Theoretical Analysis of Nonlinear Differential Equations

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THEORETICAL ANALYSIS OF NONLINEAR DIFFERENTIAL EQUATIONS

by

Emily Weymier, B.S.

Presented to the Faculty of the Graduate School of
Stephen F. Austin State University
In Partial Fulfillment
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For the Degree of
Master of Science

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THEORETICAL ANALYSIS OF NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT

Nonlinear differential equations arise as mathematical models of various phenomena. Here, various methods of solving and approximating linear and nonlinear differential equations are examined. Since analytical solutions to nonlinear differential equations are rare and difficult to determine, approximation methods have been developed. Initial and boundary value problems will be discussed. Several linear and nonlinear techniques to approximate or solve the linear or nonlinear problems are demonstrated. Regular and singular perturbation theory and Magnus expansions are our particular focus. Each section offers several examples to show how each technique is implemented along with the use of visuals to demonstrate the accuracy, or lack thereof, of each technique. These techniques are integral in applied mathematics and it is shown that correct employment allows us to see the behavior of a differential equation when the exact solution may not be attainable.
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1 INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

1.1 Introduction

This section provides a brief overview of differential equations and two solution techniques for nonlinear differential equations. Several types of initial value problems and boundary value problems are discussed. Examples and solutions are calculated to provide a clear picture of the solution techniques and how they are implemented.

1.2 A Brief Overview

An ordinary differential equation (ODE) is an equation that involves derivatives of an unknown function. Ordinary differential equations are used to model change over a single independent variable (it is usually \( t \) for time). These equations do not involve any partial derivatives. Differential equations contain three types of variables: an independent variable, at least one dependent variable (these will be functions of the independent variable), and the parameters. ODE’s can contain multiple iterations of derivatives and are named accordingly (i.e. if there are only first derivatives, then the ODE is called a first order ODE).

1.3 A Common Example: Population Growth

Population growth is commonly modeled using differential equations [8]. Consider the following equation

\[
\frac{dP}{dt} = kP,
\]

where \( t \) represents time, \( P \) is the population density at time \( t \), and \( k \) is the proportionality constant. The constant \( k \) represents the ratio between the growth rate of the population and the size of the population. In this particular equation, the left hand side represents the growth rate of the population being proportional to the size of the population, \( P \). This is a very simple example of a first order ordinary differential equation. The equation contains first order derivatives and no partial derivatives.
1.4 Initial Value Problems

An initial value problem consists of a differential equation and an initial condition [9]. Referencing the population example, the following is an example of an initial value problem:

\[
\frac{dP}{dt} = kP, \quad P(0) = P_0.
\]

The solution to this set of equations is a function, call it \( P(t) \), that satisfies both equations. The standard form for a first order differential equation is

\[
\frac{dy}{dt} = f(t, y),
\]

where the right hand side represents the function \( f \) that depends on the independent variable, \( t \), and the dependent variable, \( y \).

1.4.1 General Solution

Begin with the following equation:

\[
\frac{dy}{dt} = (ty)^2.
\]

We will “separate” the variables then integrate both sides of the equation to find the general solution [9]. Namely,

\[
\frac{dy}{dt} = t^2 y^2, \\
\frac{1}{y^2} dy = t^2 dt, \\
\int \frac{1}{y^2} dy = \int t^2 dt, \\
-\frac{1}{y} = \frac{t^3}{3} + c.
\]

Solving for \( y \) yields,

\[
y(t) = -\frac{3}{t^3 + c_1},
\]

where \( c_1 \) is any real number determined by the initial condition of \( y \). Given \( y(0) = y_0 \) then \( c_1 = -3/y_0 \). So, when \( y(0) = y_0 \), \( y(t) = -\frac{3}{t^3 - \frac{3}{y_0}} \).
1.4.2 Second Order Differential Equations

Next, we will examine second order differential equations. These contain a second derivative of the dependent variable. The following is a common example of a second order differential equation that models a simple harmonic oscillator [9].

\[
\frac{d^2y}{dt^2} + \frac{k}{m}y = 0, \tag{1.1}
\]

where \( m \) and \( k \) are determined by the mass and spring involved. Recalling some calculus knowledge, when \( v(t) \) is velocity, \( v = \frac{dy}{dt} \). Thus, the above equation can be rewritten as the following first order differential equation.

\[
\frac{dv}{dt} = -\frac{k}{m}y, \\
\frac{d^2y}{dt^2} = -\frac{k}{m}y,
\]

where \( v \) denotes velocity. So the original second order differential equation (1.1) has been recast as a first order differential equation. This creates the following system:

\[
\frac{dy}{dt} = v, \\
\frac{dv}{dt} = -\frac{k}{m}y.
\]

For further reading on harmonic oscillators, the reader can refer to Chapter 2 of [9].

Next we will look at the following initial value problem that begins with a second order differential equation.

\[
\frac{d^2y}{dt^2} + y = 0, \tag{1.2}
\]

\[
y(0) = 0, \\
y'(0) = v(0) = 1.
\]

Instead of finding a solution, we will verify that \( y(t) = \sin(t) \) is a solution to the initial value problem. That is, verify that \( y(t) = \sin(t) \) satisfies (1.2) and its initial conditions. Let \( v = \frac{dy}{dt} \), then \( \frac{dv}{dt} = \frac{d^2y}{dt^2} \) and (1.2) becomes the following system,

\[
\frac{dy}{dt} = v, \\
\frac{dv}{dt} = -y.
\]
Next, we substitute \( y(t) = \sin(t) \) into the system

\[
\begin{align*}
v &= \frac{dy}{dt} = \frac{d}{dt} y = \frac{d}{dt} \sin(t) = \cos(t), \\
-y &= \frac{dv}{dt} = -\sin(t).
\end{align*}
\]

So, we have the following

\[
\begin{align*}
v &= \cos(t), \\
-y &= -\sin(t).
\end{align*}
\]

Remember that we are ultimately trying to reconstruct \( \frac{d^2y}{dt^2} + y = 0 \) and show that \( y(t) = \sin(t) \) satisfies the differential equation. Next we need \( \frac{d^2y}{dt^2} = -\sin(t) \).

\[
\frac{d^2y}{dt^2} + y = \frac{d^2(\sin(t))}{dt^2} + \sin(t) = -\sin(t) + \sin(t) = 0.
\]

Therefore, \( y(t) = \sin(t) \) satisfies \( \frac{d^2y}{dt^2} + y = 0 \) and is a solution to the second order differential equation (1.2).

### 1.4.3 Recasting High Order Differential Equations as a System of First Order Differential Equations

Next we will move on to a problem that is a little more complex. Begin with the following second order differential equation:

\[
\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x + x^3 = 0. \tag{1.3}
\]

Again, let \( v = \frac{dx}{dt} \), then (1.3) becomes

\[
\frac{dv}{dt} + 2\frac{dx}{dt} - 3x + x^3 = 0.
\]

Now (1.3) can be written as the following system of first order differential equations,

\[
\begin{align*}
\frac{dv}{dt} &= -x^3 + 3x - 2\frac{dx}{dt} = -x^3 + 3x - 2v, \\
\frac{dx}{dt} &= v.
\end{align*}
\]
This example is used to show that higher order differential equations can be recast into a system of first order differential equations. Of course, this idea can be extended to higher order differential equations. In fact, any higher order differential system of differential equations can recast into a system of first order equations [9].

1.5 Boundary Value Problems

Boundary value problems (BVP) are essentially initial value problems that offer two constraints rather than one. Boundary value problems are ubiquitous in the separation of variables technique in partial differential equations [7, 11]. In order to solve a BVP, we will need to solve for the general solution before utilizing the boundary constraints.

Next we will look at the following example

\[
\frac{d^2y}{dx^2} + 4y = 0, \quad (1.4)
\]

\[
y(0) = 1,
\]

\[
y\left(\frac{\pi}{2}\right) = 2.
\]

These boundary conditions are called Dirichlet and Neumann, respectively. The general solution to the second order differential equation can be found using the characteristic polynomial, Euler’s Formula, and a few trigonometric tricks. The general solution is

\[
y = a_1 \cos(2x) + a_2 \sin(2x). \quad (1.5)
\]

Given the general solution (1.5), we will use the boundary conditions to find the exact solution. First we will use the initial condition \(y(0) = 1\). That is,

\[
y = a_1 \cos(2x) + a_2 \sin(2x),
\]

\[
y(0) = a_1 \cos(0) + a_2 \sin(0)
\]

\[
= a_1 = 1
\]

Now that we have \(a_1 = 1\), we can solve for \(a_2\) using \(y\left(\frac{\pi}{2}\right) = 2\).

\[
y' = -2a_1 \sin(2x) + 2a_2 \cos(2x),
\]

\[
y'\left(\frac{\pi}{2}\right) = -2 \sin(\pi) + 2a_2 \cos(\pi)
\]

\[
= -2a_2 = 2,
\]

\[
a_2 = -1.
\]
Therefore,

\[ y = a_1 \cos(2x) + a_2 \sin(2x) \]
\[ = \cos(2x) - \sin(2x). \]

So, the particular solution given \( y(0) = 1 \) and \( y'(\frac{\pi}{2}) = 2 \) is \( y = \cos(2x) - \sin(2x) \).

### 1.6 Solution Techniques For Nonlinear Differential Equations

In the last section techniques were used to solve linear differential equations. This next section outlines two different techniques utilizing expansions to solve nonlinear differential equations. Power series solutions and perturbation techniques will be used to solve or approximate nonlinear differential equations.

#### 1.6.1 Power Series Solutions

We will begin with the following nonlinear differential equation known as Hermite’s Equation [7]:

\[
\frac{d^2y}{dt^2} - 2t \frac{dy}{dt} + 2py = 0. \tag{1.6}
\]

We will utilize the following power series and its first and second derivatives to make an educated guess of the solution

\[
y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots = \sum_{n=0}^{\infty} a_n t^n, \tag{1.7}
\]

\[
\frac{dy}{dt} = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \ldots = \sum_{n=1}^{\infty} n a_n t^{n-1}, \tag{1.8}
\]

\[
\frac{d^2y}{dt^2} = 2a_2 + 6a_3 t + 12a_4 t^2 + \ldots = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}. \tag{1.9}
\]

From the previous equations we can conclude that

\[
y(0) = a_0, \]
\[
y'(0) = a_1. \]
Next, we will substitute (1.7), (1.8) and (1.9) into Hermite’s Equation and collect like terms,

\[
\frac{d^2 y}{dt^2} - 2t \frac{dy}{dt} + 2py = (2a_2 + 6a_3 t + 12a_4 t^2 + \ldots) \\
-2t(a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \ldots) \\
+2p(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots) \\
= (2pa_0 + 2a_2) + (2pa_1 - 2a_1 + 6a_3) t + \\
(2pa_2 - 4a_2 + 12a_4) t^2 + (2pa_3 - 6a_3 + 20a_5) t^3 + \ldots = 0.
\]

Then from here, we will set all coefficients equal to 0 since the equation is equal to 0 and \( t \neq 0 \). We get the following sequence of equations:

\[
2pa_0 + 2a_2 = 0, \\
2pa_1 - 2a_1 + 6a_3 = 0, \\
2pa_2 - 4a_2 + 12a_4 = 0, \\
2pa_3 - 6a_3 + 20a_5 = 0.
\]

Then with several substitutions we arrive at the following set of equations:

\[
a_2 = -pa_0, \\
a_3 = \frac{p - 1}{3} a_1, \\
a_4 = \frac{p - 2}{6} a_2 = \frac{(p - 2)p}{6} a_0, \\
a_5 = \frac{p - 3}{10} a_3 = \frac{(p - 3)(p - 1)}{30} a_1.
\]

Notice that we can write all coefficients as functions of \( a_0 \) or \( a_1 \), which leads to the following general solution to Hermite’s Equation.

\[
y(t) = a_0 \left(1 - pt^2 + \frac{(p - 2)p}{6} t^4 + \ldots\right) \\
+ a_1 \left(t - \frac{p - 1}{3} t^3 + \frac{(p - 3)(p - 1)}{30} t^5 + \ldots\right).
\]

The above is the general solution after substitutions have been made and like terms have been collected. Notice that an exact solution can be determined if values were provided.
### 1.6.2 Perturbation Theory

Perturbation theory is used when a mathematical equation involves a small perturbation, usually $\epsilon$. Weakly nonlinear problems, that is problems for which the nonlinearities are at a smaller order than that of the linear terms, are common in many physical models. For instance in the study of fluid mechanics viscous terms are often viewed as a small perturbation from Euler’s equations [1]. Perturbation theory’s applications are primarily motivated from continuum mechanics, in particular fluid mechanics [1,15,16]. Techniques such as regular and singular perturbation theory as well as matched asymptotics and multiscales analysis have been at the backbone of approximations nonlinear models, see [10,14,15] and references therein.

Here we consider a differential equation with a small parameter and then create a solution, $y(x)$ such that it is an expansion in terms of $\epsilon$. For example

$$y(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \ldots$$

This summation is called a perturbation series and it has a nice feature that allows each $y_i$ to be solved using the previous $y_i$’s. Consider the equation,

$$x^2 + x + 6\epsilon = 0, \quad \epsilon \ll 1. \quad (1.10)$$

Next, consider using perturbation theory to determine approximations for the roots of equation (1.10). In the following example we will be using the technique on a quadratic equation rather than a differential equation to demonstrate the technique.

Notice this equation is a perturbation of $x^2 + x = 0$. Let $x(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n$. This series will be substituted into (1.10) and powers of $\epsilon$ will be collected. Once the substitution is made the equation will become

$$(a_0 \epsilon^0 + a_1 \epsilon^1 + a_2 \epsilon^2 + \cdots)^2 + (a_0 \epsilon^0 + a_1 \epsilon^1 + a_2 \epsilon^2 + \cdots) + 6\epsilon = 0.$$  

Next, we will calculate the first term of the series by setting $\epsilon = 0$ in (1.10). So the leading order equation is

$$a_0^2 + a_0 = 0, \quad (1.11)$$

with solutions $x = -1, 0$. Thus $x(0) = a_0 = -1, 0$. Now the perturbation series are as follows

$$= 1 - a_1 \epsilon - a_2 \epsilon^2 - a_1 \epsilon + a_1^2 \epsilon^2 + a_1 a_2 \epsilon^3 - a_2 \epsilon^2 + a_1 a_2 \epsilon^3$$

$$+ a_2^2 \epsilon^4 - 1 + a_1 \epsilon + a_2 \epsilon^2 + 6\epsilon$$

$$= (1 - 1) + (-2a_1 + a_1 + 6)\epsilon + (-2a_2 + a_1^2 + a_2)\epsilon^2 + \mathcal{O}(\epsilon^3). \quad (1.12)$$

The following equations represent the two solution branches, (1.14) is when $a_0 = -1$ and (1.15) is when $a_1 = 0$.

$$x_1(\epsilon) = -1 + a_1 \epsilon + a_2 \epsilon^2 + \mathcal{O}(\epsilon^3) \quad (1.14)$$

$$x_2(\epsilon) = 0 + \epsilon + \mathcal{O}(\epsilon^2). \quad (1.15)$$
and

\[ x_2(\epsilon) = 0 + b_1 \epsilon + b_2 \epsilon^2 + \mathcal{O}(\epsilon^3). \]  

(1.15)

From here we will need to solve these two equations separately and the result will be two different sets of coefficients.

Next, we will substitute (1.14) into (1.10) while ignoring powers of \( \epsilon \) greater than 2. Since we are only approximating the solution to the second order, we can disregard the powers of \( \epsilon \) greater than 2.

\[ x^2 + x + 6 \epsilon = (-1 + a_1 \epsilon + a_2 \epsilon^2)^2 + (-1 + a_1 \epsilon + a_2 \epsilon^2) + 6 \epsilon \]

\[ = (-a_1 + 6) \epsilon + (-a_2 + a_1^2) \epsilon^2 + \mathcal{O}(\epsilon^3). \]

From here we take the coefficient of each power of \( \epsilon \) and set it equal to zero. This step is justified because (1.10) is equal to zero and \( \epsilon \neq 0 \) so each coefficient must be equal to zero. Thus we have the following equations

\[ \mathcal{O}(\epsilon^1) : -a_1 + 6 = 0, \]

\[ \mathcal{O}(\epsilon^2) : a_1^2 - a_2 = 0. \]

These equations will be solved sequentially. The results are \( a_1 = 6 \) and \( a_2 = 36 \). Thus the perturbation expansion for the root \( x_1 = -1 \) is:

\[ x_1(\epsilon) = -1 + 6 \epsilon + 36 \epsilon^2 + \mathcal{O}(\epsilon^3). \]

The same process can be repeated for \( x_2 \) with the perturbation expansion for the root \( x_2 = 0 \) resulting in

\[ x_2(\epsilon) = -6 \epsilon - 36 \epsilon^2 + \mathcal{O}(\epsilon^3). \]

Since we began with a quadratic equation, we can simply use the quadratic formula to find the exact roots.

\[ x^2 + x + 6 \epsilon = 0, \]

\[ x = \frac{1 \pm \sqrt{1 - 24 \epsilon}}{2}. \]

If we take each of these exact solutions and expand them around \( \epsilon = 0 \), we get the following equations:

\[ x_3(\epsilon) = -1 + 6 \epsilon + 36 \epsilon^2 + 432 \epsilon^3 + \cdots, \]

\[ x_4(\epsilon) = -6 \epsilon - 36 \epsilon^2 - 432 \epsilon^3 - \cdots. \]
These look very similar to the equations that were presented above that gave an approximation for the roots of the quadratic equation using a power series expansion:

\[
x_1(\epsilon) = -1 + 6\epsilon + 36\epsilon^2 + O(\epsilon^3),
\]
\[
x_2(\epsilon) = -6\epsilon - 36\epsilon^2 + O(\epsilon^3).
\]

Based on the exact solutions for the quadratic equation we began with, we can see that our approximations for the roots are exact up to the $\epsilon^2$ term.

The following provides a visual representation of the quadratic function along with our approximated roots for various values of $\epsilon$.

![Figure 1.1:](image)

*Figure 1.1:* The figure above shows the graph of $x^2 + x + 6\epsilon = 0$ for $\epsilon = 0.01$. The x intercepts plotted on the graph show the values of $x_1(0.01)$ and $x_2(0.01)$.

Again, in the example given above, we jumped to a quadratic equation rather than a differential equation to demonstrate the technique. The perturbation technique can be applied to any type of equation where a small parameter exists.

1.7 Conclusion

This section provided a brief overview of differential equations and solution techniques for linear and nonlinear differential equations. These solution techniques are fairly easy to implement, but there are instances when these techniques are not powerful enough to solve nonlinear problems. This lack of power leads us into the next section that discusses a procedure called linear analysis that enables us to use linear solution techniques to understand local behavior around equilibrium solutions to differential equations. This is particularly useful for nonlinear differential equations.
Figure 1.2: The figure above shows the graph of $x^2 + x + 6\epsilon = 0$ for $\epsilon = 0.1$. The x intercepts plotted on the graph show the values of $x_1(0.1)$ and $x_2(0.1)$. 
2 LINEAR ANALYSIS

2.1 Introduction

This section continues with the theme of differential equations, but instead of solving them this section uses techniques to determine the behavior of a differential equation around its equilibrium solution(s). There are no known solutions for the majority of nonlinear differential equations. Some nonlinear equations are tougher to solve or unsolvable. In the case where an exact solution cannot be found or only a general picture is needed, eigenvalues and eigenvectors of the characteristic polynomial of a system of differential equations can provide the behavior around the equilibrium solutions. This section provides a few examples along with phase diagrams to provide a visual of our findings.

2.2 Real Eigenvalues

2.2.1 Saddle

Given the system of equations

\[ \begin{align*}
\dot{x} &= 3x + 5y, \\
\dot{y} &= x + y.
\end{align*} \tag{2.1, 2.2} \]

This can be written in matrix form as,

\[ \begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
3 & 5 \\
1 & 1
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}. \]

If we let \( \vec{y}(t) = \exp(\lambda t) \vec{v} \), then we generate the eigenvalue and eigenvector equation,

\[ \lambda \exp(\lambda t) \vec{v} = A \exp(\lambda t) \vec{v}. \]

Canceling the exponentials results in the familiar eigenvalue program,

\[ \begin{bmatrix}
3 - \lambda & 5 \\
1 & 1 - \lambda
\end{bmatrix} \vec{v} = 0. \]
The characteristic polynomial is $\lambda^2 - 4\lambda - 2$. Hence the eigenvalues and associated eigenvectors are simply,

$$\lambda_{1,2} = 2 \pm \sqrt{6}, \quad \vec{v}_{1,2} = \begin{bmatrix} -5 \\ 1 \mp \sqrt{6} \end{bmatrix}.$$ 

Therefore the general solution is

$$\vec{y}(t) = c_1 \exp(\lambda_1 t) \vec{v}_1 + c_2 \exp(\lambda_2 t) \vec{v}_2,$$

(2.3)

where the coefficients $c_1$ and $c_2$ are set with an initial condition for $x$ and $y$. Notice $\lambda_1 > 0$ while $\lambda_2 < 0$. Therefore

$$\lim_{t \to \infty} c_2 \exp(\lambda_2 t) \vec{v}_2 = 0.$$ 

(2.4)

This means that $\vec{y}(t)$ approaches $c_1 \exp(\lambda_1 t) \vec{v}_1$ and that the equilibrium solution represents a saddle. The vector $\vec{v}_1$ represents the direction of exponential growth. At the equilibrium position this vector is tangent to the unstable manifold. Likewise, the eigenvector $\vec{v}_2$ represents the direction of exponential decay and at the equilibrium position it is tangent to the stable manifold. The phase portrait for various initial conditions is shown in figure 2.1.

### 2.2.2 Source

Now we will look at an example that has two positive eigenvalues. Begin with the following system of equations.

$$\begin{align*}
\dot{x} & = x + y, \\
\dot{y} & = 2y.
\end{align*}$$

(2.5) (2.6)

These can be written in matrix form as follows.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$ 

The eigenvalues can be calculated using the determinant of

$$\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} - I\lambda.$$

$$\det \begin{bmatrix} 1 - \lambda & 1 \\ 0 & 2 - \lambda \end{bmatrix} = \lambda^2 - 3\lambda + 2.$$ 

Then the eigenvalues are $\lambda = 1, 2$ with eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. 

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Figure 2.1: The phase portrait is shown for the system in Eqs. (2.1)-(2.2). Notice that it is easy to see each eigenvectors direction. For any initial condition we see the contributions in the eigenvectors. The initial contribution decays in the stable eigenvector $\vec{v}_2$ direction while the contribution of $\vec{v}_2$ begins to grow. Hence, any perturbation from the equilibrium solution that contains a $\vec{v}_1$ contribution will grow indefinitely. Therefore, we classify the equilibrium solution at $(0,0)$ as unstable.

2.2.3 Sink

The next example begins with the following system of equations

\[
\begin{align*}
\dot{x} &= -x, \\
\dot{y} &= -2y.
\end{align*}
\]

(2.7) (2.8)

Then converting the above into matrix form we get the following

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} =
\begin{bmatrix}
-1 & 0 \\
0 & -2
\end{bmatrix}
\begin{bmatrix} x \\ y \end{bmatrix}.
\]
Now let $\vec{y}(t) = \exp(\lambda t)\vec{v}$. Then we will substitute into $\lambda \vec{y} = A\vec{y}$ to get the following:

$$\lambda \exp(\lambda t)\vec{v} = A \exp(\lambda t)\vec{v},$$

$$\lambda \vec{v} = A\vec{v},$$

$$\begin{bmatrix} -1 - \lambda & 0 \\ 0 & -2 - \lambda \end{bmatrix} \vec{v} = 0.$$ 

Next we will find the characteristic polynomial (find the determinant)

$$(-1 - \lambda)(-2 - \lambda) = 0.$$ 

So, the eigenvalues are $\lambda = -1, -2$. With the eigenvalues $\lambda = -1$ and $-2$, we can compute the eigenvectors,

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
Remember that $A\vec{v}_i = \lambda_i \vec{v}_i$.

Figure 2.3: The phase portrait is shown for the system in Eqs. (2.7)-(2.8). Notice that it is easy to see each eigenvectors direction. For any initial condition we see the contributions in the eigenvectors. The initial contribution decays in the stable in the direction of both eigenvectors. Therefore, we classify the equilibrium solution at $(0, 0)$ as stable.

2.2.4 Repeated Eigenvalues

There is another scenario that falls under the two real eigenvalue case. In the instance when there are repeated eigenvalues, the equilibrium solution will either be a stable or unstable node. A saddle will not form due to the sign of the eigenvalues. Even though there is only one distinct eigenvalue, there can potentially be two distinct eigenvectors. This can result in not having a complete set of linearly independent eigenvectors. The repeated eigenvalues create an interesting phase portrait where either all vectors are drawn into the eigenvector or all vectors are moving directly away from the eigenvector. To solidify this idea, we will go through two examples.
Consider the linear system of equations
\begin{align}
\dot{x} &= x + y, \tag{2.9} \\
\dot{y} &= y. \tag{2.10}
\end{align}
Now it can be shown that the exact solution is,
\begin{align*}
x(t) &= (y_0 t + x_0) e^t, \\
y(t) &= y_0 e^t,
\end{align*}
where $x_0$ and $y_0$ are the initial conditions of $x$ and $y$, respectively. Clearly, for any small perturbation away from the equilibrium solution at the origin we see that $x(t)$ and $y(t)$ grow without bound. Therefore, we see that the equilibrium solution is unstable. The sole eigenvalue of the underlying matrix is $\lambda = 1$ and has only one eigenvector in the direction of $(1, 0)$. This means the solutions exponentially grows in this direction. However, the solution still grows in the y-direction. Although, it has a tendency to grow faster in the $x$ direction since the coefficient grows linearly in $t$. This gives the appearance of rotation without complex eigenvalues, but again the sole eigenvalue is $\lambda = 1$.

In the present situation we can create a *generalized eigenvector* that satisfies $(A - \lambda I) \vec{v}_1 = 0$. In essence the generalized eigenvector creates the other direction that we are missing, namely $(0, 1)$. Therefore, these degenerate and generalized eigenvectors gives us the visual aid to know the vectors are being pulled more weakly in this direction.

To understand the behavior of a dynamical system with a repeated eigenvalue that only has one eigenvector we consider,
\begin{align}
\dot{x} &= (1 + \epsilon)x + y, \tag{2.11} \\
\dot{y} &= y. \tag{2.12}
\end{align}
where $\epsilon \ll 1$. This system represents a small perturbation to our original problem and has eigenvalues of,
\begin{align*}
\lambda_1 &= 1, \\
\lambda_2 &= 1 + \epsilon,
\end{align*}
with corresponding eigenvectors
\begin{align*}
\vec{v}_1 &= \begin{bmatrix} 1 \\ -\epsilon \end{bmatrix}, \\
\vec{v}_2 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\end{align*}
Notice since both eigenvalues are positive, the equilibrium solution will be unstable. Also, as $\epsilon$ approaches 0, the behavior and phase portrait will begin to look similar to the first example in this subsection. As $\epsilon$ approaches 0, the eigenvalues and eigenvectors will also begin to look like the previous example’s eigenvalue and eigenvector.
Figure 2.4: The phase portrait is shown for the system in Eqs. (2.9)-(2.10). Notice that there is just one eigenvector direction. The equilibrium solution at (0, 0) is classified as unstable. This is expected since the sole eigenvalue is positive.

2.3 Complex Eigenvalues

2.3.1 Unstable Spiral

Given the following system of equations

\[
\begin{align*}
\dot{x} &= 3x - 5y, \\
\dot{y} &= x + y.
\end{align*}
\]  

(2.13) \hspace{1cm} (2.14)

The above system can then be written in matrix form as,

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
3 & -5 \\
1 & 1
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}.
\]
Then we can find the characteristic polynomial,

\[
\begin{bmatrix}
3 & -5 \\
1 & 1
\end{bmatrix} - \lambda I = \begin{bmatrix}
3 - \lambda & -5 \\
1 & 1 - \lambda
\end{bmatrix},
\tag{2.15}
\]

\[
\det \begin{bmatrix}
3 - \lambda & -5 \\
1 & 1 - \lambda
\end{bmatrix} = \lambda^2 - 4\lambda + 8. \tag{2.16}
\]

Equation (2.16) is the characteristic polynomial. Then the eigenvalues and eigenvectors are as follows,

\[
\lambda = 2 \pm 2i,
\]

\[
\vec{v} = \begin{bmatrix}
5 \\
1 \mp 2i
\end{bmatrix}.
\]

Now remember,

\[
(A - \lambda I)\vec{v} = 0.
\]

Our choice of \(\lambda\) ensures that the matrix \(A - \lambda I\) is singular, that is, not invertible. Therefore, we will have a nontrivial null space. To get a nontrivial null space we try to determine the null vectors that satisfy the above equation for each \(\lambda\). Consider, \(\lambda = 2 + 2i\). This generates,

\[
(A - (2 + 2i)I) = \begin{bmatrix}
3 - (2 + 2i) & -5 \\
1 & 1 - (2 + 2i)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 - 2i & -5 \\
1 & -1 - 2i
\end{bmatrix}.
\]

Notice that if we choose

\[
\vec{v}_1 = \begin{bmatrix}
5 \\
(1 - 2i)
\end{bmatrix}.
\]

Now for \(\lambda = 2 - 2i\) the eigenvector is

\[
\vec{v}_2 = \begin{bmatrix}
5 \\
(1 + 2i)
\end{bmatrix}.
\]
The general solution is

\[ y(t) = c_1 \exp((2 + 2i)t) \vec{v}_1 + c_2 \exp((2 - 2i)t) \vec{v}_2 \]  
(2.17)

\[ = \exp(2t) (c_1 \exp(2it) \vec{v}_1 + c_2 \exp(-2it) \vec{v}_2) \]  
(2.18)

\[ = \exp(2t) (c_1(\cos(2t) + i \sin(2t)) \vec{v}_1 + c_2(\cos(2t) - i \sin(2t)) \vec{v}_2). \]  
(2.19)

Euler’s Formula was used to achieve (2.19). Then collecting like terms we get the following:

\[ y(t) = \exp(2t) (c_1 \vec{v}_1 + c_2 \vec{v}_2) \cos(2t) + (ic_1 \vec{v}_1 - ic_2 \vec{v}_2) \sin(2t)). \]  
(2.20)

Let us examine the coefficients for \(\cos(2t)\) and \(\sin(2t)\).

\[ c_1 \vec{v}_1 + c_2 \vec{v}_2 = c_1 \begin{bmatrix} 5 \\ (1 - 2i) \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ (1 + 2i) \end{bmatrix} = \begin{bmatrix} 5(c_1 + c_2) \\ (c_1 + c_2) + 2i(c_2 - c_1) \end{bmatrix}. \]

Likewise,

\[ ic_1 \vec{v}_1 - ic_2 \vec{v}_2 = ic_1 \begin{bmatrix} 5 \\ (1 - 2i) \end{bmatrix} - ic_2 \begin{bmatrix} 5 \\ (1 + 2i) \end{bmatrix} = \begin{bmatrix} -5i(c_2 - c_1) \\ ic_1(1 - 2i) - ic_2(1 + 2i) \end{bmatrix} = \begin{bmatrix} -5i(c_2 - c_1) \\ 2(c_1 + c_2) - i(c_2 - c_1) \end{bmatrix}. \]

Let \(d_1 = c_1 + c_2\) and \(d_2 = i(c_2 - c_1)\). Therefore we have,

\[ c_1 \vec{v}_1 + c_2 \vec{v}_2 = \begin{bmatrix} 5d_1 \\ d_1 + 2d_2 \end{bmatrix}, \]

\[ ic_1 \vec{v}_1 - ic_2 \vec{v}_2 = \begin{bmatrix} -5d_2 \\ 2d_1 - d_2 \end{bmatrix}. \]
Notice that,
\[
\begin{bmatrix}
5d_1 \\
d_1 + 2d_2
\end{bmatrix} = d_1 \begin{bmatrix} 5 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix}.
\]
Likewise,
\[
\begin{bmatrix}
-5d_2 \\
2d_1 - d_2
\end{bmatrix} = -d_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix} + d_1 \begin{bmatrix} 0 \\ 2 \end{bmatrix}.
\]

Figure 2.5: The phase portrait is shown for the system in Eqs. (2.13)-(2.14). Since the eigenvalues have a nonzero imaginary component, the phase portrait has a rotational or spiral aspect. Whether the spiral rotates into the equilibrium solution or away is based on the real component of the eigenvalue. In this instance, the real components are positive, thus the spiral rotates away from the equilibrium solution. Therefore, we classify the equilibrium solution at (0, 0) as unstable. The direction of the rotation is determined by the original system (2.13)-(2.14). If we find the vector associated with (1, 1), we get (−2, 2). The vector associated with (−4, −2) is (−2, −6). So the rotation is moving away from the equilibrium solution in a counterclockwise direction.
2.3.2 Center

Beginning with the following system of equations

\begin{align}
\dot{x} &= y, \\
\dot{y} &= -x.
\end{align}

(2.21) (2.22)

Then the above system can put into matrix form as follows

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}.
\]

Next, we will find the characteristic polynomial and then the eigenvalues and eigenvectors. Consider the determinant of \( A - \lambda I \), namely,

\[
\det\begin{bmatrix}
-\lambda & 1 \\
-1 & -\lambda
\end{bmatrix} = \lambda^2 + 1,
\]

(2.23)

Therefore the eigenvalues and eigenvectors are,

\[
\lambda = \pm i,
\]

(2.24)

\[
\vec{v} = \begin{bmatrix} 1 \\ \pm i \end{bmatrix}.
\]

(2.25)

Equation (2.23) is the characteristic polynomial, (2.24) are the eigenvalues and (2.25) are the corresponding eigenvectors. The general solution is as follows:

\[
\vec{y}(t) = c_1 \exp(\lambda_1 t)\vec{v}_1 + c_2 \exp(\lambda_2 t)\vec{v}_2
\]

\[
= c_1 \exp(it)\vec{v}_1 + c_2 \exp(-it)\vec{v}_2
\]

\[
= c_1(\cos(t) + i \sin(t))\vec{v}_1 + c_2(\cos(t) - \sin(t))\vec{v}_2
\]

\[
= c_1(\cos(t) + i \sin(t))\vec{v}_1 + c_2(\cos(t) - i \sin(t))\vec{v}_2
\]

\[
= \cos(t)[c_1 \vec{v}_1 + c_2 \vec{v}_2] + \sin(t)[c_1 \vec{v}_1 i - c_2 \vec{v}_2 i].
\]

2.3.3 Stable Spiral

Begin with the following system of equations

\begin{align}
\dot{x} &= -6x - 2y, \\
\dot{y} &= 4x - 6y.
\end{align}

(2.26) (2.27)
Figure 2.6: The phase portrait is shown for the system in Eqs. (2.21)-(2.22). Since the eigenvalues are purely imaginary, a spiral effect occurs but since the real component is zero nothing is pulling the vectors into or away from the equilibrium solution. Therefore, we classify the equilibrium solution at (0, 0) as a center. The direction of the rotation is determined by the original system (2.21)-(2.22). If we find the vector associated with (1, 1), we get (1, -1). The vector associated with (-4, -2) is (-2, 4). So the rotation is in a clockwise direction.

This is cast into matrix form

$$\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
-6 & -2 \\
4 & -6
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}.$$  \hspace{1cm} (2.28)

The characteristic polynomial of the matrix is

$$\det \begin{bmatrix}
-6 - \lambda & -2 \\
4 & -6 - \lambda
\end{bmatrix} = \lambda^2 + 12\lambda + 44.$$  \hspace{1cm} (2.29)
Consequently the eigenvalues and eigenvectors are,

\[ \lambda = -6 \pm 2i\sqrt{2}, \quad (2.30) \]
\[ \vec{v} = \begin{bmatrix} 1 \\ \pm i\sqrt{2} \end{bmatrix}. \quad (2.31) \]

Now the general solution is as follows:

\[ \vec{y}(t) = c_1 \exp(\lambda_1 t)\vec{v}_1 + c_2 \exp(\lambda_2 t)\vec{v}_2 \]
\[ = c_1 \exp((-6 + 2i\sqrt{2})t)\vec{v}_1 + c_2 \exp((-6 - 2i\sqrt{2})t)\vec{v}_2 \]
\[ = \exp(-6) \left( c_1 \exp(2i\sqrt{2}t)\vec{v}_1 + c_2 \exp(-2i\sqrt{2}t)\vec{v}_2 \right) \]
\[ = \exp(-6) \left( c_1 \vec{v}_1 \cos(2\sqrt{2}t) + i \sin(2\sqrt{2}t) + c_2 \vec{v}_2 \cos(-2\sqrt{2}t) - i \sin(-2\sqrt{2}t) \right) \]
\[ = \exp(-6) \left( \cos(2\sqrt{2}t)(c_1 \vec{v}_1 + c_2 \vec{v}_2) + i \sin(-2\sqrt{2}t)(c_1 \vec{v}_1 - c_2 \vec{v}_2) \right). \]

Here the coefficients of \( \cos(t) \) and \( \sin(t) \) are as follows:

\[ (c_1 \vec{v}_1 + c_2 \vec{v}_2), \]
\[ (c_1 \vec{v}_1 - c_2 \vec{v}_2). \]

As a reminder, \( c_1 \) and \( c_2 \) are complex conjugates of each other and \( \vec{v}_1 \) and \( \vec{v}_2 \) are complex conjugates of each other.

### 2.4 Conclusion

This section continued discussing differential equations, but used techniques to determine the behavior of a differential equation around its equilibrium solutions. In some cases eigenvalues and eigenvectors of differential equations can be useful and provide insight into the behavior around the equilibrium solutions. We observe the real part of the eigenvalues completely determines the long-term dynamics of a small perturbation around the equilibrium solution. In particular, the eigenvalues fall into two categories:

1. Eigenvalues are purely real there are three scenarios. If the eigenvalues are both positive this means that the equilibrium solution is unstable. If they are both negative this means the equilibrium solution is stable and all initial conditions will return to the equilibrium solution. If they are opposite signs, then in
Figure 2.7: The phase portrait is shown for the system in Eqs. (2.26)-(2.27). Since the eigenvalues have a nonzero imaginary component, the phase portrait has a rotational or spiral aspect. Whether the spiral rotates into the equilibrium solution or away is based on the real component of the eigenvalue. In this instance, the real components are negative, thus the spiral rotates towards the equilibrium solution. Therefore, we classify the equilibrium solution at \((0, 0)\) as stable. The direction of the rotation is determined by the original system (2.26)-(2.27). If we find the vector associated with \((1, 1)\), we get \((-8, -2)\). The vector associated with \((-4, -2)\) is \((28, -4)\). So the rotation is moving towards the equilibrium solution in a counterclockwise direction.

one eigenvalue direction the phase portrait will be pulled into the equilibrium solution while in the other eigenvalue direction the vectors will be moving away from the equilibrium solution.

2. Eigenvalues are complex there are also three scenarios. Since the eigenvalues are complex conjugates the real part is the same for both. Hence the stability is determined by the sign of the real portion of the eigenvalue. If the real portion is positive, then the equilibrium solution is unstable. If the real component is negative, then the equilibrium solution is stable. If the real component is 0, then the equilibrium solution is unstable and the phase portrait will create a series of rings because there is no real component pushing or pulling away from
the equilibrium solution.

This section provided a few examples along with phase diagrams to give the reader a visual of our findings. The following section will tie the use of eigenvalues and eigenvectors to perturbation theory to analyze behavior around equilibrium solutions.
3 LINEAR ANALYSIS OF NONLINEAR EQUATIONS

3.1 Introduction

This section will tie the use of linear analysis to analyze behavior around equilibrium solutions. Nonlinear analysis has been used throughout the development of applied mathematics. This particular technique can be implemented in numerous areas of science that utilize mathematical modeling [9]. Some of the same procedures will be followed with the addition of perturbing the equilibrium solutions with a small function, meaning that the function is small in magnitude. Eigenvalues will also be calculated in the section as they have been previously, two methods are discussed to solve for the eigenvalues.

3.2 The Nonlinear DE and Equilibrium Solutions

Let

\[ y^{''}(x) + ay(x) - b^{3}y^{3}(x) = 0. \] (3.1)

Then \( y^{''}(x) = -ay(x) + b^{3}y^{3}(x) \). First, we need to find the equilibrium solutions. The work is as follows:

\[
\begin{align*}
-ay(x) + b^{3}y^{3}(x) &= 0 \\
y(x) \left[-a + b^{3}y^{2}(x)\right] &= 0, \\
y(x) &= 0, \pm \sqrt{\frac{a}{b^{3}}}. 
\end{align*}
\]

3.3 Perturbation Technique

Then let these equilibrium solutions be \( \alpha_{n} \)'s to get the following \( y_{n} \)'s:

\[
\begin{align*}
y_{1}(x) &= 0 + g_{1}(x), \\
y_{2}(x) &= \sqrt{\frac{a}{b^{3}}} + g_{2}(x), \\
y_{3}(x) &= -\sqrt{\frac{a}{b^{3}}} + g_{3}(x). 
\end{align*}
\]

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Where \( g_n(x) \ll 1 \) is a perturbation on the equilibrium solutions for all \( n \). Then each of the \( y_n \)'s will be substituted back into (3.1). The following represents \( y_1 \) substituted back into (3.1).

\[
\begin{align*}
y''(x) + ay(x) - b^3y^3(x) &= 0 \\
g_1''(x) + ag_1(x) - b^3(g_1(x))^3 &= 0 \\
g_1''(x) + ag_1(x) + O(g_1(x)^2) &= 0.
\end{align*}
\]  

(3.2) The following represents \( y_2 \) substituted back into (3.1).

\[
\begin{align*}
\left( \sqrt{\frac{a}{b^3} + g_2(x)} \right)'' + a \left( \sqrt{\frac{a}{b^3} + g_2(x)} \right) - b^3 \left( \sqrt{\frac{a}{b^3} + g_2(x)} \right)^3 &= 0 \\
g_2''(x) + a\sqrt{\frac{a}{b^3} + ag_2(x)} - b^3 \left( \sqrt{\frac{a}{b^3} + g_2(x)} \right)^3 &= 0.
\end{align*}
\]

After some algebraic simplification we arrive at the following equation.

\[
g_2''(x) - 2ag_2(x) + O(g_2^2(x)) = 0. \quad (3.3)
\]

Lastly, we will plug \( y_3(x) = -\sqrt{\frac{a}{b^3} + g_3(x)} \) into the original equation (3.1),

\[
\begin{align*}
\left( -\sqrt{\frac{a}{b^3} + g_3(x)} \right)'' + a \left( -\sqrt{\frac{a}{b^3} + g_3(x)} \right) - b^3 \left( -\sqrt{\frac{a}{b^3} + g_3(x)} \right)^3 &= 0 \\
y''(x) + ay(x) - b^3y^3(x) &= 0
\end{align*}
\]

As previously, we obtain,

\[
g_3''(x) - 2ag_3(x) + O(g_3^2(x)) = 0. \quad (3.4)
\]

Notice that (3.3) and (3.4) are the same equation. Now we have the following three equations:

\[
\begin{align*}
g_1''(x) + ag_1(x) + O(g_1^2(x)) &= 0, \\
g_2''(x) - 2ag_2(x) + O(g_2^2(x)) &= 0, \\
g_3''(x) - 2ag_3(x) + O(g_3^2(x)) &= 0.
\end{align*}
\]

3.4 Direct Technique: First Equilibrium Solution

Using the direct method we let \( g_1(x) = Ce^{\lambda_1x} \) and plug this back into (3.2) while disregarding the powers of \( g_1(x) \) that are greater than 1. This will produce the
following characteristic polynomial. The work is as follows,

\[ g''_1(x) + ag_1(x) = 0 \]
\[ (Ce^{\lambda_1 x})'' + a(Ce^{\lambda_1 x}) = 0 \]
\[ \lambda^2 Ce^{\lambda_1 x} + aCe^{\lambda_1 x} = 0 \]
\[ \lambda^2 + a = 0. \]

Then we will use the quadratic formula to solve for \( \lambda_1 \), the eigenvalues,

\[ \lambda_1 = \frac{-0 \pm \sqrt{0^2 - 4(1)(a)}}{2(1)} \]
\[ = \frac{\pm \sqrt{-4a}}{2} \]
\[ = \pm \sqrt{-a}. \]

If \( a < 0 \), then we have two real eigenvalues, one positive and one negative (saddle). If \( a > 0 \), then we have two solely imaginary eigenvalues (center).

### 3.5 Matrix Method: First Equilibrium Solution

Next we will use the matrix method to find the characteristic polynomial (this is simply another method to arrive at the same conclusion). Let \( u = g_1(x) \) and \( v = g'_1(x) \). Then \( u' = g'_1(x) \) and \( v' = g''_1(x) \). The following is a matrix representation of the previous statements:

\[
\begin{bmatrix}
  u' \\
  v'
\end{bmatrix} =
\begin{bmatrix}
  0 & 1 \\
  -a & 0
\end{bmatrix}
\begin{bmatrix}
  u \\
  v
\end{bmatrix}.
\]

The characteristic equation is,

\[
\det \begin{bmatrix}
  -\lambda_1 & 1 \\
  -a & -\lambda_1
\end{bmatrix} = \lambda^2 + a.
\]

We have arrived at the same characteristic polynomial using the matrix method. Thus the eigenvalues are the same \( \lambda_1 = \pm \sqrt{-a} \).
3.6 Direct Method: Second and Third Equilibrium Solutions

We will repeat the same steps for (3.3) and (3.4), but again these are identical equations. Begin with the following equation since we are disregarding powers of \( g_2(x) \) that are greater than 1 and then let \( g_2(x) = C e^{\lambda_2 x} \),

\[
\begin{align*}
  g_2''(x) - 2a g_2(x) + O(g_2^2(x)) &= 0 \\
  (C e^{\lambda_2 x})'' - 2a C e^{\lambda_2 x} &= 0 \\
  \lambda_2^2 C e^{\lambda_2 x} - 2a C e^{\lambda_2 x} &= 0 \\
  \lambda_2^2 - 2a &= 0. \\
\end{align*}
\]

Where (3.5) is the characteristic polynomial. Next we will compute the eigenvalues, \( \lambda_2 = 0 \pm \sqrt{0^2 - 4(1)(-2a)} \)

\[
\begin{align*}
  &= 0 \pm \frac{\sqrt{8a}}{2} \\
  &= \pm \frac{2\sqrt{2a}}{2} \\
  &= \pm \sqrt{2a}.
\end{align*}
\]

If \( a < 0 \), then the eigenvalues will be purely imaginary and the result is a center. If \( a > 0 \), then the eigenvalues will be real and opposite signs, resulting in a saddle.

3.7 Matrix Method: Second and Third Equilibrium Solutions

Next we will use the matrix method to find the characteristic polynomial. Let \( v = g_3' \) and \( v' = g'' = 2ag \). Then

\[
\begin{bmatrix} v' \\ g' \end{bmatrix} = \begin{bmatrix} 0 & 2a \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ g \end{bmatrix}.
\]

The characteristic polynomial is \( \lambda_2^2 - 2a \) and is used to find the eigenvalues. Using the quadratic formula the eigenvalues are \( \lambda_2 = \pm \sqrt{2a} \). The corresponding eigenvectors are \( \begin{bmatrix} \sqrt{2a} \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} -\sqrt{2a} \\ 1 \end{bmatrix} \). If \( a > 0 \), then the eigenvalues are real and opposite
3.8 The Big Picture

In order to get a visual, let’s assume the \( a, b = 1. \) Then \( y''(x) + ay(x) - b^2y(x)^3 = 0 \) becomes \( y''(x) + y(x) - y(x)^3 = 0. \) Then from the work above we have concluded that there is one center and two saddles around the equilibrium solutions. The resulting phase portrait is shown in figure 3.1.

![Phase Portrait](image)

**Figure 3.1:** The phase portrait shows the center around the equilibrium solution of \( y(x) = 0 \) and the saddles are around \( y(x) = \pm 1. \)

3.9 Conclusion

This section utilized eigenvalues and eigenvectors along with perturbation theory to analyze behavior around equilibrium solutions. This provided a brief overview of a
linear analysis technique for nonlinear problems. Although an exact solution was not calculated, we have a general idea of the behavior of the original differential equation around its equilibrium solutions. The following section provides another technique called Magnus expansion that develops an approximation for nonlinear differential equations.
4 INTRODUCTION TO MAGNUS EXPANSION

4.1 Introduction

This section will provide a brief overview of Magnus expansions which was pioneered by Magnus in 1954 [12] to solve initial value problems of the form $Y'(t) = A(t)Y(t)$ where $Y$ and $A$ are $N \times N$ matrices. For a thorough review of the developments of the Magnus expansion technique and theory as well as its applications in mathematical physics the interested reader should read [17]. The development of the Magnus technique for nonlinear initial value problems is relatively new, developed in 2006 by Iserles and Casas [4]. Only recently has this technique been the focus of developing higher order structure preserving numerical approximations [2, 5, 6]. The essential idea is to approximate the evolution operator of the initial value problem. This in turn, preserves many of the important solution characteristics, in particular the finite term approximations lie within the same Lie group [17]. This is an important property that holds because Lie groups maintain group properties and require that the group operations are differentiable. This latter fact is an incredible byproduct of the Magnus expansion. In a sense it means that any finite term Magnus expansion approximate solution lives in the same solution space of the differential equation. Therefore, if the true solution has various properties, such as invariant quantities, then the approximate solution will as well [12, 17]. This is a potentially important characteristic that Magnus expansion has over perturbation theory. In some instances it is important to preserve certain characteristics and properties when approximating a solution. In other cases, it may be more important to find any solution regardless of preservation of properties. Hence, the evolution operator itself is approximated [13].

Here, we shall go through two examples that walk through the procedure of using the expansions to find an approximate solution to a nonlinear differential equation. In these cases, we have exact solutions and can compare the approximations to the exact solutions to see how “good” the approximations are. Several visuals will be provided to depict the exact solutions against the approximations. These equations can become complex while the graphs provide a succinct picture of the results found.
4.2 Magnus Expansion: An Introduction

This section will provide a brief overview of the Magnus expansion technique as well as a few examples of how to use it. In this section both examples begin with differential equations of the form:

\[ y'(t) = A(t, y(t))y(t), \]  

with initial conditions \( y(0) = y_0 \). Here, we shall restrict ourselves to the case \( y(t) : \mathbb{R}^+ \mapsto \mathbb{R}^N \) and \( A : \mathbb{R}^+ \times \mathbb{R}^N \mapsto \mathbb{R}^{N \times N} \). Using the ansatz, \( y(t) = \exp(\Omega(t))y_0 \) we arrive\(^1\) at the differential equation for \( \Omega(t) \),

\[ \Omega' = \text{dexp}_{\Omega}^{-1}(A(t, \exp(\Omega)y_0)), \quad \Omega(0) = \mathcal{O}, \]  

where \( \text{dexp}_{\Omega}^{-1} \) is a function that was developed for Magnus expansions and is further defined in [3] and

\[ \text{dexp}_{\Omega}^{-1}(C) = \sum_{k=0}^{\infty} \frac{B_k}{k!} \text{ad}^k_{\Omega} C, \]

\( B_k \) are the Bernoulli numbers and \( \text{ad}^m \) is the commutator defined recursively as

\[ \text{ad}^0_{\Omega} A = A, \quad \text{ad}^{m+1}_{\Omega} A = [\Omega, \text{ad}^m_{\Omega} A]. \]

Although we do not directly use all of the formulations above, they are an integral part of the construction of the formulas that we will use in this section. This is very similar in technique to letting \( y = \exp(\lambda t) \) as we have done in the past.

Let us look at a linear example with the equations listed above. Let \( y' = 2y \). Then \( A(t, y) = 2 \) and \( y = \exp(2t)y_0 \) where \( \Omega(t) = 2t \) and \( \Omega'(t) = 2 \). Using (4.2),

\[ \Omega' = \text{dexp}_{\Omega}^{-1}(A(t, \exp(\Omega)y_0)) \]
\[ = \text{dexp}_{\Omega}^{-1}(2) \]
\[ = \sum_{k=0}^{\infty} \frac{B_k}{k!} \text{ad}^k_{\Omega} 2 \]
\[ = \frac{B_0}{0!} \text{ad}^0_{\Omega} 2 + \frac{B_1}{1!} \text{ad}^1_{\Omega} 2 + \cdots \]
\[ = \text{ad}^0_{\Omega} 2 \]
\[ = 2 \quad \text{by definition of } \text{ad}^0_{\Omega} A = A. \]

The result is \( \Omega' = 2 \), just as we expected.

\(^1\)After some serious work.
The essential idea of a Magnus expansion is to find solutions to differential equations of the form \( \exp(\Omega(t))y_0 \). Hence, we wish to determine \( \Omega(t) \). While there are other solution techniques, this solution technique has the additional advantage of staying within the correct solution space and therefore maintains the same symmetries as the real solution. Hence, approximations using finite term Magnus expansions will remain in the correct Lie algebra. Unfortunately, this does not mean that our series expansion will converge faster than say a regular power series solution or some other technique. Therefore, we may be sacrificing preserving symmetries over creating approximate solutions with only a few terms. The idea of applying this to nonlinear differential equations is a relatively new idea and was first developed in 2005 by Casas and Iserles. Casas followed by considering the sufficient criteria for convergence of the Magnus expansion [3]. In this latter paper Casas provided the criteria to ensure that:

1. The differential equation has a solution of the form \( \exp(\Omega(t)) \)
2. The exponent \( \Omega(t) \) lies in the same Lie Algebra as that of the operator \( A(t, y) \)
3. The exponent \( \Omega(t) \) is continuously differentiable and satisfies the nonlinear differential equation (4.2)

More recently, in the last 5 years there has been development of reliable numerical approximations based on the Magnus expansions. Magnus expansions can be useful in numerical approximations since it creates sympletic integrators, for which preserve many of the important properties of the fundamental solution [2]. To date, the applications of Magnus expansions have mainly been focused on nonlinear differential equations in mathematical physics [17].

We will represent the exponent in a series and seek to determine explicit formulations for each term,

\[
\Omega(t) = \sum_{k=1}^{\infty} \Omega_k(t) = \lim_{m \to \infty} \Omega[m](t). \tag{4.3}
\]

For the purposes of this thesis, we will be looking at the first several terms of the summation in equation (4.3) to approximate \( y(t) \). The equations for the first two terms of the series are:

\[
\Omega_1(t) = \int_{0}^{t} A(s) \, ds, \tag{4.4}
\]
\[
\Omega_2(t) = \frac{1}{2} \int_{0}^{t} \int_{0}^{s} [A(s), A(r)] \, dr \, ds, \tag{4.5}
\]

where \([A, B] = AB - BA\).
We note, that if $A$ and $B$ are simply scalars, then $[A, B] = 0$. In [4] a compact formulation for the finite term approximation to $\Omega(t)$ is given:

$$\Omega^0 = \mathcal{O},$$

$$\Omega^1 = \int_0^t A(s, y_0) \, ds,$$  \hspace{1cm} (4.6)

$$\Omega^2 = \int_0^t A(s, \exp(\Omega^1(s)) y_0) \, ds,$$  \hspace{1cm} (4.7)

$$\Omega^m = \sum_{k=0}^{m-2} \frac{B_k}{k!} \int_0^t \text{ad}^{k}_{\Omega^{m-1}(s)} A(s, \exp(\Omega^{m-1}(s)) y_0) \, ds, \quad m \geq 2. \quad (4.8)$$

4.3 A Nonlinear Differential Equation

First consider the following nonlinear differential equation with its exact solution:

$$y' = ty^2, \quad y(0) = 1, \quad (4.10)$$

$$y(t) = \frac{2}{2 - t^2}, \quad t < \sqrt{2}. \quad (4.11)$$

The exact solution can be found by the following steps:

$$\frac{dy}{dt} = yt^2$$

$$\frac{1}{y^2} = t dt$$

$$\int \frac{1}{y^2} = \int t \, dt$$

$$\frac{-1}{y} = \frac{t^2}{2} + C_1$$

$$y = -\frac{1}{\frac{t^2}{2} + C_2},$$

where $C_1$ and $C_2$ are constants. Upon using the initial condition, the exact solution can be determined,

$$y(0) = 1$$

$$1 = -\frac{1}{\frac{0^2}{2} + C_2}$$

$$C_2 = -2$$

$$y(t) = \frac{2}{2 - t^2}.$$
Now that we have the exact solution for $y(t)$, we can see that $t \neq \sqrt{2}$ and since the initial condition is $y(0) = 1$, $t \in [0, \sqrt{2})$. (4.10) is in the form $y' = A(t, y)y$ such that $A(t, y) = ty$.

Our goal is to create the second order Magnus expansion. This is accomplished by:

1. Determine $\Omega^{[2]}$ which is the sum of $\Omega_1$ and $\Omega_2$.

2. Construct the exponential $\exp(\Omega^{[2]}(t))y_0$ which determines an approximation to $y(t)$ to second order.

In order to approximate a solution to (4.10), we will use the following second order Magnus expansion:

$$y(t) \approx \exp(\Omega_0(t) + \Omega_1(t) + \Omega_2(t)).$$

In order to use the Magnus expansion above, we will need to calculate the three components of the exponent, equations (4.7), (4.8) and (4.9) from above. Their formulas again are as follows with the general formula for $m \geq 2$:

$$\Omega^{[0]} = \mathcal{O},$$

$$\Omega^{[1]} = \int_0^t A(s, y_0) \, ds,$$  

$$\Omega^{[2]} = \int_0^t A(s, \exp(\Omega^{[1]}(s))y_0) \, ds,$$

$$\Omega^{[m]} = \sum_{k=0}^{m-2} \frac{B_k}{k!} \int_0^t \text{ad}_{\Omega^{[m-1]}(s)}^k A(s, \exp(\Omega^{[m-1]}(s))y_0) \, ds, \quad m \geq 2. \tag{4.15}$$

In order to use the above formulas, let’s define the following:

$$\text{ad}_{\Omega}^0 A = A, \tag{4.16}$$

$$\text{ad}_{\Omega}^{m+1} A = [\Omega, \text{ad}_{\Omega}^m A]. \tag{4.17}$$

Along with the Bernoulli numbers, $B_k$, that are a defined set of numbers. For our purposes we need the following:

$$B_0 = 1,$$

$$B_1 = \frac{1}{2}.$$
Also, define the following:

\[
\Omega^m(t) = \sum_{k=0}^{m} \Omega_m(t), \quad (4.18)
\]

\[
\Omega^3(t) = \Omega_1(t) + \Omega_2(t) + \Omega_3(t), \quad (4.19)
\]

\[
\Omega_1(t) = \int_0^t A(s)ds, \quad (4.20)
\]

\[
\Omega_2(t) = \frac{1}{2} \int_0^t \int_0^s [A(s), A(r)]drds. \quad (4.21)
\]

Notice that \( \Omega_m(t) \) and \( \Omega^m \) are distinct values. With the definitions from above we will compute \( \Omega^1 \) and \( \Omega^2 \):

\[
\Omega^0 = 0, \quad (4.22)
\]

\[
\Omega^1 = \int_0^t s \, ds = \frac{1}{2} t^2, \quad (4.23)
\]

\[
\Omega^2 = \int_0^t A \left( s, \exp \left( \frac{1}{2} s^2 \right) \right) ds = \int_0^t s \exp \left( \frac{1}{2} s^2 \right) ds = \exp \left( \frac{1}{2} t^2 \right) - 1. \quad (4.23)
\]

Based on the function \( A(t, y) = ty, A \left( s, \exp \left( \frac{1}{2} s^2 \right) \right) = s \exp \left( \frac{1}{2} s^2 \right) \). In this example, the function \( A \) is defined as the multiplication of the components. Then this function is integrated from 0 to \( t \). The following is an approximation for \( y(t) \) using a second order Magnus expansion:

\[
y(t) \approx \exp \left( -1 + \exp \left( \frac{1}{2} t^2 \right) \right). \quad (4.24)
\]

This is a second order expansion. To further analyze we could utilize Taylor expansions to obtain a different representation of this (we take the Taylor expansion of the exponent of the exponential so Taylor expand \( -1 + \exp \left( \frac{1}{2} t^2 \right) \)). The result is as follows:

\[
\Omega^2(t) = \frac{1}{2} t^2 + \mathcal{O}(t^4). \quad (4.25)
\]

Just a reminder, \( \Omega^2(t) = \Omega_1(t) + \Omega_2(t) \) so there is no need to sum anything, the equation for \( \Omega^2(t) \) takes care of it. The above is an approximation of \( \Omega^2(t) \). Then,
similarly to other assignments, we will ignore $O(t^4)$. This approximation will be substituted back into (4.24). Resulting in the following $y(t)$ approximation:

$$y(t) \approx \exp\left(\frac{1}{2} t^2\right).$$

(4.25)

At this point we have the exact solution, Equation (4.26), and two different approximations, Equations (4.27), (4.28):

$$y(t) = \frac{2}{2 - t^2},$$

(4.26)

$$y(t) \approx \exp\left(-1 + \exp\left(\frac{1}{2} t^2\right)\right),$$

(4.27)

$$y(t) \approx \exp\left(\frac{1}{2} t^2\right).$$

(4.28)

To better see how close each approximation comes to the exact solution Figure 4.1(b) shows the absolute error between the exact solution and the two approximations at each order. Notice that the error in higher order Magnus approximation has been reduced. Since there exists a vertical asymptote at $t = \sqrt{2}$ it is unreasonable to expect continuous approximate solutions to exhibit this feature. Hence, we expect to see pointwise convergence to the exact solution, that is, the number of terms to include in the expansion to ensure an the error between the approximation and the exact solution depends on the specific location for $t \in (0, \sqrt{2})$.

![Figure 4.1:](image)

Figure 4.1: (a) The exact solution $y(t)$ for Eq. 4.10 is represented by the red curve with its Magnus approximations at first (green) and second (blue) orders. We can see that the approximate equation of $y(t)$ using $\Omega^{[2]}$ is closer to the exact solution than when $\Omega^{[1]}$ is used. (b) The absolute error is shown for the two approximations. Notice that the solution at first order (red) has a higher error than that at second order (blue).
Consider the differential equation
\[
\frac{d}{dt}y = y - y^3, \quad (4.29)
\]
\[
0 < y(0) = y_0 < 1. \quad (4.30)
\]

Similar to the last example, we will put \( y'(t) \) in the form \( A(t,y)y \).
\[
\frac{d}{dt}y = (1 - y^2)y, \quad (4.31)
\]
\[
A(t,y) = 1 - y^2. \quad (4.32)
\]

Therefore we can use our nonlinear Magnus expansion to create approximations to the exact solution. The exact solution can be determined to be,
\[
y(t) = \frac{1}{\sqrt{1 - \alpha \exp(-2t)}}
\]
where
\[
\alpha = 1 - \frac{1}{y_0^2}.
\]

Before using a Magnus expansion to estimate \( y(t) \), we can find the equilibrium solutions for \( y(t) \). Once the equilibrium solutions are calculated, the behavior around each can be determined.
\[
y' = (1 - y^2)y, \quad 0 = (1 - y^2)y, \quad y = 0, \pm 1.
\]

Given the equilibrium solutions \( y = 0 \) and \( y = \pm 1 \), we can determine the behavior around each equilibrium solution. In order to do this we will place a small perturbation, call it \( g_n(t) \ll 1 \), on each equilibrium solution. By placing this small perturbation on each equilibrium solution, we can determine the behavior around each equilibrium solution. Ultimately, we will find the eigenvalues and determine the stability of each equilibrium solution.
\[
y_1 = 0 + g_1(t), \quad (4.33)
\]
\[
y_2 = 1 + g_2(t), \quad (4.34)
\]
\[
y_3 = -1 + g_3(t). \quad (4.35)
\]
The three equations above will be substituted back into the original equation \( 0 = -y' + y - y^3 \) and let \( g_n(t) = Ce^{\lambda t} \), then \( g'_n(t) = \lambda Ce^{\lambda t} \).

\[
0 = -g'_1 + g_1 - g_1^3 \\
= -g'_1 + g_1 + O(g_1^2) \\
= -\lambda Ce^{\lambda t} + Ce^{\lambda t} \\
= -\lambda + 1, \\
\lambda = 1.
\]

Since the eigenvalue is real and positive, the equilibrium solution \( y = 0 \) is unstable.

\[
0 = -(1 + g_2)' + (1 + g_2) - (1 + g_2)^3 \\
= -g'_2 + 1 + g_2 - (1 + 3g_2 + 3g_2^2 + g_2^3) \\
= -g'_2 - 2g_2 + O(g_2^3) \\
= -\lambda Ce^{\lambda t} - 2Ce^{\lambda t} \\
= -\lambda - 2, \\
\lambda = -2.
\]

Since the eigenvalue is real and negative, the equilibrium solution \( y = 1 \) is stable.

\[
0 = -(-1 + g_2)' + (-1 + g_2) - (-1 + g_2)^3 \\
= -g'_2 - 1 + g_2 - (-1 + 3g_2 - 3g_2^2 + g_2^3) \\
= -g'_2 - 2g_2 + O(g_2^3) \\
= -\lambda Ce^{\lambda t} - 2Ce^{\lambda t} \\
= -\lambda - 2, \\
\lambda = -2.
\]

The result is the same for the equilibrium solution \( y = -1 \) as it was for \( y = 1 \). Since the eigenvalue is real and negative, the equilibrium solution \( y = -1 \) is stable. Thus, this differential equation has 3 equilibrium solution, 2 of which are stable and 1 is unstable.

Now that we have the exact solution and an idea of what the phase portrait and differential equation may look like around the equilibrium solutions, we can approximate \( y(t) \) using Magnus expansions and Taylor expansions. We want to develop \( \Omega^{[2]} \), but first we need to find \( \Omega^{[1]} \). We were given \( y' = (1-y^2)y \), so \( A(t, y) = \)
$1 - y^2$. Notice that $A$ is autonomous and does not depend on $t$.

$$\Omega^{[1]} = \int_0^t A(s, y_0) \, ds$$
$$= \int_0^t 1 - y_0^2 \, ds$$
$$= (1 - y_0^2) \bigg|_0^t$$
$$= (1 - y_0^2)t.$$ 

So, now that we have $\Omega^{[1]}$, we can solve for $\Omega^{[2]}$,

$$\Omega^{[2]} = \int_0^t A(s, \exp(\Omega^{[1]}(s))y_0) \, ds$$
$$= \int_0^t A(s, \exp((1 - y_0^2)s)y_0) \, ds$$
$$= \int_0^t 1 - \exp(2(1 - y_0^2)s)y_0^2 \, ds$$
$$= -\frac{y_0^2}{2(y_0^2 - 1)} + \frac{(2(y_0^2 - 1)te^{2y_0^2t} + y_0^2e^{2t})e^{-2y_0^2t}}{2(y_0^2 - 1)}.$$ 

The integral above can also be written in the following form:

$$\int_0^t 1 - b \exp(as)ds,$$

where $a$ and $b$ are constants. The integral then is

$$t - \frac{y_0^2}{2(1 - y_0^2)} \left( \exp(2(1 - y_0^2)t) - 1 \right).$$

So the approximation for $y(t)$ becomes:

$$y(t) \approx \exp(\Omega^{[2]}(t))y_0,$$
$$y(t) \approx \exp \left( -\frac{y_0^2}{2(y_0^2 - 1)} + \frac{(2(y_0^2 - 1)te^{2y_0^2t} + y_0^2e^{2t})e^{-2y_0^2t}}{2(y_0^2 - 1)} \right) y_0.$$
Next we will calculate \( \Omega^{[3]}(t) \) using \( \Omega^{[1]}(t) \) and \( \Omega^{[2]}(t) \).

\[
\Omega^{[3]} = \sum_{k=0}^{3-2} \frac{B_k}{k!} \int_0^t \text{ad}^{k}_{\Omega^{[3-1]}(s)} A(s, \exp(\Omega^{[3-1]}(s)y_0)) ds
\]

\[
= 1 \int_0^t A(s, \exp(\Omega^{[2]}(s)y_0)) ds + \frac{1}{2} \int_0^t \text{ad}^1_{\Omega^{[2]}(s)} A(s, \exp(\Omega^{[2]}(s)y_0)) ds
\]

\[
= \int_0^t 1 - \left( \exp(\Omega^{[2]}(s)y_0) \right)^2 ds + 0
\]

\[
= \int_0^t 1 - \left( \exp \left[ \left( -\frac{y_0^2}{2(y_0^2 - 1)} + \frac{(2(y_0^2 - 1)se^{2(y_0^2)} + y_0^2e^{2s})e^{-2y_0^2s}}{2(y_0^2 - 1)}y_0 \right) \right] \right)^2 ds.
\]

The reader can appreciate the difficulty of the integral above along with the Taylor expansion used to approximate the exponential.

### 4.5 Conclusion

This section provided a very brief overview of Magnus expansions as well as two examples and the procedures used to find an approximate solution to the differential equation. Several visuals were provided to depict the exact solutions against the approximations. The graphs, more than the sometimes lengthy equations, help the reader visualize how close these approximations are to the actual solutions. Although
these are helpful examples, more often than not when a Magnus expansion technique is implemented, the exact solution is not known and cannot be compared to the approximations.
5 SUMMARY AND FUTURE WORK

Nonlinear differential equations are incorporated in mathematical models in numerous instances. We have discussed various methods of solving and approximating linear and nonlinear differential equations. Due to the difficulty of solving nonlinear differential equations, approximation methods have been developed. Initial and boundary value problems have been discussed along with several linear and nonlinear techniques to approximate or solve the linear or nonlinear problems. Regular and singular perturbation theory and Magnus expansions were our particular focus with examples of each technique shown. A brief history of Magnus expansions was discussed and highlighted the recent developments and applications that these expansions provide, [2]. Visuals were used to demonstrate the accuracy, or lack thereof, of each technique. These techniques are integral in applied mathematics and the correct employment allows us to see the behavior of a differential equation when the exact solution may not be attainable. Overall, this paper has demonstrated a few techniques that can be used to approximate linear and nonlinear differential equations that can be implemented when the exact solution may not be achievable.

Extensions to this work could consist of:

1. Synthesis between different methods of approximation, in particular perturbation theory and Magnus expansions.

2. Numerical extensions developed from Magnus expansions. How do these new methods compare to standard ones?

3. In depth look at the systems of systems of equations that are created using Magnus expansions.
BIBLIOGRAPHY


VITA

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