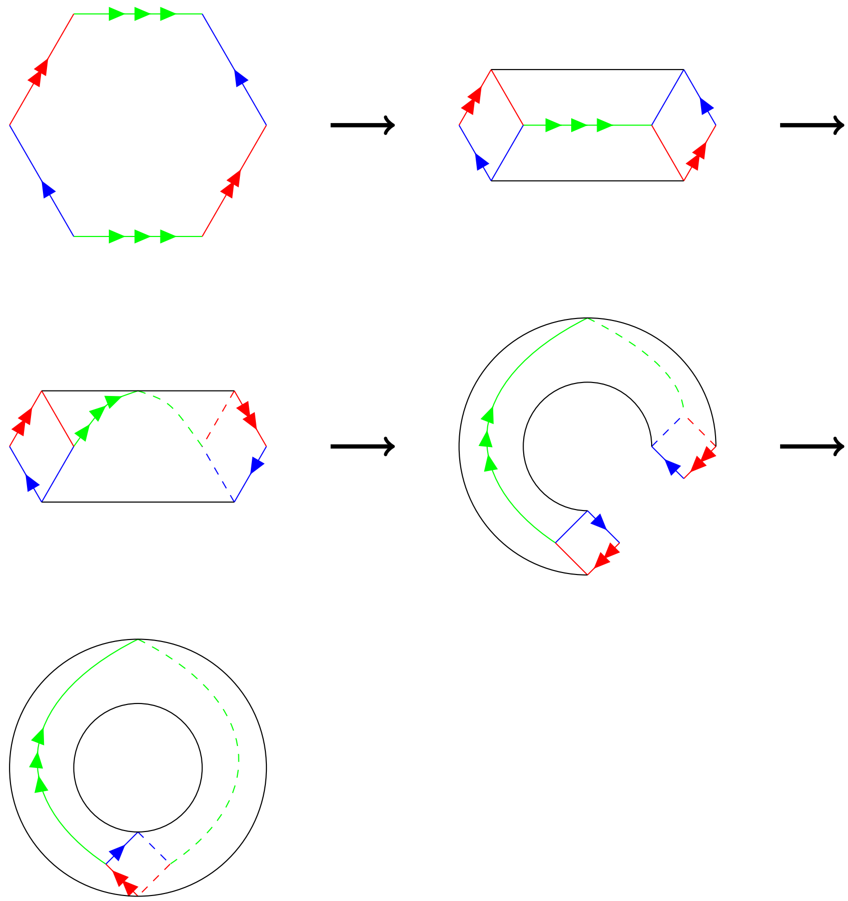

Figure 2.4: Construction of the torus from the unit hexagon

There are other ways to define a coordinate system on a regular hexagon. We consider only one other coordinate system that we call the *unit hexagon with alternate coordinates*. Figure 2.5 shows the unit hexagon with the alternate coordinates. Notice that the torus could be formed as an identification space from the unit hexagon with alternate coordinates in a similar way to the unit hexagon.

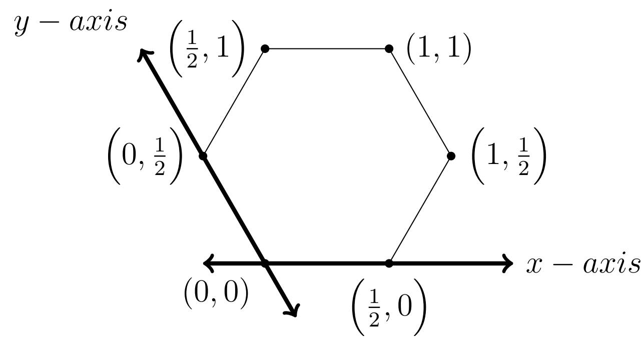


Figure 2.5: The unit hexagon with alternate coordinates

For our work it is useful to construct the torus as an identification space from $\mathbb{R}^2$. Previously, we defined the torus as an identification space from the unit square. However, in chapter 3 we transform our unit square so that the unit square is not always taken back to itself. So, we have an alternative definition of the torus as an identification space. We identify $(x, y) \in \mathbb{R}^2$ with $(x + m, y + n)$ where $m, n \in \mathbb{Z}$. Then, we construct the torus as an identification from the unit square by the same definition as before.

**Definition 2.4.** For every $(x, y) \in \mathbb{R}^2$ send $(x, y)$ to $(x + m, y + n)$ where $m, n \in \mathbb{Z}$ so that $0 \leq x + m \leq 1$ and $0 \leq y + n \leq 1$.

Take $X$ to be the unit square $[0, 1] \times [0, 1]$ in $\mathbb{R}^2$ with the subspace topology, and partition $X$ into the following subsets:

1. the set $\{(0,0), (0,1), (1,0), (1,1)\}$ of four corner points;

2. sets consisting of pairs of points $(x, 0), (x, 1)$, where $0 < x < 1$;

3. sets consisting of pairs of points $(0, y), (1, y)$, where $0 < y < 1$;

4. sets consisting of a single point $(x, y)$ where $0 < x < 1$ and $0 < y < 1$.

The resulting identification space is the *torus*.

Although the details of identifying $\mathbb{R}^2$ to the torus through the unit hexagon are different, we can similarly define the torus as an identification space from all of $\mathbb{R}^2$ using the hexagon tiling.

### 2.2 Homeomorphisms of the Torus

In this section we discuss which subgroups of $S_6$ could possibly be orientation preserving topological symmetry groups of the torus. We construct this list after considering the correspondence of homeomorphisms of the torus to elements in $GL_2(\mathbb{Z})$.

**Definition 2.5.** The *general linear group* $GL_2(\mathbb{Z})$ is the set of all $2 \times 2$ matrices with integer entries and a nonzero determinant.

**Definition 2.6.** The *special linear group* $SL_2(\mathbb{Z})$ is the set of all $2 \times 2$ matrices with integer entries and determinant 1.

According to Casson and Bleiler, the homeomorphisms of the torus correspond up to isotopy to the elements of the general linear group $GL_2(\mathbb{Z})$ [2]. Furthermore, a homeomorphism preserves orientation if and only if it corresponds to an element of the special linear group $SL_2(\mathbb{Z})$ [2]. Since we are interested only in the orientation-preserving topological symmetry groups, we can rule out all subgroups of $S_6$ that are not isomorphic to subgroups of $SL_2(\mathbb{Z})$.

In [10], Newman states that the finite subgroups of $GL_2(\mathbb{Z})$ are $\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$, $D_2, D_3, D_4, D_6$. Furthermore, the finite subgroups of $SL_2(\mathbb{Z})$ are isomorphic to the cyclic groups $\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$. We consider only the non-trivial subgroups. This means that $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ are the only groups that could possibly be nontrivial orientation preserving topological symmetry groups of the 3-rung Möbius ladder embedded on the torus.

### 2.3 Correspondence of Homeomorphisms

In this section we describe the correspondence between the elements of $SL_2(\mathbb{Z})$ and the homeomorphisms of the torus. The homeomorphisms correspond to rotations of the unit square, unit hexagon, and unit hexagon with alternate coordinates as shown in Figure 2.6 before the torus is constructed as an identification space.

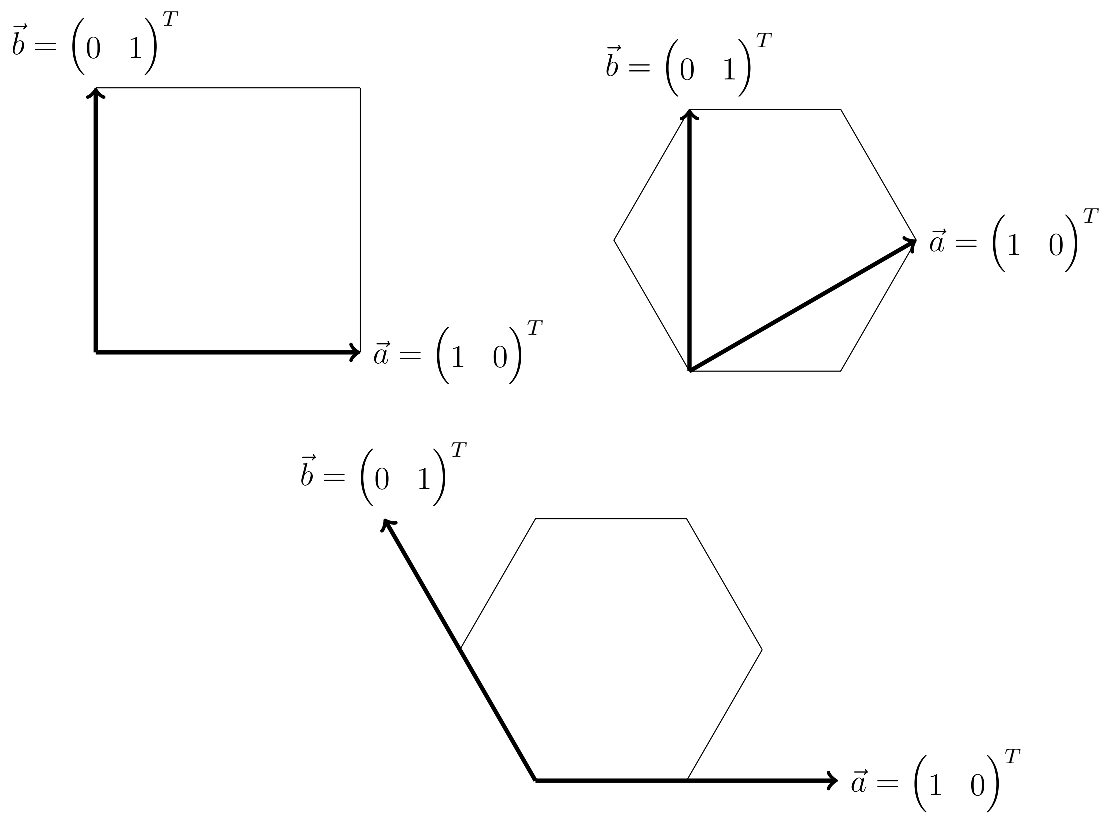


Figure 2.6: Identity homeomorphism

We write $\vec{a} = \begin{pmatrix} a_x & a_y \end{pmatrix}^T$ and $\vec{b} = \begin{pmatrix} b_x & b_y \end{pmatrix}^T$ where $a_x, a_y, b_x, b_y$ are integers that correspond to the magnitude of the vector along the $x-$ and $y - axes$. For our purposes, $a_x, a_y, b_x, b_y \in \{-1, 0, 1\}$. Then, we create a matrix from vectors $\vec{a}, \vec{b}$ as follows

$$\begin{pmatrix} a_x & b_x \\ a_y & b_y \end{pmatrix}.$$

The identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ corresponds to the identity homeomorphism of the torus. That is, no transformation occurs. Figure 2.6 shows the unit square, unit hexagon, and unit hexagon with alternate coordinates with the vectors $\vec{a}, \vec{b}$ that correspond to the identity matrix.

There are twelve elements of finite order in $SL_2(\mathbb{Z})$. They are listed in Table 2.1 with their order and corresponding rotation of the unit square, unit hexagon, or unit hexagon with alternate coordinates.

The matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

naturally correspond to homeomorphsims of the torus defined as the identification space of the unit square since rotations of $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ send the boundary of the unit square to itself. The matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

naturally correspond to homeomorphsims of the torus defined as the identification space of the unit hexagon since rotations of $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$ send the boundary of the unit hexagon to itself.

The matrices

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

correspond to homeomorphsims of the torus defined as the identification space of the unit hexagon with alternate coordinates. It is most natural to examine these homeomorphisms as matrix transformations.

| Matrix | Order | Rotation | Matrix | Order | Rotation |
|---|---|---|---|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | 1 | 0 | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ | 2 | $\pi$ |
| $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ | 6 | $\frac{\pi}{3}$ | $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ | 3 | $\frac{4\pi}{3}$ |
| $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ | 6 | $\frac{\pi}{3}$ | $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ | 3 | $\frac{4\pi}{3}$ |
| $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ | 4 | $\frac{\pi}{2}$ | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ | 4 | $\frac{3\pi}{2}$ |
| $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ | 3 | $\frac{2\pi}{3}$ | $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ | 6 | $\frac{5\pi}{3}$ |
| $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ | 3 | $\frac{2\pi}{3}$ | $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ | 6 | $\frac{5\pi}{3}$ |

Table 2.1: The finite order $SL_2(\mathbb{Z})$ elements

To find an order-2 automorphism, we can examine embeddings of the 3-rung Möbius ladder on the unit square or the unit hexagon. To find an order-4 automorphism, it is most natural to consider embeddings on the unit square and to find automorphisms of order 3 or 6 it is most natural to examine embeddings on the unit hexagon. We can also use a change of coordinates to examine an embedding on the unit square, the unit hexagon, and the unit hexagon with alternate coordinates.

## 2.4 The Loop of the 3-rung Möbius Ladder

In this section we discuss a technique for determining that a homeomorphism does not induce an automorphism of the 3-rung Möbius ladder. Since our 3-rung Möbius ladder has sides and rungs that are chemically distinct, our automorphisms must send rungs to rungs and sides to sides.

**Definition 2.7.** A *loop* in a topological space $X$ is a continuous function $\alpha : [0, 1] \to X$ such that $\alpha(0) = \alpha(1)$, and we shall say the loop is *based* at the point $\alpha(0)$ [1].

We define the loop of our molecule on the torus as the sides of the ladder based at 1, so that our loop is 123456 as shown in Figure 2.7. So, our automorphisms must send the loop to itself through continuous deformation.

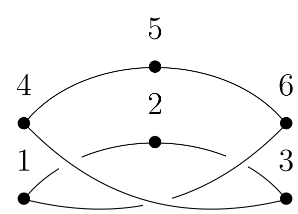


Figure 2.7: A loop of the 3-rung Möbius ladder

**Definition 2.8.** Let $X, Y$ be topological spaces and $f, g : X \to Y$ be continuous functions. Then $f$ is *homotopic* to $g$ if there exists a continuous function $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all points $x \in X$. Furthermore, the function $F$ is called a *homotopy* from $f$ to $g$ [1].

Notice that isotopy is a special case of homotopy since, by definition every isotopy is a homotopy. We do not need to meet all of the requirements of an isotopy, as a homotopy provides us with the continuous deformation that we need. Also, the relation of homotopy is an equivalence relation [1].

**Definition 2.9.** The equivalence class of a loop $L$ under the equivalence relation of homotopy is denoted $[L]$ and called the *homotopy class* of $L$ [8].

**Definition 2.10.** Let $\Sigma$ be a compact connected orientable surface. The *mapping class group* of $\Sigma$, $\mathcal{M}(\Sigma)$, is the group of orientation preserving homeomorphisms from $\Sigma$ to $\Sigma$ up to isotopy among orientation preserving homeomorphisms [9].

Two homeomorphisms from $\Sigma$ to $\Sigma$ are homotopic if and only if they are isotopic [9]. Since the torus is a compact connected orientable surface, we can use homotopy and isotopy interchangeably. Massuyeau states the mapping class group of the torus is isomorphic to $SL_2(\mathbb{Z})$ [9], that is, $\mathcal{M}(T^2) \cong SL_2(\mathbb{Z})$. Furthermore, if a homeomorphism $h \in \mathcal{M}(T^2)$ and $L$ is a loop such that $h(L)$ is isotopic to $L$, then $[h(L)] = [L]$. We can also say that if $[h(L)] \neq [L]$ then $h(L)$ is not isotopic to $L$ and in this case $h$ would not provide us with an automorphism of the 3-rung Möbius Ladder.

We use the fundamental group of the torus to describe the homotopy class of the loop. It is well known that the fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$ [1]. Homotopy classes can be seen as ordered pairs where the first coordinate represents the number of times the loop circles the longitude, and the second coordinate represents the number of times the loop circles the meridian.

## 3 Orientation Preserving Topological Symmetry Group

In this chapter we present a proof that $\mathbb{Z}_2$ occurs as an orientation preserving topological symmetry group of the 3-rung Möbius ladder with sides and rungs distinct, embedded in the torus. First, we introduce an embedding of the 3-rung Möbius ladder and show that it supports an order-2 automorphism. Then we prove that this embedding does not support any higher-order automorphisms so that we can conclude $\mathbb{Z}_2$ is the topological symmetry group of this particular embedding.

**Theorem 3.1.** *The finite cyclic group $\mathbb{Z}_2$ occurs as an orientation preserving topological symmetry group of the 3-rung Möbius ladder with sides and rungs distinct, embedded in the torus.*

Proof: Recall, that the only element of order 2 in $SL_2(\mathbb{Z})$ is

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This matrix corresponds to a rotation of the unit square by $\pi$. Consider the embedding, $f$, of the 3-rung Möbius ladder in the unit square as shown in Figure 3.1. The loop, L, is 123456 and is represented by the dashed line and the rungs $\overline{14}$, $\overline{25}$, and $\overline{36}$ are represented by the dotted lines.

A rotation of the unit square by $\pi$ takes the unit square to the square $[-1, 0] \times [0, -1]$. Recall that we defined the torus as an identification space from $\mathbb{R}^2$. So, the torus we construct as an identification space before and after we rotate the embedding in $\mathbb{R}^2$ is the same torus. Notice, this rotation takes vertices to vertices and edges to edges as seen in Figure 3.2. Also, notice that the loop is still 123456 and we have not reversed the orientation.

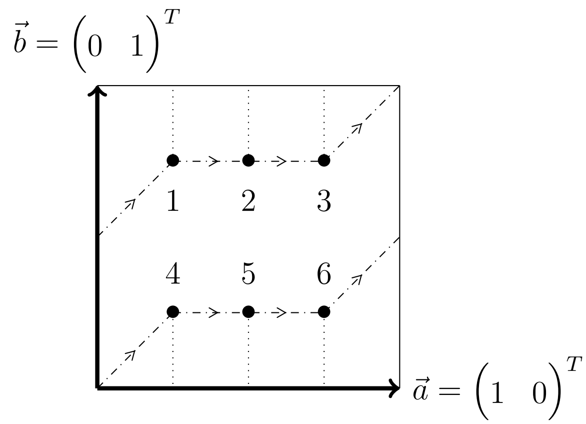


Figure 3.1: The 3-rung Möbius ladder embedded in the unit square

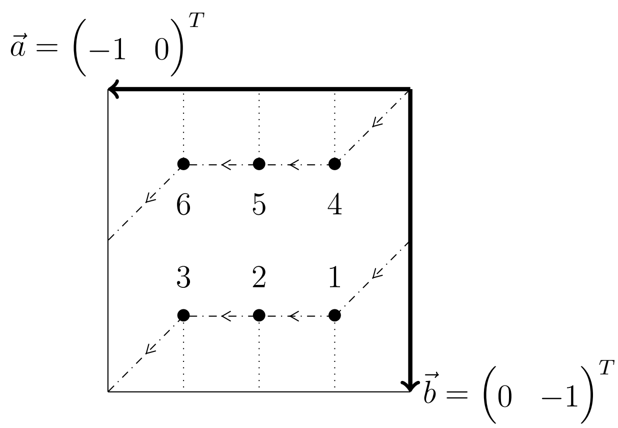


Figure 3.2: Rotation of the 3-rung Möbius ladder embedded in the unit square

So, this homeomorphism $h$ of the torus taking the embedding $f$ of the 3-rung Möbius ladder to itself induces an order 2 automorphism (16)(25)(34). Since this embedding supports an orientation preserving order-2 automorphism, the topological symmetry group must be $\mathbb{Z}_2, \mathbb{Z}_4$, or $\mathbb{Z}_6$. We now show that $\mathbb{Z}_4$ and $\mathbb{Z}_6$ are not possible for this embedding. In order to rule out $\mathbb{Z}_4$, we examine rotations of the unit square by $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

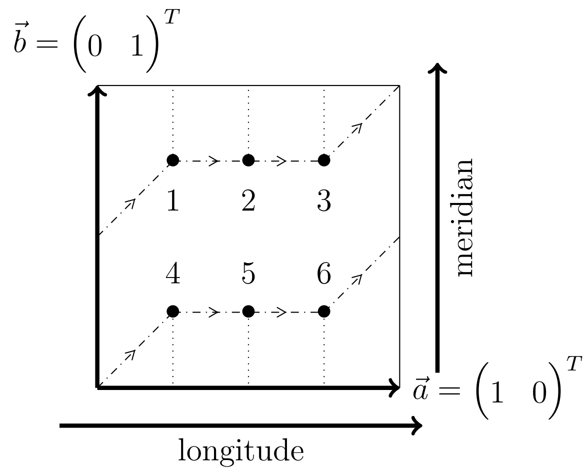


Figure 3.3: $f$ embedded in the unit square

We start with the embedding $f$ in the unit square as shown in Figure 3.3 with the longitude and meridian labeled. The homotopy class of the loop $L$ is $[L] = (2, 1)$ since the loop circles the longitude twice and the meridian once.

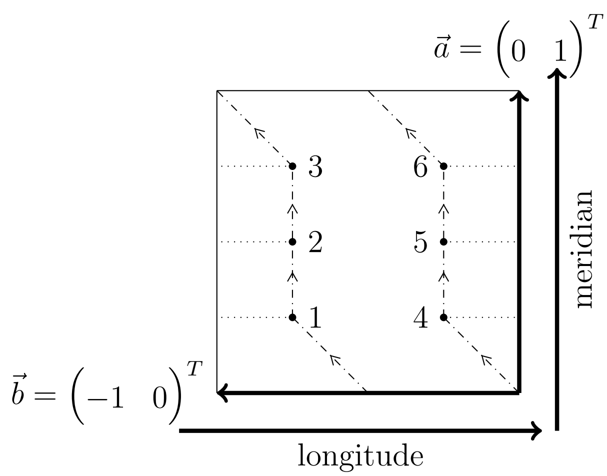


Figure 3.4: $h_1(f)$ embedded in the unit square

Now, when we apply the homeomorphism $h_1$ to the torus that corresponds to a rotation of the unit square by $\frac{\pi}{2}$ the loop becomes $h_1(L)$ as seen in Figure 3.4. It is clear that the homotopy class of $h_1(L)$ is $[h_1(L)] = (1, 2)$ since the loop circles the longitude once and the meridian twice. Since, $[h_1(L)] \neq [L]$ then $h_1(L)$ is not isotopic to $L$ and thus this homeomorphism, $h_1$, does not induce an automorphism.

Applying the homeomorphism $h_2$ to the torus that corresponds to a rotation of the unit square by $\frac{3\pi}{2}$ and the loop becomes $h_2(L)$ as seen in Figure 3.5. The homotopy class of $h_2(L)$ is $[h_2(L)] = (1, 2)$ since the loop circles the longitude once and the meridian twice. Since, $[h_2(L)] \neq [L]$ then $h_2(L)$ is not isotopic to $L$ and thus the homeomorphism, $h_2$, does not induce an automorphism.

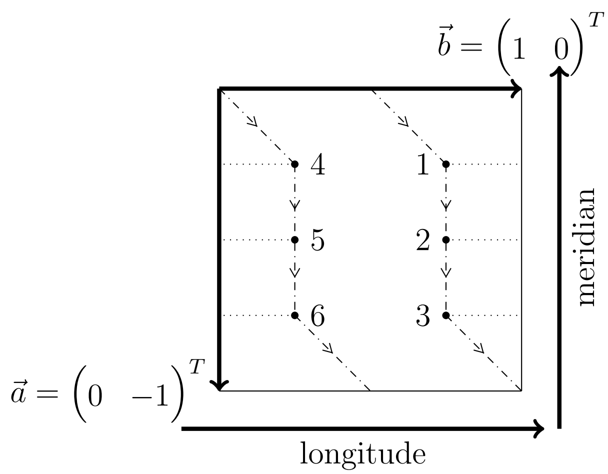


Figure 3.5: $h_2(f)$ embedded in the unit square

We have examined both order 4 homeomorphisms, $h_1$ and $h_2$. Neither one is an automorphism of the 3-rung Möbius ladder, so the topological symmetry group of the embedding $f$ cannot be $\mathbb{Z}_4$.

We have two methods when examining the homeomorphisms of the torus to rule out $\mathbb{Z}_6$. For the elements $\begin{pmatrix}0 & -1\\1 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 1\\-1 & 0\end{pmatrix}$ we examine rotations of the unit hexagon by $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. First, consider the embedding, $g$, in the unit hexagon as shown in the top left corner of Figure 3.6.

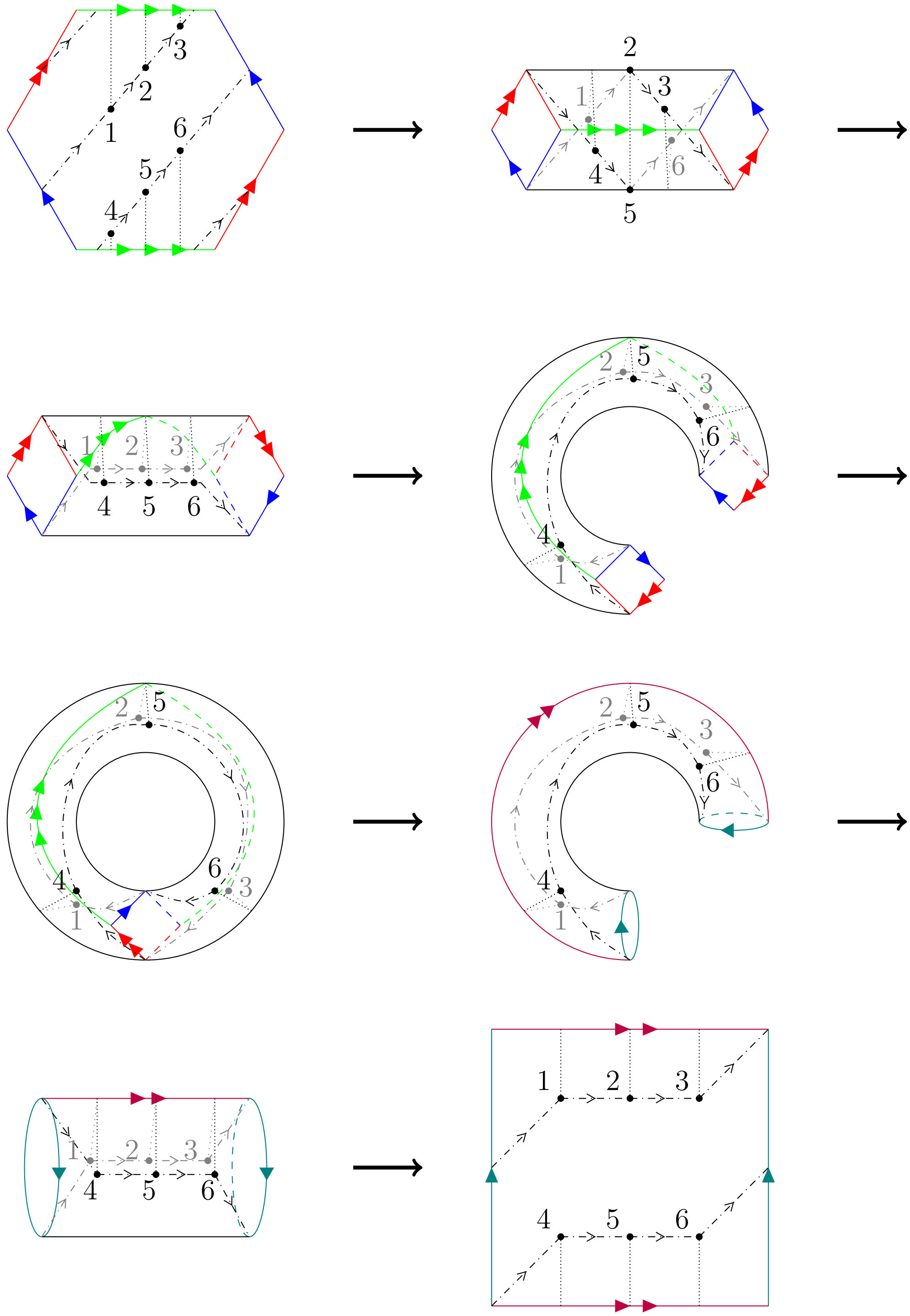

Figure 3.6: Transformation of an embedding

We can form the torus as an identification space from the unit hexagon and then take it apart again so that we have $f$ embedded in the unit square as shown in the bottom right corner of Figure 3.6. This means that the embeddings $f$ and $g$ are identical when we form the torus as an identification space.

So, we start with the embedding $g$ in the unit hexagon as shown in Figure 3.7. The homotopy class of the loop $L$ is $[L] = (2, 1)$ since $g$ gives us the same embedding of the 3-rung Möbius ladder in the torus as $f$.

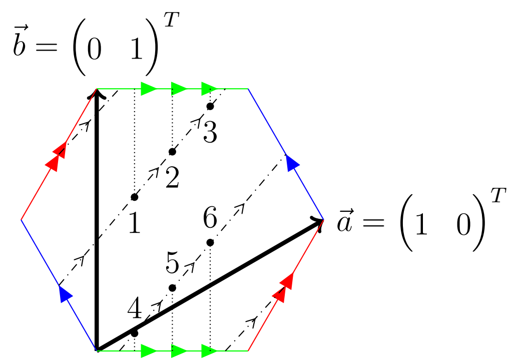


Figure 3.7: $g$ in the unit hexagon

Applying the homeomorphism $h_3$ to the torus that corresponds to a rotation of the unit hexagon by $\dfrac{\pi}{3}$ and the loop becomes $h_3(L)$ as seen in Figure 3.8. The homotopy class of $h_3(L)$ is $[h_3(L)] = (1, 3)$ since the loop circles the longitude once and the meridian three times. Since, $[h_3(L)] \neq [L]$ then $h_3(L)$ is not isotopic to $L$ and thus this homeomorphism does not induce an automorphism.

We then apply the homeomorphism $h_4$ to the torus that corresponds to a rotation of the unit hexagon by $\dfrac{5\pi}{3}$ the loop becomes $h_4(L)$ as seen in Figure 3.9. The homotopy class of $h_4(L)$ is $[h_4(L)] = (3, 2)$ since the loop circles the longitude three times and the meridian twice. Since, $[h_4(L)] \neq [L]$ then $h_4(L)$ is not isotopic to $L$ and thus this homeomorphism does not induce an automorphism.

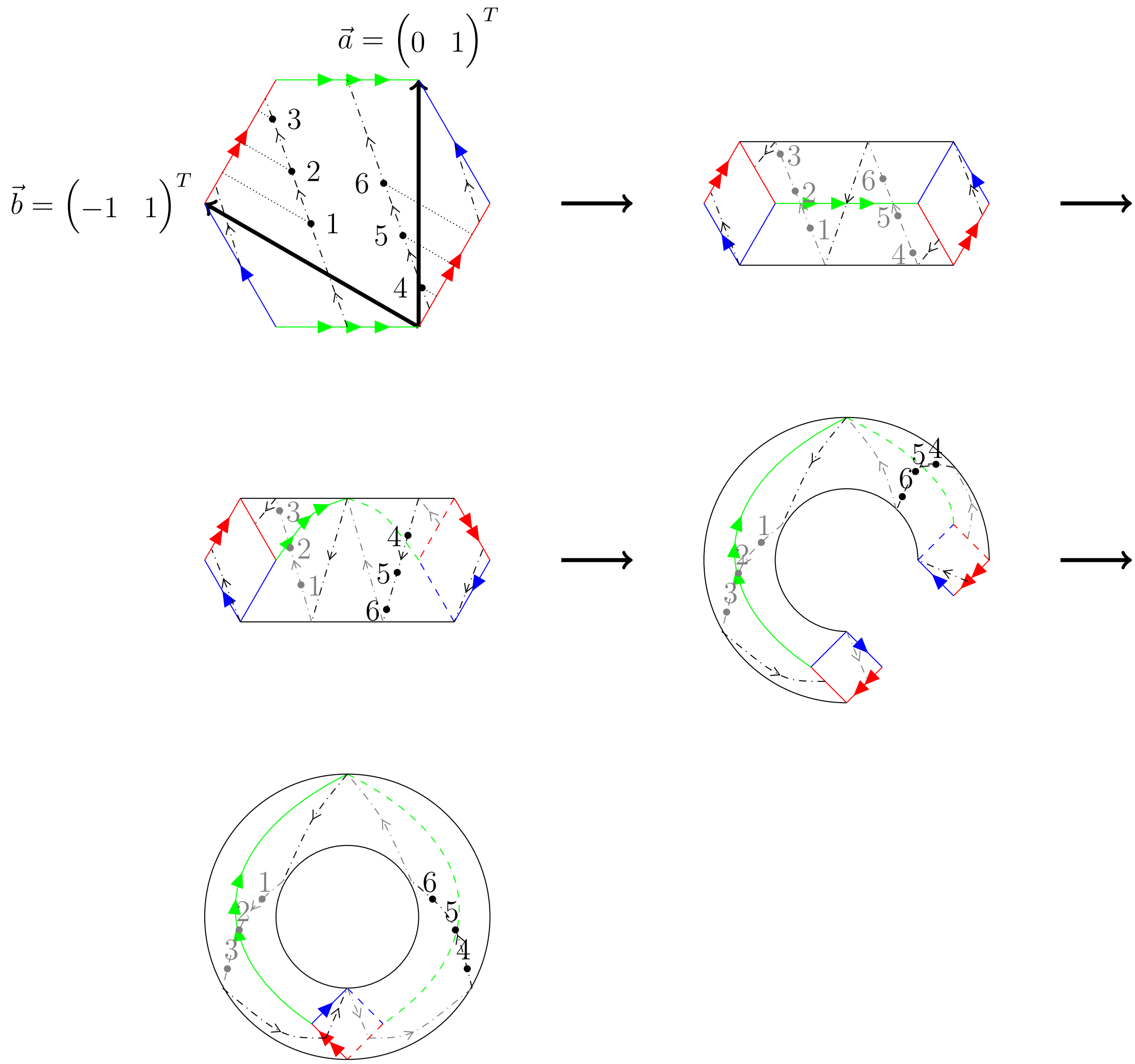


Figure 3.8: $h_3(L)$ embedded in the torus

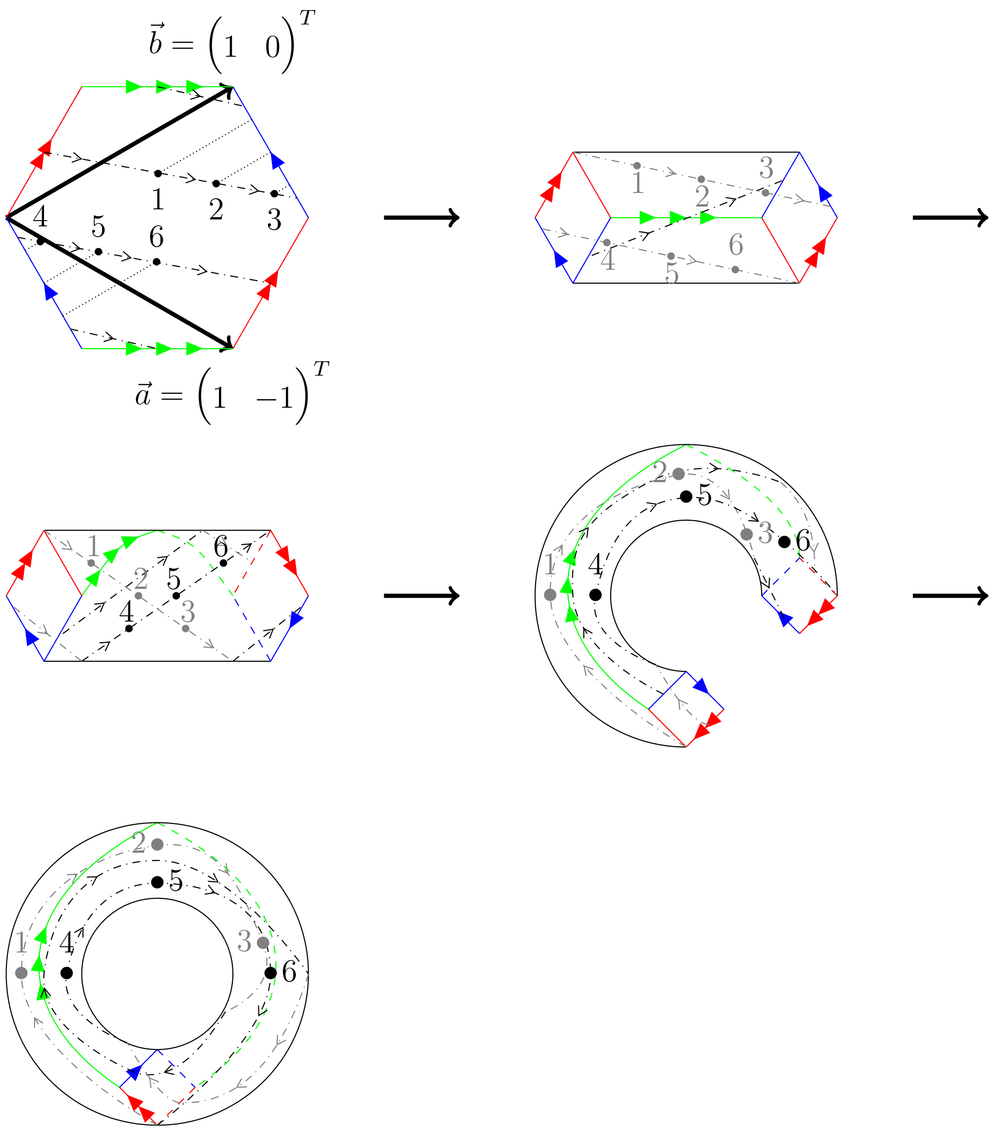


Figure 3.9: $h_4(L)$ embedded in the torus

For the elements $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ we examine the homeomorphisms as matrix transformations. The matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ transforms rectangular coordinates into the alternate hexagonal coordinates. Then we apply the $SL_2(\mathbb{Z})$ element that corresponds to a rotation in the alternate hexagonal coordinates. Finally the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ transforms the alternate hexagonal coordinates back into rectangular coordinates.

For example, the embedding $f$ has vertex 1 located at the point $\left(\frac{1}{4}, \frac{3}{4}\right)$. We treat this ordered pair like the vector $\begin{pmatrix} \frac{1}{4} & \frac{3}{4} \end{pmatrix}^T$. First we perform the multiplication that takes vertex 1 to the alternate hexagonal coordinates.

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{4} \end{pmatrix}$$

Then we perform the multiplication that rotates vertex 1 by $\frac{\pi}{3}$ in the alternate hexagonal coordinates.

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{4} \end{pmatrix} = \begin{pmatrix} -\frac{5}{4} \\ -\frac{1}{2} \end{pmatrix}$$

Finally, we perform the multiplication that takes vertex 1 back to rectangular coordinates.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{5}{4} \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{7}{4} \\ -\frac{1}{2} \end{pmatrix}$$

Notice that vertex 1 is no longer in the unit square. However, it identifies with the point $\left(\frac{1}{4}, \frac{1}{2}\right)$ when we form the torus as an identification space from $\mathbb{R}^2$. Figure 3.10 depicts this process for the embedding $f$.

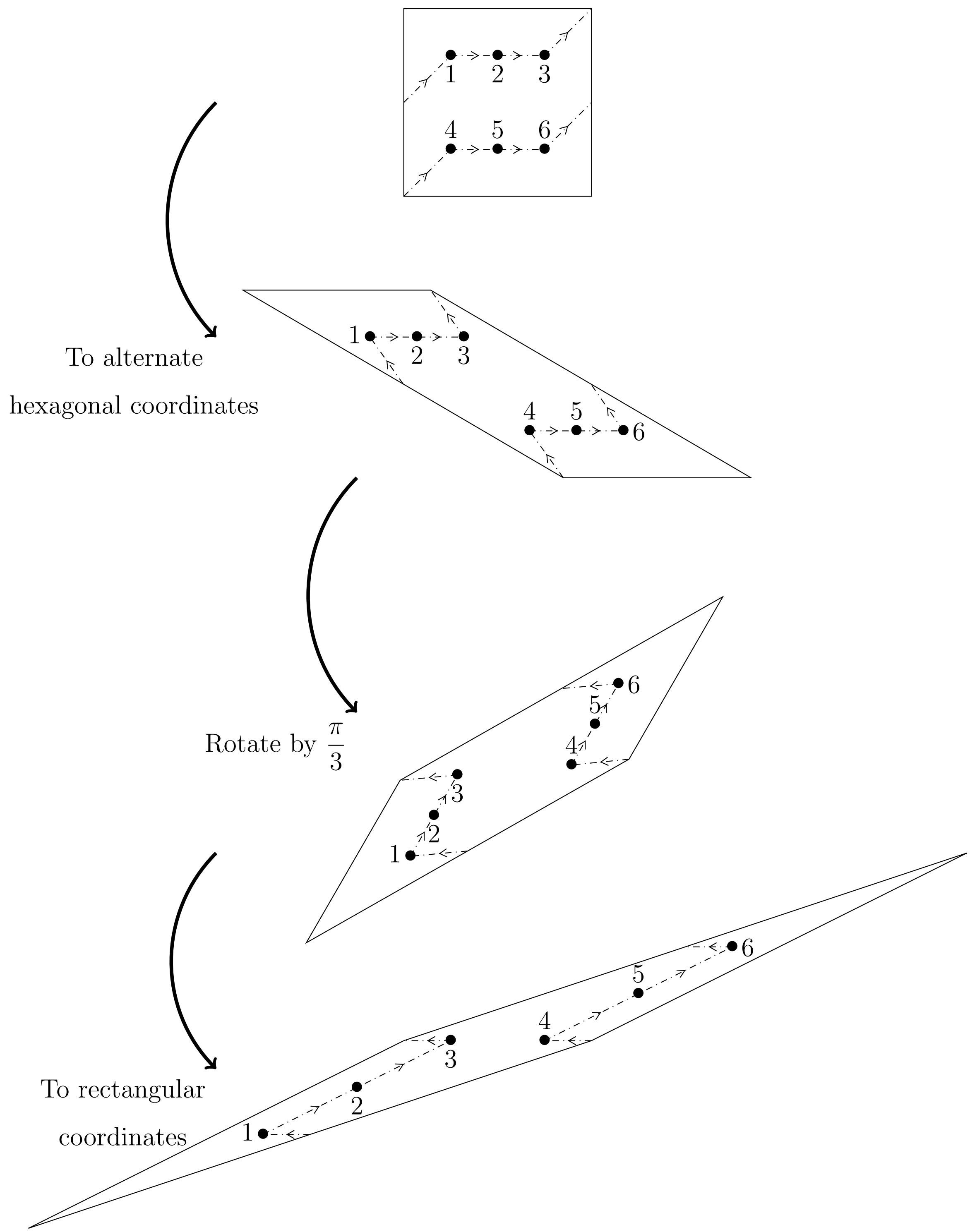


Figure 3.10: $h_5(L)$ embedded in $\mathbb{R}^2$

Figure 3.11 illustrates the construction of the torus and the embedding $h_5(L)$ as an identification space from the unit square after $\mathbb{R}^2$ has be identified to the unit square. The homotopy class of $h_5(L)$ is $[h_5(L)] = (1, 1)$ since the loop circles the longitude once and the meridian once. Since, $[h_5(L)] \neq [L]$ then $h_5(L)$ is not isotopic to $L$ and thus this homeomorphism does not induce an automorphism.

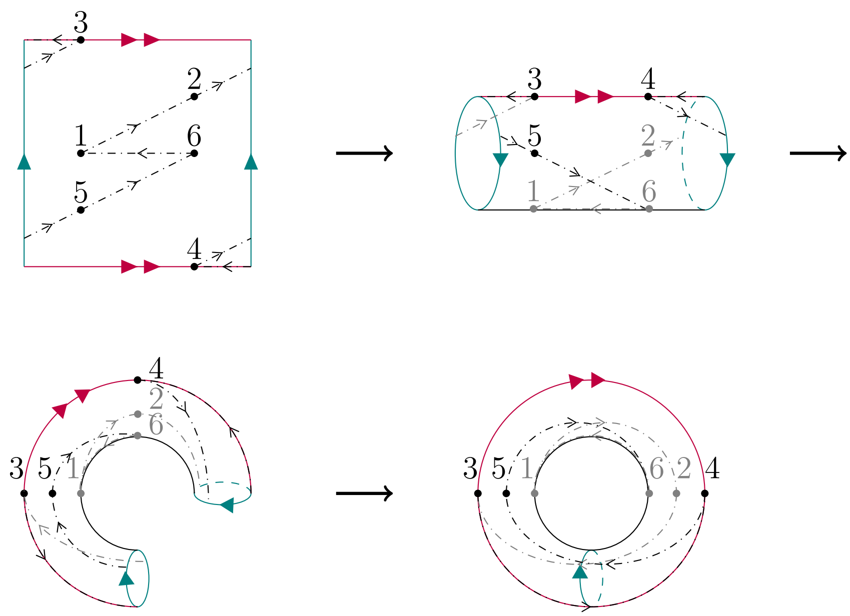


Figure 3.11: $h_5(L)$ embedded in the torus

Now, we apply the homeomorphism $h_6$ to $f$ as shown in Figure 3.12. Then we construct the torus as an identification space from $\mathbb{R}^2$ as shown in Figure 3.13. The homotopy class of $h_6(L)$ is $[h_6(L)] = (1, 0)$ since the loop circles the longitude once and never makes a complete circle of the meridian. Since, $[h_6(L)] \neq [L]$ then $h_6(L)$ is not isotopic to $L$ and thus this homeomorphism does not induce an automorphism.

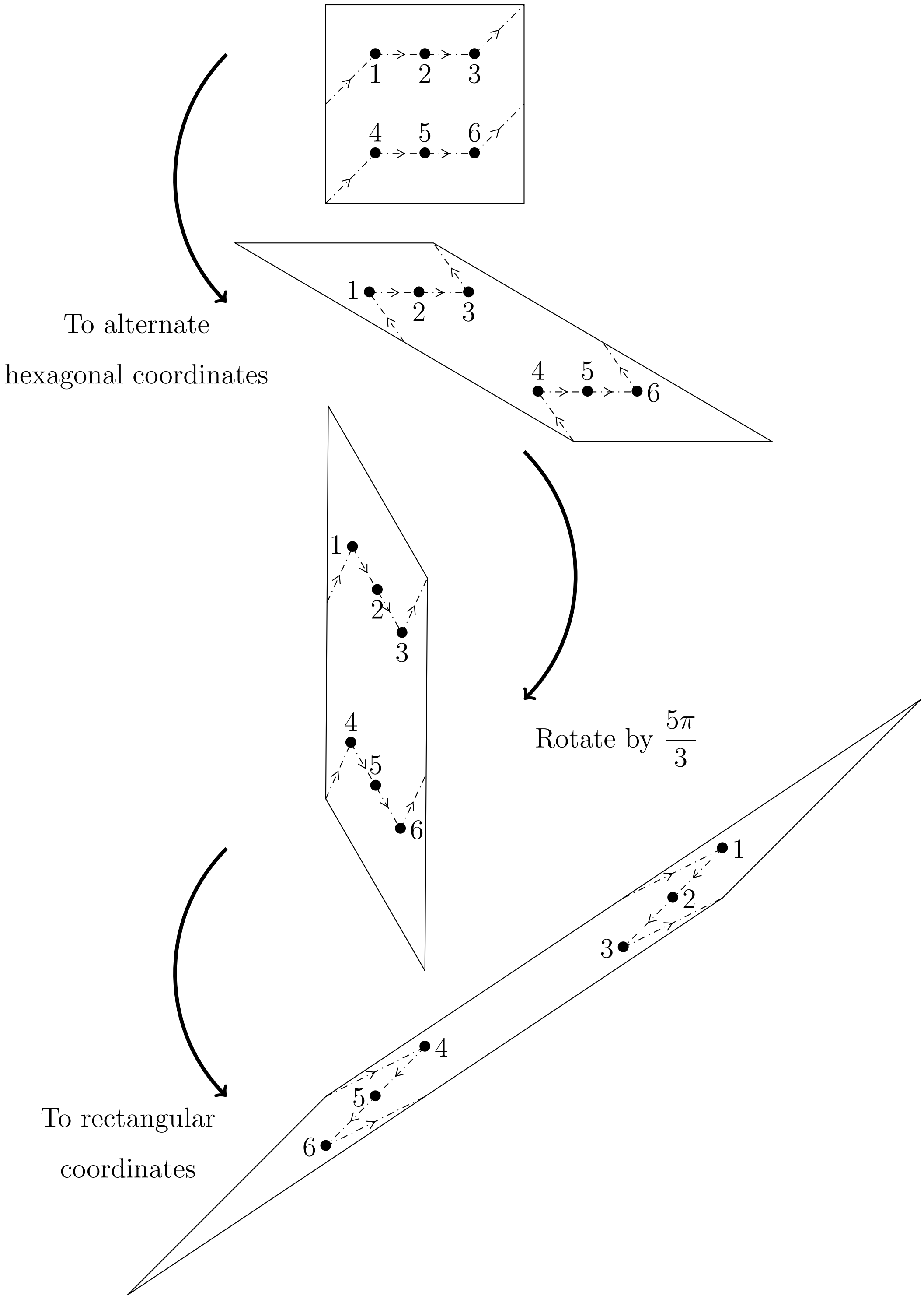


Figure 3.12: $h_6(L)$ embedded in $\mathbb{R}^2$

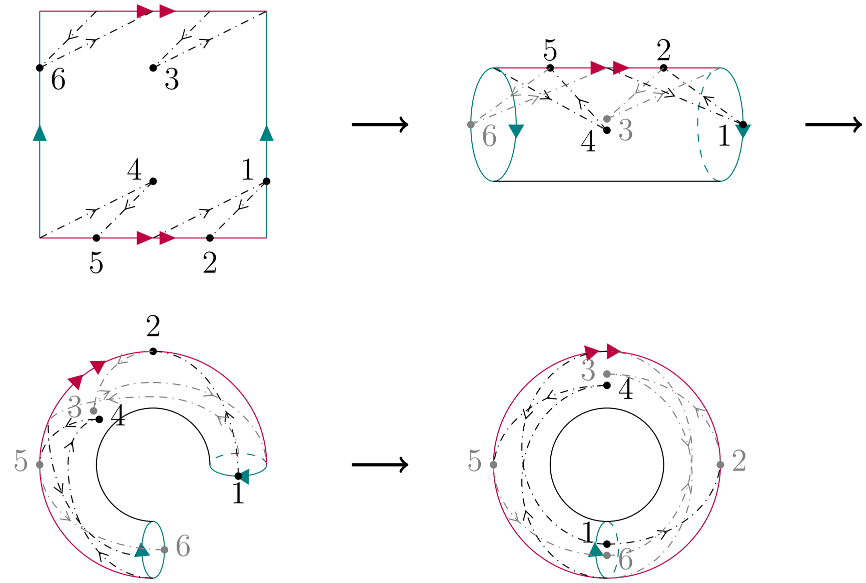


Figure 3.13: $h_6(L)$ embedded in the torus

We have examined all four of the order 6 homeomorphisms: $h_3$, $h_4$, $h_5$, and $h_6$. None of them are an automorphism of the 3-rung Möbius ladder, so the topological symmetry group of the embedding $f$ cannot be $\mathbb{Z}_6$.

Thus, $f$ is an embedding that supports an orientation preserving order-2 automorphism, but not an order-4 or order-6 automorphism. So, $\mathbb{Z}_2$ is an orientation preserving topological symmetry group of the 3-rung Möbius ladder embedded in the torus. $\square$

## 4 Future Work

We hope to determine whether $\mathbb{Z}_1, \mathbb{Z}_3, \mathbb{Z}_4$, and $\mathbb{Z}_6$ can occur as topological symmetry groups of some embedding of the 3-rung Möbius ladder. If so, we hope to construct the embedding. Similar to the work in [6], we would also like to find an embedding that has $\mathbb{Z}_2$ as a topological symmetry group without the need to declare that rungs must go to rungs.

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## VITA

Rebecca Woods received a Bachelor degree in Mathematics in 2016 from Stephen F. Austin State University. She began work towards a Master degree in Mathematics at Stephen F. Austin State University in the Fall of 2016 and is expected to graduate in December 2018.

The style manual used in this thesis is A Manual For Authors of Mathematical Papers published by the American Mathematical Society.

This thesis was prepared by Rebecca Woods using LaTeX.