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MIXED-HARDWOOD
THINNING OPTIMIZATION

by

Steven H. Bullard

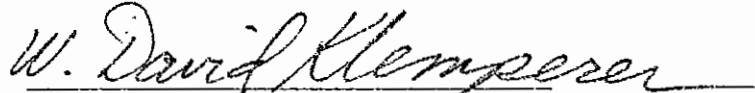
Dissertation submitted to the Graduate Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

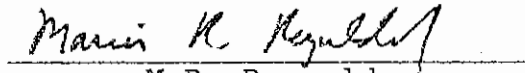
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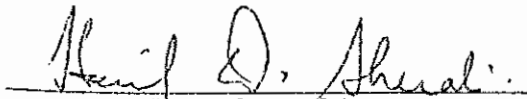
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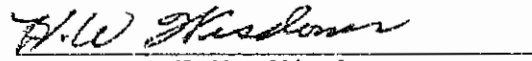
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ACKNOWLEDGEMENTS

Throughout the research in this study, generous assistance was provided by graduate committee members to whom I will always be indebted. Sincere thanks are extended to Dr. W. David Klemperer, chairman, Dr. Harold E. Burkhart, Dr. Marion R. Reynolds, Dr. Hanif D. Sherali, and Dr. Harold W. Wisdom. I would also like to express sincere gratitude to Dr. Layne T. Watson for unsolicited assistance at a crucial stage of the study.

The research summarized in this report was funded by a grant from the U.S. Forest Service, Southeastern Forest Experiment Station. I would also like to acknowledge Resources for the Future, Inc., for partially sponsoring the research with a dissertation fellowship grant.

Family support during my graduate program was often needed and never failing. I will forever be grateful for the encouragement of my parents and the love and sacrifice of my wife Susie and son Jacob. Such support can never be repaid but will always be cherished.

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I. INTRODUCTION

Upland hardwood forest types are by far the most widespread in the United States. Stands of the oak-hickory forest type alone include 109 million acres, 23 percent of the Nation's commercial timberland (U.S. Forest Service 1982). Many even-aged upland hardwood stands developed on nonindustrial private lands through hardwood invasion after pine stands were harvested. In 1973, half of the hardwood timber in the South was determined to be on upland sites which formerly supported pine stands (Murphy and Knight 1974).

Many nonindustrial private landowners passively permit the biologically better adapted hardwoods to increase after the harvest of pines. These landowners may be pursuing their best interests as perceived through prevailing social and economic conditions (Boyce and Knight 1980). The resulting even-aged hardwood stands are often poorly stocked and consist of mixed-species with differential growth rates.

Rates of return to landowners are typically low from even-aged upland hardwoods. These stands can often be converted to higher return softwood forest types but landowners frequently reject the investment because of the high costs and long time periods involved. Past market conditions favored the production of higher quality hardwood

products but prospects are good for expanded market opportunities for lower grade hardwood raw materials. These new or expanded market opportunities should improve the future profitability of currently low value upland hardwoods and provide more economic incentives for active forest management. Partial harvests are particularly attractive forest management activities for most landowners because of the returns generated.

Past studies have applied mathematical programming techniques to the optimization of harvest schedules in softwood stands. Stand-level hardwood harvesting models designed to optimize economic objectives, however, may depend on different relationships than softwood models, e.g., the relationships between stumpage price and stem quality may be more pronounced for hardwood stands. This study will focus on the theory and application of mathematical programming to the problem of optimizing harvests over time in mixed-species, even-aged upland hardwoods. Operations research methods and mathematical programming techniques have been developed as analytical tools in management science. Several studies have been done in the area of stand-level softwood harvest schedules but little application of these powerful tools has been made to the problem of hardwood harvest scheduling.

Objectives

The objectives of this study are:

1. To mathematically define the problem of deriving economically optimal stand-level harvest schedules for even-aged upland forest types of mixed-species.
2. To select an applicable operations research method for solving the mathematical model.
3. To review the growth and yield information currently available for even-aged, mixed-species stands with an application of the model if adequate response information is available.

Justification

Hardwood forest management has received much less attention in the past than management of softwood forest types. Comparatively low growth rates and values, as well as relatively few markets for hardwood raw materials have resulted in very little active hardwood forest management. With an estimated 255 billion cubic feet of hardwoods, covering over 260 million acres in the United States (U.S. Forest Service 1982), the problems of managing this resource cannot be ignored. While many upland hardwood stands are currently of low value, expanding market opportunities should enhance the possibilities for upgrading the quality and value of such stands through intermediate harvests

(Schropshire 1977, Sims 1981).

Commercial thinning has not been widely practiced in hardwood stands in the past, chiefly due to inadequate markets for the material removed (Baumgras 1981). Future price increases and expanded markets for lower quality hardwood raw materials are expected, however. Assuming base-level price trends, the medium projection of timber demand by the U.S. Forest Service (1982) indicates softwood demand will increase by 80 percent by 2030. Hardwood demand, however, is projected to more than triple over the same period. A significant portion of the increased hardwood demand reflects increased requirements for hardwood pulpwood and hardwood lumber for pallets. Beyond the next few decades, stumpage prices for lower-grade hardwoods are expected to rise (U.S. Forest Service 1982). Future competition for available hardwood supplies is expected to be particularly intense in the South-Central Region.

Market opportunities for hardwood raw materials are expected to increase due to greater energy-wood demands as well as technological advances in pulping and the development of new products. Changes in the economic relationships of energy sources in the past decade have led to an increased market for industrial and home fuel (Curtis 1980). As hardwood is generally a more efficient fuel than softwood, the fuel market should provide new opportunities

for intermediate harvests in hardwood stands at lower net costs or with immediate net gains, in addition to the longer term potential gains in tree quality.

In other areas, hardwoods are increasingly being used in the manufacture of pulp and paper (Malac 1978). These increases should continue with further refinements in high-yield pulping processes. Hardwoods are also increasingly being used in the production of particleboard products (McLintock 1979), as well as organic chemicals (Glasser 1981). Prospects for hardwood fiberboard and flakeboard are particularly bright, with 80 percent of the market east of the Mississippi River (Thielges 1980). Further enhancing fiberboard and flakeboard prospects are the favorable raw materials costs compared to softwood chips, which will be experiencing increased demand and rising prices for pulping uses during the next 20 years (Thielges 1980). As an indication of future market expansion, the first two hardwood flakeboard plants in the South are scheduled to begin operations in 1983 (Koch and Springate 1983).

While a significant amount of research is being devoted to developing new and better ways of utilizing the hardwood resource in the United States, increasing emphasis is also being placed on the problems of managing natural hardwood stands. Enhanced opportunities for upland hardwood management are almost certain and intermediate harvests

should be an important factor in hardwood management strategies. The problem of intermediate harvest decisions is particularly difficult where upland stands are comprised of mixed-species with differential growth rates. Theoretically sound models are needed for hardwood conditions if forest landowners or managers are to achieve stand-level and forest-wide objectives through their intermediate harvest decisions.

Literature Review

The ability of decision-makers to answer stand-level questions about the timing and intensity of thinnings has been greatly enhanced through the application of operations research techniques to such problems. A broad class of these techniques will be considered with respect to applications that have been made to softwood stands. The literature concerning the special problems of thinnings in the management of hardwood stands will also be reviewed.

Operations Research Applications

Simulation and Stand-Level Decisions. Simulation techniques basically involve a specification of treatment regimes for stands. The impacts of various treatments and timing of treatments are then assessed. The selection of a preferred regime is made based on a common criterion of performance. Either physical or economic criteria may be used, but no assurance is made that the management regime

selected will be globally optimal where complex relationships are involved. Problems with the simulation method may also arise through the stochastic nature of the models. Methods for statistical validation of stochastic simulation systems were presented by Goçhenour and Johnson (1973), and Reynolds et al. (1981).

Simulation methods have been applied to stand-level decisions in several studies. Examples summarized by Hann and Brodie (1980) include the work of Hamilton and Christie (1974), Myers (1969, 1973), and Hoyer (1975). Each method employs a stand development model enabling the user to alter thinnings and rotation length in the evaluation of specific management programs.

In a study of maximum volume production, Walker (1981) used a modified version of a computer simulation model developed by Daniels and Burkhart (1975) to determine optimal management regimes in loblolly pine plantations. Optimization techniques were used to determine regimes which maximized the mean annual increment predicted by the stochastic stand simulation model. Management factors examined included rotation length, planting density, and timing and intensity of a single thinning. Response surface analysis and a simplex search technique presented by Ollson (1974) were used to determine management regimes which maximized mean annual increment.

The Daniels and Burkhardt simulation model was used in a deterministic manner by Broderick et al. (1982) to estimate economically optimal management regimes for loblolly plantations. Management regimes which maximized soil expectation values were determined by evaluating the model for various combinations of planting spacing, rotation, and frequency, timing, and intensity of thinnings. The impacts of assumed interest rates, prices, and product mixes on optimal management regimes were also examined.

Optimization Techniques and Stand-Level Decisions. The forestry literature is replete with applications of optimization methods to stand-level decisions. Maximizing mean annual increment or soil expectation value (SEV) were early methods used in determining optimal rotations. Much of the recent work has concentrated on the simultaneous determination of optimal thinning schedules and rotation length. Mathematical programming techniques have been applied extensively in this area.

The following discussion of stand-level decision models is confined to deterministic analyses. Presentations have also been made of stochastic stand-level decision analyses using operations research techniques. These studies include Hool (1966), Lembersky and Johnson (1975), Lembersky (1976), and Kao (1982). The stand-level decision models reviewed are also similar in that only timber values are used in the

analyses. Studies which address the complications arising when non-timber values are considered include Hartman (1976), Calish et al. (1978), Nguyen (1979), and Riitters et al. (1982).

Optimal management plans were derived by Hardie (1977) for loblolly pine plantations in the Mid-Atlantic Region. Rotation length and thinning timing and intensity were varied to determine the regimes which maximized per acre present net values for a single rotation. The effects of various economic assumptions were also compared. The solution technique employed by Hardie was complete enumeration and comparison of results under a highly constrained set of thinning and rotation alternatives.

An early study by Chappelle and Nelson (1964) made use of marginal analysis to jointly determine optimal thinning and rotation length. With profit maximization as the guiding criterion, optimal stocking levels were determined using the alternative rate of return as the marginal unit cost and value growth percent as the marginal unit revenue. After determining the optimal stocking level, the volume removed by thinnings in each period was determined for specified rotation lengths, given the initial stocking and a volume growth procedure. This information was then used to determine the SEV maximizing rotation length.

The question of optimal growing stock levels was

addressed from the standpoint of inventory theory by Pelz (1977). Expected total costs of inventory were defined as the sum of inventory holding costs and the costs associated with deviating from the optimal stocking level. By minimizing the expected total costs of inventory, Pelz demonstrated a correspondence of optimal stocking level results with those of Chappelle and Nelson (1964), when similar assumptions were made. Optimal rotation lengths were not discussed.

Several attempts to determine optimal thinning and rotation length have been presented which use dynamic programming. With time defined as a discrete rather than a continuous variable, dynamic problems, or multi-stage optimization problems, can be solved by discrete dynamic programming. This technique involves dividing the problem into discrete stages and then making decisions recursively at each stage. The recursion may involve moving forward from initial time or backward from terminal time. At each stage, decisions are made based on the recursive equation. This process employs Bellman's Principle of Optimality, i.e., given an initial state and decision, the remaining decisions must constitute an optimal policy with respect to the state resulting from the first decision (Bellman and Dreyfuss 1962). This principle may be paraphrased in terms of the optimal growing stock problem as follows: once the

optimal thinning schedule has been specified to a given stand age and structure, the optimal plan for the next older stand age depends only on the older stand's age/structure combinations not yet analyzed (Hann and Brodie 1980). This greatly reduces the number of calculations necessary to determine the optimal path, as various possibilities at each stage are only considered once (Cawrse 1979). The recursion equation is based on the contribution of the stage variable and the optimal contribution of all preceding variables. The results of decisions at each stage of the problem are combined to generate the overall solution.

In applying discrete dynamic programming to determine optimal thinning and rotation length, Amidon and Akin (1968) obtained the same solutions as Chappelle and Nelson (1964). A two dimensional network was defined using volume stocking and stand age as the state descriptors. The objective of the dynamic problem was to determine the optimal stocking level at each age class using 1,000 board foot and 5-year intervals between stages. In this problem, the optimal stocking level at each age class was determined using backward recursion, examining the objective function value for all possible points. The backward recursion method will only solve the problem of optimal thinning plan for one rotation at a time. Amidon and Akin therefore obtained solutions for alternate rotation lengths, following

Chappelle and Nelson in using the SEV maximizing rotation as optimal.

The approaches of Chappelle and Nelson (1964) and Amidon and Akin (1968) were discounted by Schreuder (1971). Schreuder proposed that these approaches did not allow for possible interdependencies between stocking and rotation and that the cost of land should be included when determining optimal economic stocking levels. Schreuder's approach was to determine the jointly optimal thinning plan and rotation by defining the harvest cut as an extreme thinning. Schreuder formulated the problem as a continuous function of time using the calculus of variations form but found that explicit solutions could only be obtained for trivial examples. The problem was then cast as a discrete dynamic programming problem with backward recursion. Schreuder concluded that solutions could be easily obtained using the dynamic programming technique but did not present examples.

Naslund (1969) also presented a formulation of the optimal thinning and rotation problem using the calculus of variations form. Both time and removals were continuous in value. The approach assumed certain specific, differentiable functions, e.g., a function relating the effects of the timing and intensity of thinnings to sales value of the final harvest. No examples were presented by Naslund although solution techniques were discussed.

Subsequent efforts by Kao and Brodie (1980) failed to obtain a solution to Naslund's formulation.

More recent studies have reported practical applications of dynamic programming to the joint optimality problem of thinning and rotation length. Brodie et al. (1978) analyzed the economic impacts of thinning and rotation in Douglas-fir using dynamic programming. The major goal of their study was to assess the effects of regeneration costs, initial stocking, quality differences, site, and logging costs on thinning intensity and rotation age. The approach of Brodie et al. differed from that of Amidon and Akin by incorporating a mortality estimator into the stand growth model, allowing more realistic potential stocking for each age class, and by using the forward recursion method. Brodie et al. demonstrated that the approaches of Chappelle and Nelson (1964) and Amidon and Akin (1968) actually do determine the jointly optimal stocking level and rotation age (contrary to Schreuder's (1971) findings). A major problem with their approach, recognized by Brodie et al., was the lack of diameter growth acceleration in the stand model after thinning.

Accelerated diameter growth should be reflected in thinning analyses, especially where logging costs are reduced and income increases with the size and quality of harvested material. A study by Brodie and Kao (1979)

accounted for this problem by using a more complex stand model and using three state descriptors. These descriptors were stand age, basal area, and number of trees. Solutions generated with this framework are the optimal number of trees and basal area to maintain in each time period, i.e., for each age class.

A related approach for deriving optimal stand density over time was presented by Chen et al. (1980). This method involves using a calculus approach to search for optimal solutions stage by stage. Chen et al. used this approach to derive a set of optimal stand densities and an optimal rotation where the criterion used was the maximization of volume harvested. The technique proposed by Chen et al. incorporates the advantages of both forward and backward recursion methods. The approach is not readily applicable to optimization with an economic criterion, however. The incorporation of price and cost functions prevents the derivation of a generalized solution because of differentiability requirements. In such cases, the thinning problem can be solved for the discrete case but solutions are only optimal over the possible solutions simulated in the discrete formulation (Chen et al. 1980).

A nonlinear programming approach for the simultaneous optimization of thinning and rotation was presented by Kao and Brodie (1980). This approach allows continuous values

for both the timing and intensity of thinnings. The optimal frequency of thinning was determined by solving the model with no thinnings, with one thinning, with two thinnings, etc., until the present net worth criterion decreased. Decision variables in the nonlinear formulation were the age for each thinning, the percent normality of the residual stand after each thinning (defining the amount harvested), and the age of final harvest. A comparison of this approach was also made to a discrete dynamic programming formulation of the same problem. The dynamic programming solution using narrow state intervals required much more storage and computation time. Another advantage cited by Kao and Brodie for the nonlinear programming formulation was that additional constraints such as minimum removals could be imposed.

Thinning Hardwood Stands

Even-aged hardwood stands in the South are most often high in density. Stands referred to as poorly stocked are usually understocked in terms of trees of high quality or preferred species, rather than stems per acre (Gingrich 1970). Thinnings are usually administered to concentrate growth on the more desirable stems, and remove trees with poor form or slower growth rates. In this manner, thinnings affect both the quality and quantity of wood produced in a stand. Of the information published on hardwood thinning,

little is based on long-term experimental results and even less on the economics involved in thinning decisions.

General Considerations. Through the timing and intensity of thinnings, emphasis can be placed on present benefits or future benefits. The relative condition of the residual stand may or may not be of primary concern. In hardwood stands, thinnings must be balanced between volume and quality. Heavy thinnings may provide too much growing space and result in epicormic branching (Evans et al. 1975). In many cases, the price differential between high and low quality hardwood timber may be the only justification for thinning.

The effects of density, thinning, and species composition in eastern hardwoods were summarized by Gingrich (1970). Most of the general discussion in this section is presented in Gingrich's work. The three factors which most affect hardwood thinning results are species composition, tree vigor, and potential stem quality.

Even-aged upland hardwood stands are typically composed of a mixture of species. These stands often appear uneven-aged due to a wide distribution of diameters. This characteristic is due in part to differential species growth (Gingrich 1967, Oliver 1980). Very little data is available on the biological performance of various species mixtures after thinning due to the large number of possible mixtures.

Differences in growth rates are generally known, however, and thinning plans in mixed hardwood stands must take into account the initial composition.

The effect of relative tree vigor on thinning results must also be considered. The growth capabilities of residual trees often depend on the degree of past competition through the ability of crown and root systems to respond to release. The tree vigor aspect presents a sound basis for thinning hardwoods from below as the subdominant classes exhibit characteristics of greater competition (Gingrich 1971).

Potential stem quality is another important factor in hardwood thinnings. Hardwood quality largely depends on the proportion of clear bole. In a study of even-aged red oak stands, Ward (1964) presented evidence for maintaining higher densities to encourage natural pruning. A study of the influence of stand density on stem quality in pole-size northern hardwoods (Godman and Books 1971) classed bole defects as live limbs, dead limbs, bumps, and epicormic branches. This study reported that differences in the number and retention of defects among species after thinning were primarily influenced by shade tolerance, i.e., the more tolerant species exhibited the greatest incidence of defects. Indications that some hardwood species produce clear bole more rapidly than others under common age and

size conditions were presented by Weitzman and Trimble (1957). This suggests the existence of differences in grade potential similar to previously discussed differences in growth potential (Gingrich 1970).

Another factor affecting the quality of hardwood timber is stem form. A recent study using two measures of stem form provided evidence that post-thinning stocking levels do not significantly affect the stem form of upland oaks (Hitt and Dale 1979). Stem form changes were found to be correlated to pre-thinning form, however. Regardless of the residual stocking level, better formed stems deteriorated in form after thinning while more poorly formed stems improved in form.

Studies have also been presented which attempt to quantify the quality of hardwood growing stock. A system based on the correlation between the number of surface defects and the probability that the future butt log will be a certain grade was presented by Boyce and Carpenter (1968). A quality classification system for young hardwood trees has also been developed (Sonderman and Brisbin 1978, Sonderman 1979). In this system, external tree measurements are used as a basis for predicting the future product potential of young hardwood stands. The system is proposed as a possible aid to managers in making decisions on cultural treatment investments.

Yield Information. Physical response data related to thinning in even-aged hardwood stands is sparse. It is unlikely that data will ever be gathered for all combinations of thinning schedules, species mixtures, site quality, etc. Work that has been published in this area is often for certain species under localized conditions.

For predominantly oak stands in the Central States Region, Gingrich (1971) presented per acre yield results using a fixed 10-year thinning interval. Results were presented where thinnings were initiated at different points in the lives of even-aged stands. The age at which thinning was started was a primary factor determining maximum production. Per acre yields were more than 50 percent higher in stands where thinning began at age 10 rather than at age 60. Gingrich also found that without precommercial thinning, the latest effective age for beginning thinning was between 30 and 40 years for pulpwood production, and between 50 and 60 years for sawtimber production.

Growth and yield information for upland oak stands 10 years after initial thinning was presented by Dale (1972). Thinning intensity varied up to removal of 70-80 percent of the original stand basal area. The thinning procedure was designed to remove trees in all crown classes, with the residual stand composed of evenly spaced desirable stems. The differential effects of species composition on thinning

response were not incorporated in the presentation of results.

A study has also been installed in the Boston mountains of Arkansas to evaluate the growth response of upland hardwoods to thinning (Graney 1980). Although thinning response data are not yet available from this study, comparisons of initial stand conditions were made to Schnur's (1937) yield tables for unthinned oak stands, and to stand conditions reported by Gingrich (1971). Comparisons were also made of post-thinning stand volumes to the predicted volumes for thinned upland oak stands in the Central States Region reported by Dale (1972). One goal of such comparisons is to help determine if the results of thinning studies in the Central States can be applied to other regions.

Interim results of a continuing study of thinning effects on even-aged yellow-poplar stands in the southern Appalachians have been reported by Beck and Della-Bianca (1970, 1972, 1975). The findings presented by Beck and Della-Bianca for yellow poplar are the most comprehensive available for any even-aged hardwood forest type. In the 1975 report, equations and tables are presented for estimating board-foot growth and yield, and residual quadratic mean diameter growth for a range of site indexes, ages, residual basal areas, and residual quadratic mean

diameters. Individual tree responses to thinning are also discussed.

Computerized hardwood growth simulation has also received attention in recent years. Simulation methods for estimating growth and yield are often the most feasible in light of the impracticality of field studies covering all possible combinations of factors affecting responses to management. Stiff (1979) modeled the growth dynamics of natural, mixed-species Appalachian hardwood stands. In this study, a generalized modeling system for the projection of diameter distributions through time was developed to predict growth and yield in such stands. Possibilities for thinning were not incorporated, however.

A more general growth projection simulator, applicable to the Lake States Region, has been developed at the North Central Forest Experiment Station (U.S. Forest Service 1979). The system is designed to project forest growth and mortality, with or without harvesting, for any species mix or stand structure. The basic components of the model are a procedure for estimating potential diameter growth, a procedure for modifying potential growth to actual growth, a rule to allocate the total projected growth to individual trees, and a mortality function (Leary 1979). The model provides for three possible resolution levels; differentiation by species alone, by tree size and species,

or by individual trees. Data for estimation of the model parameters were from even and uneven-aged natural stands and plantations in the Lake States Region.

II. GROWTH MODEL DEVELOPMENT

Prior to the development of a hardwood thinning model, a means of projecting the growth of such stands must be available. The model must be capable of projecting the growth of existing mixed-species stands, and must incorporate responses to thinning. Considering the important factors in modeling such stands will aid in determining the necessary growth model resolution. This factor in turn affects the joint considerations necessary to interface the mixed-species growth model with optimization procedures.

Resolution Level

Resolution level is a primary factor in determining whether or not a stand model can adequately meet particular users' needs. Models yielding information on total volume, volume by size class, volume by size class and species, etc., all have specific applications in forest management.

Recent studies concerned with optimal thinning and rotation have recognized a need to account for diameter class distributions in making such stand-level decisions. Hann and Brodie (1980) report that diameter distribution data is important in the planning of milling facilities as well as applying specified treatments to field conditions.

Hardie (1977) notes that diameter distribution information is necessary to fully evaluate the benefits of thinning in loblolly pine stands, when multiple product values occur. That is, pulpwood, sawtimber, and pole and piling values can be assigned based on diameter.

Discrimination by size class is particularly important in modeling the benefits from hardwood thinning as price differentials between size classes may be pronounced. A further consideration is that hardwood stands are usually comprised of mixed-species, each with different growth rates and value-by-size-class relationships. For a mixed-species hardwood thinning model to adequately reflect these relations, the underlying growth model must provide information by size class and species over time. This level of resolution will allow the model to closely reflect actual conditions, and will result in thinning prescriptions with more realistic application in the field.

Growth Modeling Approaches

The method selected to model mixed-species growth must be combined with a method of determining optimal thinning schedules. Joint considerations are therefore required to ensure that the necessary interface can be achieved. These considerations will be discussed in conjunction with two approaches to stand modeling for mixed-species.

Diameter Distribution Approach

One approach to stand modeling which has been combined with optimization over time involves the use of probability density functions to describe diameter distributions. The parameters describing the distribution, e.g., the scale, shape, and location parameters of the Weibull distribution, are used as decision variables in an optimization procedure. Optimal values of these parameters describe the optimal residual diameter distributions for each period. This procedure was used by Martin (1982) in deriving optimal management guides for uneven-aged northern hardwoods.

The diameter distribution approach to stand modeling, however, is not readily applicable to mixed-species stands unless species are aggregated. That is, while the diameter distributions of entire stands may be described by such functions, the post-thinning distributions for separate species would be unlikely to follow smooth, continuous patterns.

Stand-Table Projection Approach

Another approach to stand modeling which has been used with optimization procedures is stand-table projection. This approach simplifies the complex nature of modeling stand growth and thinning response by isolating certain growth components. Stand-tables are projected through time by predicting upgrowth for each size class, i.e., the

proportion of trees in each size class that will grow into the next larger class, during a fixed time period. The required level of resolution may be obtained with this approach by predicting such proportions for each species and diameter class.

As described by Wahlenberg (1941), three factors affect the upgrowth of trees from a given diameter class during a fixed time interval: diameter growth, diameter class size, and the distribution of the number of trees within the diameter class. Upgrowth may be modeled by treating each of the three components separately, or by predicting upgrowth directly. Examples of the two approaches may be found in Hann (1980) and Ek (1974), respectively. A modified version of Ek's (1974) model was used by Adams and Ek (1974) to derive optimal management strategies for uneven-aged hardwood stands.

Adams and Ek (1974) addressed certain aspects of uneven-aged management, treating mixed-species as aggregates. The general approach to stand modeling and subsequent combination with optimization techniques, however, provides a basis for modeling the even-aged hardwood thinning problem. That is, Adams and Ek used a stand model comprised of ingrowth, upgrowth, and mortality functions. Nonlinear programming was then used to derive an optimal size class distribution, and an optimal cutting

policy for achieving the desired distribution. Assuming an even-aged stand-table projection method which accounts for individual species, similar techniques could be used to derive optimal thinning and rotation for even-aged, mixed-species stands. Such a formulation would entail achieving a distribution of zero trees in each diameter class, for each species, in an optimal manner.

Developing optimal thinning strategies with the approach outlined above requires a stand-table projection system for even-aged, mixed-species hardwoods. Concepts used to develop such a system and the subsequent specification of equations will be discussed.

Mixed-Species Modeling Concepts

As previously noted, Adams and Ek (1974) dealt with management problems in mixed-species stands, treating species as aggregates. These authors also considered the problems of recognizing individual species groups, however, concluding that a stand simulator at the individual tree level of resolution would be required (Adams and Ek 1975). A more recent study concerned with uneven-aged management concluded that a stand-table projection method could be designed to incorporate species (Hann and Bare 1979). These authors base their conclusion on work involving uneven-aged ponderosa pine. Hann (1980) presented a projection system for ponderosa pine which recognizes two vigor classes.

These vigor classes were modeled in a manner similar to recognizing two distinct species.

While Hann's (1980) approach for modeling uneven-aged ponderosa pine is significant, the number of equations required would severely limit attempts at optimization. An even-aged stand-table projection model comprised of two equations, upgrowth and mortality, for each species/diameter class combination could be more easily interfaced with optimization procedures. Concepts used to model mixed-species' hardwoods at the North Central Forest Experiment Station (U.S. Forest Service 1979) were used in the present study to develop a two equation stand-table projection model.

The growth projection system developed at the North Central Station was designed to estimate forest growth and mortality, with or without harvesting, for any species mix or stand structure. The model is comprised of a potential diameter growth procedure, a process to adjust potential growth to actual growth, a method of allocating projected growth to individual trees, and a mortality function (Leary 1979).

One of the most significant concepts employed in the North Central Station study is the approach of estimating diameter growth by first bracketing the estimate between zero and an upper potential. The upper potential represents

diameter growth under ideal circumstances, e.g., open-grown conditions. This potential is then adjusted downward to an estimate of actual diameter growth. The downward adjustment is a function of stand conditions reflecting competition, e.g., stand density measures. Thinnings or other harvests are incorporated since cuttings reduce stand density, decreasing the downward adjustment of potential growth, thereby increasing the diameter growth estimate for the residual stand. The effects of cutting different species are incorporated by including stand density measures related to each species. That is, both total stand and separate species density measures are included.

This general approach to modeling growth was used by the U.S. Forest Service (1979) in estimating total diameter growth on mixed-species plots. A similar approach is used in the present study to model the diameter upgrowth component of an even-aged stand-table projection system for mixed-species stands. The development and specification of the necessary equations will be discussed, including the assumptions, advantages, and disadvantages inherent in the model specification.

Model Specification

Stand-table projection models for uneven-aged stands must incorporate ingrowth, upgrowth, and mortality processes. The ingrowth process allows trees to grow into

the smallest diameter class represented, and is not necessary to model even-aged conditions. That is, while even-aged hardwoods may appear uneven-aged by diameter distribution, the appearance is attributed to differential species growth rather than ingrowth of younger trees into the stand (Oliver 1980). Even-aged stand-table projection may therefore be accomplished by modeling the upgrowth and mortality processes alone.

Upgrowth

As previously discussed, the approach used to model the upgrowth component in the present study includes estimating a potential proportion of upgrowth, and an adjustment to reduce the potential to an actual estimate. The estimated upgrowth occurs during a fixed growth period, e.g., 5 or 10 years, and is estimated for each species and diameter class. The upgrowth relation may be represented symbolically as:

$$UPG_{ijk} = (PP_{ijk})(ADJ_{ijk})(QTY_{ijk-1}) \quad (1)$$

where:

Subscripts represent species *i*, and diameter

class *j*, after growth period *k*,

UPG is upgrowth (in units projected),

PP is potential proportion of upgrowth,

ADJ is a downward adjustment (also a proportion), and

QTY is quantity (in units projected).

All symbols used in the present study are defined in alphabetical order in Appendix A. Prior to considering the potential and adjustment portions of relation (1) in detail, two important considerations will be discussed: the units projected, and the relationship between diameter class size and the length of the growth period.

Stand-tables yield information on the number of trees per unit area by diameter class, and as usually applied, stand-table projection involves projecting numbers of trees. As the growth model is to be combined with an optimization procedure, however, other projection units were considered. Both basal area and volume were evaluated as alternatives to numbers of trees as projection units because of their continuous nature, possible use as measures of stand density, and in the case of volume, the ability to assign per unit values. Number of trees per unit area was selected as the projection unit, however, for reasons to be discussed following the upgrowth and mortality specifications.

Another consideration regarding the upgrowth component is the relationship between diameter class size and the length of the growth period. Recognizing the periodic nature of much forest growth data, relation (1) represents upgrowth over a fixed time interval. As presented in relation (1), a single upgrowth equation would be required

for each diameter class and species, at the end of each growth period. The specification therefore assumes that the growth period is short enough, or the diameter classes large enough, that no trees will advance two or more size classes. Providing for other relations would require more upgrowth equations, e.g., an equation for the proportion moving up one diameter class, an equation for the proportion moving up two diameter classes, etc. The specification of additional equations should only be of concern in cases where extremely fast growing species are modeled, or where remeasurement data were obtained after a very long growth period.

Potential Proportion. The purpose of estimating a potential proportion of upgrowth is to provide an upper limit on the actual estimate. The potential proportion moving up one diameter class is related to stand age, site quality, and past competition, but is unaffected by present harvesting decisions. This estimated upper limit is therefore a constant with respect to optimization. Harvesting affects the degree to which the estimated potential is realized, but not the estimated potential itself. For this reason, specification of a functional form for estimating potential upgrowth is not required prior to developing a formulation for thinning optimization.

Although functional specification is not required at this stage, several factors affecting the estimation of

potential upgrowth may be considered. Open-grown conditions, for example, have been judged unsuitable for diameter growth studies due to differences (compared to stand-grown trees) in the distribution of increment between the tree bole and branches (Hahn and Leary 1979). Forest-grown conditions in which trees of a particular diameter class hold dominant and codominant positions in the canopy are favored. Under these conditions, stand age and site quality are factors which should affect the potential diameter growth of trees of a given species, in a particular diameter class. That is, information on tree diameter, species, age, crown position, and site quality should be sufficient to predict potential diameter growth over a fixed time interval. These variables should reflect the degree of suppression experienced, and therefore the potential ability to respond to release.

Adjustment Procedure. The adjustment process provides an estimate of the proportion of potential that is actually realized. The proportion realized therefore reflects the growth rate of trees of the relevant diameter class and species. As thinning affects competition and therefore diameter growth rate, prior to formulating a problem to derive optimal thinning schedules, the functional form of the adjustment procedure must be specified. Due to a lack of data, an adjustment function was tentatively specified

based entirely on joint biological and optimization considerations.

The diameter growth rate of a given tree should be inversely related to stand density. The adjustment value predicted in the present study corresponds to diameter growth rate, with higher proportions of potential realized as stand density approaches zero. The marginal effects of density on growth rate should also decrease as density increases. These relations, as well as the criterion that the proportion realized must lie between zero and one, were modeled with a negative exponential specification of the adjustment process, as presented in relation (2).

$$ADJ_{ijk} = \text{EXP}[b_{1j}^{ij} (V_{T,k-1}) + \sum_{m=1}^S b_{m+1}^{ij} (V_{m,\geq j,k-1})] \quad (2)$$

where:

ADJ_{ijk} is the adjustment value for species i ,
diameter class j , after growth period k ,

$V_{T,k-1}$ is total volume after period $k-1$,

$V_{m,\geq j,k-1}$ is volume of each species ($m=1, \dots, S$) in
diameter classes greater than or equal to j ,
after period $k-1$, note that m is used as an
index or counter in relation (2),

S is the number of species, and

$b_m^{ij} \leq 0, m=1, \dots, S+1$ are parameter estimates for species
 i , diameter class j .

Relation (2) incorporates the necessary properties for the adjustment process, using stand volume as a measure of density. As volume approaches zero, the proportion of upgrowth potential realized approaches one. Increasing the residual volume after period $k-1$ reduces the adjustment value for period k , i.e., less upgrowth potential will be realized. Also, the marginal reduction for period k decreases at a decreasing rate, as density measures increase.

Although different measures of density were proposed, the general form of relation (2) was used by Hann (1980) in modeling basal area growth in uneven-aged ponderosa pine. The density variables specified in the present study were based on considerations of both thinning response and optimization. That is, as thinning should not reduce diameter growth rate, measures were chosen such that all partial derivatives with respect to density were strictly negative. This condition resulted in rejecting measures which might better reflect the relative position of each diameter class within the stand. For example, Stage (1973) defined variables reflecting the proportion of total stand basal area which occurred in diameter classes smaller than the class being modeled. Variables representing the proportion of stand volume in greater diameter classes were considered in the present study, but were rejected due to

the indeterminate algebraic sign of the first derivatives with respect to density.

Variables indicating the volume of each species in diameter classes greater than or equal to the class modeled were chosen for two reasons. The first is that the direction of change implied by changes in these variables is the same as for total volume. That is, if trees in a greater diameter class are cut, both total volume and the volume in greater diameter classes are reduced. This relationship is indicated by the strictly negative first derivatives with respect to volume. The second reason for choosing volumes in larger diameter classes is to provide for a greater impact on growth rate when trees in these classes are cut. When trees in lower diameter classes are harvested, for example, only total volume is reduced and the adjustment value for a particular species/diameter class combination increases accordingly. When the same volume is cut from trees in larger diameter classes, however, the increase in the adjustment value is greater. This results because the same reduction in total volume is augmented by a reduction in the appropriate variables for larger diameter classes.

Optimization aspects were also considered in specifying the adjustment process equation. These considerations dealt with the convexity of the equation, and will be discussed

following the development of an optimization procedure for mixed-hardwoods.

Another consideration regarding the adjustment process is the recognition that all relation (2) parameters cannot be estimated as the function is specified. That is, for the smallest diameter class modeled, the variables representing volumes in diameter classes greater than or equal to the smallest class comprise the total volume of the stand. From the perspective of estimating parameters, a singular matrix results for the independent variables. For this reason, in estimating the parameters of relation (2) for the smallest diameter class, it will be necessary to use volumes in diameter classes greater than but not equal to the smallest class.

Finally, although the adjustment process was analyzed in order that optimization could be considered, the specification is tentative. Final determination of an appropriate specification requires that data be available for use in analyzing and evaluating alternate forms. The proposed specification was used, however, in formulating and evaluating an optimal thinning and rotation procedure for mixed-hardwoods.

Mortality

Mortality is the second component of the even-aged stand-table projection system. The mortality referred to in

this study represents regular or noncatastrophic mortality, i.e., that resulting from resource competition (Lee 1971). As with the adjustment process in the upgrowth component, a mortality relation must be specified prior to formulating an optimization procedure. Just as harvests affect growth rates of residual trees, mortality rates are influenced by harvesting. Also, as with the adjustment process, specifying the mortality relation was influenced by both biological and optimization considerations.

Monserud (1976) predicted overstory tree mortality in northern hardwoods using diameter and diameter increment, a competition index, and the length of growth period as independent variables. In the present study, diameter and the length of growth period are fixed. Indications of diameter increment and competition were modeled in the adjustment process of the upgrowth component, however. The same variables which affect diameter growth rates were therefore used in the present study to model the proportion of mortality for each diameter class and species. The proposed expression to represent the proportion of trees dying during a particular growth period is presented in relation (3).

$$PD_{ijk} = 1 - \text{EXP} \left[b_{s+2}^{ij} (V_{T,k-1}) + \sum_{m=1}^s b_{s+2+m}^{ij} (V_{m,\geq j,k-1}) \right] \quad (3)$$

where:

PD is proportion of trees dying, and

Other variables are as defined for relation (2).

Relation (3) expresses the proportion of trees dying as a function of the same stand density measures used to model the adjustment to potential upgrowth. Using the same variables was biologically reasonable and was desirable from an optimization standpoint, as the number of variables necessary to model the optimization problem is minimized. Relation (3) also has the required property that the proportion of trees dying must lie between zero and one, with mortality approaching zero as stand density approaches zero. The proportion dying asymptotically approaches one at extremely high densities.

Again, as with the adjustment process, the mortality expression specified is tentative but was necessary for considering thinning optimization. Further study, including estimation, is necessary before a final specification can be proposed. Also, in estimating parameters for relation (3), the singularity problem discussed with respect to relation (2) would be encountered. The mortality proportion for the smallest diameter class would therefore be estimated using volume in diameter classes strictly greater than the smallest.

Discussion

The growth model presented in the present study, with the upgrowth and mortality components specified, projects future numbers of trees for each species/diameter class combination. There is no ingrowth component for even-aged stands and the total number of trees declines as stand age increases. While the total number of trees decreases, however, stand volume increases with age, as the initial diameter distribution shifts into larger diameter classes. Directly projecting stand volume or basal area by diameter class in a manner similar to that proposed for numbers of trees, however, is not as straightforward. Relationships must be incorporated into the projection model to ensure that as upgrowth occurs, stand volume or basal area also increase. If a diameter class contains 100 cubic feet of volume, for example, and upgrowth is 50 percent, the 50 cubic feet advancing into the next higher class would have to be converted to a greater volume or total volume growth would not occur during the period. No explicit consideration is required when numbers of trees are projected, however, as volume automatically increases when trees are shifted to larger diameter classes. For this reason, stand volume variables were specified as more relevant measures of density than numbers of trees. Using numbers of trees as a density measure implies lower

densities with increasing stand age, as the total number of trees decreases.

The stand-table projection approach to modeling forest growth is a difference equation method, as opposed to differential equation or instantaneous rate of change methods. By projecting growth over fixed time intervals, the approach recognizes the periodic nature of much forest growth data. Data requirements for estimation are not as severe as might be expected for mixed-species, however, due to the step-by-step development. Remeasurement data are required to estimate the potential upgrowth proportions, and the adjustment process and mortality component parameters.

Several modeling decisions must be made prior to data collection and component estimation. For example, although the projection system may be specified for any number of species, the number modeled for a given stand may be reduced by combining species with similar growth characteristics and value-by-size-class relationships. Also, although the growth period is fixed, diameter class size does not have to be the same for all species considered. Decisions concerning aggregating species, and diameter class size by species group must, however, also consider the effects on optimization. The number of variables in the formulation, for example, is directly related to the number of species group/diameter class combinations recognized.

Growth and thinning response in mixed-species hardwoods is difficult to model due to the biological diversity of such stands. Also, considering the need to integrate the growth model with an optimization procedure limits the possible approaches to those with relatively simple equation forms. The stand-table projection system proposed in this study was developed considering the necessary requirements, and was used in formulating an optimal thinning and rotation procedure.

III. THINNING MODEL FORMULATION

The thinning model formulated in the present study will enable derivation of optimal thinning schedules for mixed-species hardwood stands. The formulation will also enable determination of optimal rotation age, as final harvests will be included in the model. Implications of the growth model for the thinning model formulation will be discussed, followed by several factors regarding hardwood thinning which should be reflected by the formulation. Dynamic programming will also be considered, followed by a nonlinear programming formulation of the hardwood thinning problem. A complete statement of the hardwood thinning formulation, including variable definitions, is presented in Appendix A.

Growth Model Implications

The stand-table projection model, as previously specified, provides information on the number of trees by diameter class and species. This level of resolution will allow the thinning model to specify the number of trees to harvest over time, by diameter class and species. The specified growth model uses volume measures to reflect stand density in the upgrowth and mortality relations. Average volumes per tree for each species/diameter class combination represented are therefore necessary. Average volumes are

also necessary to derive dollar values for trees scheduled for harvest in the thinning model.

The growth model also affects the thinning model formulation in that the length of the growth period determines the thinning interval. That is, as growth is projected over fixed periods, opportunities to thin the stand are limited to fixed intervals, and rotation length is limited to discrete multiples of the growth period. With a stand currently of age 30, for example, using a 5-year growth period would result in possible rotation lengths of 30, 35, 40, etc. Final results from the thinning model should therefore be considered prior to setting the growth period length in the stand-table projection system.

An alternative to using the projection model growth interval was suggested by Adams and Ek (1975). If growth during the fixed period is assumed to accrue in a certain fashion, e.g., linearly, projections are possible for intervals other than initially implied by the growth model. This approach may be useful, for example, if growth data are available but the remeasurement period is inadequate from a thinning model standpoint.

Finally, the growth model will be used in a deterministic manner in the thinning model formulation. Possibilities for incorporating the stochastic nature of the growth model may be considered after developing a

deterministic formulation.

Hardwood Thinning Factors

The major aim in formulating a thinning model in the present study was to mathematically define the problem of deriving economically optimal harvest schedules for mixed-species hardwoods. The formulation must reflect the relevant economic and biological factors concerning harvests in such stands. Several factors which should be represented by the model will be discussed.

Harvests cannot exceed the volumes that exist and that can be grown during a given time period. The formulation must therefore limit harvests to the stand-table projections, i.e., the projection system must be an integral part of the thinning model formulation. The first phase of formulating the thinning problem will therefore be to represent the stand-table projection system in an optimization framework.

After representing the projection system in the formulation, other factors may be considered. An economic objective, for example, must be formulated. As shown by Gaffney (1960), and later by Samuelson (1976), maximization of Faustmann's (1849) soil expectation value (SEV) is the correct criterion for setting rotation length. SEV represents a present value or maximum bid price for bare land in forestry uses and in simplest form may be expressed

as:

$$SEV = HV / ((1+r)^{RL} - 1) \quad (4)$$

where:

HV=harvest value,

r=interest rate assumed, and

RL=rotation length.

Equation (4) assumes a timber income of HV dollars, every RL years in perpetuity. For typical upland hardwood stands, this assumption is untenable. As discussed by Klemperer et al. (1982), however, equation (4) may be restated for the case where only one rotation is considered, as presented in equation (5).

$$SEV = (HV + SEV) / (1+r)^{RL} \quad (5)$$

Maximizing the present value of land and timber over a finite investment period is therefore consistent with a Faustmann formulation and is used as the economic objective in the present study. Further discussion of this aspect of the hardwood model will be presented following the formulation of a mathematical programming objective function.

Another consideration in formulating the hardwood thinning model is representing tree quality. Reflecting differences in tree quality and recognizing the effects of thinning on this factor are especially important with the specification of an economic objective. That is, tree

quality is a major determinant of per unit stumpage prices, and can be adversely affected by heavy thinnings in hardwood stands.

Finally, the thinning model formulation must ensure that the results from optimization can be applied. For example, it may be necessary that volume removals exceed certain minimum levels, as landowners may be unable to market smaller quantities. Also, as per unit harvesting costs may be inversely related to volume, and as stumpage prices are directly related to harvesting costs, it may be necessary to model per unit prices in relation to volume removed.

Several factors have been discussed which should be reflected by the hardwood thinning model. The ability to incorporate these factors is a primary formulation goal. A major formulation emphasis will therefore be to develop a thinning model that is theoretically complete, i.e., a model capable of reflecting the important economic and biological relationships. If optimal thinning schedules and rotation length are to be derived, however, the feasibility of solving the model must also be considered during the formulation. A dynamic programming formulation of the problem was considered due to the many previous applications for thinning softwood forest types. Nonlinear programming was used, however, to develop a complete formulation of the

hardwood thinning model.

Dynamic Programming

As reviewed, several studies have applied dynamic programming to the problem of thinning and rotation for softwoods. The number of calculations necessary to obtain optimal thinning schedules is greatly reduced using dynamic programming, as each possibility need only be considered once. For this reason, a discrete dynamic programming formulation was considered for the mixed-species hardwood thinning problem. Formulating the thinning model as a dynamic program was rejected, however, for both modeling flexibility and dimensionality reasons.

Representing the important factors in thinning hardwood stands requires a great deal of modeling flexibility. A theoretically complete formulation must reflect the factors discussed regarding thinning in mixed-species stands. Previous applications of dynamic programming for softwood stands, however, have not shown evidence of sufficient modeling detail for the hardwood problem.

State-space dimensionality is another reason why the thinning model was not formulated as a dynamic program. Dimensionality becomes a problem for thinning studies when the resolution level involves harvests by diameter classes over time. As discussed by Hann and Brodie (1980) for a single species, let the discrete dynamic programming state

descriptors be classes of numbers of trees (TC), in each of (D) diameter classes, for each of the age periods (A) represented in the network. The network space is of dimension $D+1$, and the number of nodes in the network is $A(TC)^D$. The difficulties multiply when mixed-species are recognized. Letting S represent the number of species, D_i the number of diameter classes for the i th species, and assuming each species has a common value for TC, the number of dimensions of the network space is $\sum_{i=1}^S D_i + 1$, and the number of nodes in the network is $A(\sum_{i=1}^S (TC)^{D_i})$. For example, for a problem representing a stand with two species for five age periods, recognizing ten TC classes for each of ten diameter classes per species, the number of dimensions of the network space would be $5+5+1=11$, and the number of nodes in the network would be $5(10^{10}+10^{10})=10^{11}$, or 100 billion. As noted by Hann and Brodie (1980), the theoretically possible quickly becomes impossible in practical applications of dynamic programming to thinning problems recognizing diameter classes.

A recent study by Riitters et al. (1982) partially incorporated diameter classes in a discrete dynamic programming problem. Optimal thinning and rotation were derived for ponderosa pine, considering both timber and forage production as outputs. Diameter information was stored to enable the use of a diameter-class stand growth

model, allowing more realistic representation of the stand and of the effects of quality premiums. Thinning decisions for different diameter classes were not modeled, however, as each thinning was assumed to remove a constant proportion of trees from each diameter class. The effects of diameter distribution on thinning were thus only partially represented in the dynamic programming model for ponderosa pine.

Nonlinear Programming

Nonlinear programming was successfully applied by Adams and Ek (1974) in a study recognizing diameter classes in uneven-aged hardwoods. The formulation developed in the present study, however, must recognize species as well as diameter classes, for even-aged hardwood stand conditions and management goals. A proposed formulation will be presented and discussed, followed by convexity and problem size considerations.

Model Formulation

Selecting appropriate decision variables is a primary step in model formulation. Numbers of trees to cut from each species/diameter class combination, after each growth period were chosen for the thinning problem. Thinning guides will thus specify exact numbers to harvest from each combination, and the effects of such removals on future growth and harvest values will be considered during

optimization. The nonlinear programming constraints and objective function were formulated to represent the previously discussed hardwood thinning relationships.

Constraints. As previously discussed, the first phase in formulating the hardwood thinning problem involved representing the stand-table projection system. That is, constraints were developed to limit harvests, and to reflect the effects which cuttings would have on future growth. The following system of equation sets was developed in a manner similar to that of Adams and Ek (1974) for representing growth in uneven-aged stands.

$$N_{ijk}^R = N_{ijk}^I - N_{ijk}^C \quad (i=1, \dots, S \quad j=1, \dots, n_i \quad k=0) \quad (6)$$

$$N_{ijk}^R = N_{ijk-1}^R - N_{ijk}^U - N_{ijk}^M - N_{ijk}^C \quad (i=1, \dots, S \quad j=1 \quad k=1, \dots, G) \quad (7)$$

$$N_{ijk}^R = N_{ijk-1}^R - N_{ijk}^U - N_{ijk}^M - N_{ijk}^C + N_{i,j-1,k}^U \quad (i=1, \dots, S \quad j=2, \dots, n_i+k-1 \quad k=1, \dots, G) \quad (8)$$

$$N_{ijk}^R = N_{i,j-1,k}^U - N_{ijk}^C \quad (i=1, \dots, S \quad j=n_i+k \quad k=1, \dots, G) \quad (9)$$

$$N_{ijk}^* \geq 0 \quad (i=1, \dots, S \quad j=1, \dots, n_i+k \quad k=0, \dots, G) \quad (10)$$

* denotes R, I, C, U, and M

where:

N_{ijk} = number of trees of species i , in diameter class j after growth period k , and superscripts R, I, C, U, and M denote residual, initial, cut,

upgrowth, and mortality numbers, respectively,
 S =number of species,
 G =number of growth periods,
 n_i = initial number of diameter classes for
species i ,
 N_{ijk}^U and N_{ijk}^M are from the stand-table projection
model.

Equation sets (6) through (9) define the residual number of trees for each species/diameter class combination, after each growth period. Residual numbers are necessary for projecting growth in succeeding periods with the stand model. In this manner, thinnings affect growth during all periods after they occur. Relation set (10) merely represents non-negativity restrictions for all variables. N_{ijk}^C and N_{ijk}^R terms are variables in the formulation, while the N_{ijk}^I terms are constants, and the N_{ijk}^U and N_{ijk}^M terms are from the stand growth model.

As presented in equation set (6), the first thinning is allowed to occur now, i.e., after growth period zero. The residual numbers of trees after initial thinning, by diameter class and species, are calculated as the initial number for each combination minus the number cut. Allowing thinning to occur immediately makes possible $G+1$ harvests, i.e., now and after each of G growth periods. Values of zero for the decision variables, of course, indicate no

harvesting, and it is assumed that final harvest of the stand will occur immediately after the final growth period.

Equation set (7) defines the residual number of trees in the smallest diameter class for each species, after growth periods 1 through G. These numbers are defined by the corresponding residuals after the preceding period, minus upgrowth into the second diameter class, minus mortality during the growth interval, minus the number cut. Equation set (8) defines the residual number of trees for all diameter classes except the smallest and largest after each growth period, for each species. For diameter classes 2 through n_i+k-1 , a component must be added to reflect upgrowth from the class just smaller. Equation set (7) therefore differs from equation set (8) merely because for even-aged stands an upgrowth component is not added to the smallest diameter class for each species.

Equation set (9) defines the residual number of trees in the largest diameter class for each species, after each growth period. These residuals are comprised entirely of upgrowth from the next lower diameter class, minus the number cut. The number of diameter classes for species i after growth period k is represented by n_i+k , as the number of diameter classes recognized for each species increases by one for each period projected. This results for each species as upgrowth from the largest diameter class forms a

new highest diameter class, after each period.

In constraint sets (7), (8), and (9), upgrowth and mortality expressions occur. These terms correspond to stand-table projections, expressed as numbers of trees. Upgrowth and mortality are estimated by multiplying the estimated proportions by the appropriate residual number of trees at the start of the growth period. The projection model upgrowth and mortality expressions, written in terms of the thinning model decision variables, are presented in relations (11) and (12), respectively.

$$N_{ijk}^U = N_{ijk-1}^R \cdot (PP_{ijk}) \text{EXP} \left[b_1^{ij} (V_{T,k-1}^R) + \sum_{m=1}^s (b_{m+1}^{ij} (V_{m, \geq j, k-1}^R)) \right] \quad (11)$$

$$N_{ijk}^M = N_{ijk-1}^R \left(1 - \text{EXP} \left[b_{s+2}^{ij} (V_{T,k-1}^R) + \sum_{m=1}^s (b_{s+2+m}^{ij} (V_{m, \geq j, k-1}^R)) \right] \right) \quad (12)$$

where:

$$V_{T,k-1}^R = \sum_{i=1}^s \sum_{j=1}^{n_i+k-1} (V_{ij} N_{ijk-1}^R) = \text{total residual volume}$$

of the stand at the start of growth period k,

where V_{ij} is average volume per tree of species i, diameter class j,

$$V_{m, \geq j, k-1}^R = \sum_{q=j}^{n_i+k-1} (V_{mq} N_{mqk-1}^R) = \text{residual volume of}$$

each species ($m=1, \dots, S$) in diameter classes $\geq j$ (q is a diameter index ranging from j to n_i+k-1), at the start of growth period k, and

Other variables are as previously defined.

Relation (11) represents the number of trees of species

i , advancing from diameter class j to $j+1$ during growth period k . This number is the corresponding number at the beginning of the growth interval multiplied by the product of the appropriate potential proportion and the adjustment value (from relation (2)). Relation (12) represents the number of trees of species i , diameter class j , which are projected to die during growth period k . This number is the corresponding number at the beginning of the growth interval multiplied by the proportion dying (from relation (3)). Relations (11) and (12) may be substituted for the corresponding terms in constraint sets (7); (8), and (9). After the appropriate substitution in equation set (7), for example, and after combining terms, constraints of the form presented in relation (13) result.

$$N_{ijk}^R = (N_{ijk-1}^R) \left(\text{EXP} \left[b_{s+2}^{ij} (V_{T,k-1}^R) + \sum_{m=1}^s (b_{s+2+m}^{ij} (V_{m,>j,k-1}^R)) \right] \right) - (PP_{ijk}) \text{EXP} \left[b_1^{ij} (V_{T,k-1}^R) + \sum_{m=1}^s (b_{m+1}^{ij} (V_{m,>j,k-1}^R)) \right] - N_{ijk}^C \quad (13)$$

($i=1, \dots, S$ $j=1$ $k=1, \dots, G$)

Constraint set (13) represents the residual number of trees in the smallest diameter class for each species, after each period. Similar results are obtained upon substitution of relations (11) and (12) in constraint sets (8) and (9). These results are presented in the complete model statement in Appendix A.

The constraints expressed in equation sets (7), (8),

and (9) were specified to explicitly define residual numbers of trees. These definitions are still reflected after substituting for the growth model terms, however. Constraint set (13), for example, for the appropriate diameter class and species after each growth period, may be interpreted as:

$$[\text{Residual \#trees}] = [\text{\#Living}] - [\text{\#Upgrowth}] - [\text{\#Cut}]$$

Similar interpretations apply to the other constraint sets, after substituting and combining terms. For larger diameter classes, however, an upgrowth term is also added

Harvesting effects on quality and minimum harvest levels were also considered in formulating constraints in the thinning model. Two aspects of tree quality were considered in the model formulation. The first, reflecting differences in quality by size class and species, will be discussed in association with the objective function. The second aspect, the influence of thinning on quality, was modeled as constraining the volumes removed during thinning. That is, thinning volumes may be constrained by setting upper bounds, preventing thinnings heavy enough to result in quality losses from epicormic branching, enlarged lower limbs, etc. For upland oak stands, for example, Dale (1972) recommended that thinnings be constrained to leave at least 50 percent stocking based on Gingrich's (1964) tree-area

ratio equation. In general, such constraints should be used to ensure that residual volumes are sufficient to maintain the initially assumed value-by-size-class relationships through the final harvest. Equation set (14) represents such constraints for thinning volumes removed after each growth period.

$$\sum_{i=1}^s \sum_{j=1}^{n_i+k} (V_{ij} N_{ijk}) \leq H_{1k} \quad (k=0, \dots, G-1) \quad (14)$$

where:

H_{1k} represents a maximum harvest volume after growth period k , and

Other variables are as previously defined.

As cutting constraints should not apply to the final harvest (after growth period G), constraint set (14) allows maximum thinning levels up through period $G-1$. While constraint set (14) prevents thinning too heavily because of possible adverse effects on tree quality, constraints were also considered for marketing reasons. That is, landowners may be unable to market small thinning volumes, requiring minimum total volumes for each thinning. These constraints should only be observed, however, if harvesting occurs. Specifying minimum thinning volumes must not preclude the possibility of not cutting, i.e., choosing not to thin. Constraint sets (15) and (16) are specified to allow setting minimum levels for total volumes removed, if thinning is

performed.

$$\sum_{i=1}^s \sum_{j=1}^{n_i+k} (V_{1j} N_{ijk}^C) \geq H_{2k} X_k \quad (k=0, \dots, G-1) \quad (15)$$

$$0 \leq X_k \leq 1 \quad (k=0, \dots, G-1) \quad (16)$$

where:

H_{2k} is a minimum harvest volume after period k ,

significant only if thinning occurs,

$X_k = 1$ if thinning occurs after period k , or

equals 0 otherwise, and

Other variables are as previously defined.

Constraint set (15) represents the necessary relationship after each relevant growth period, assuming X_k equals 1 when thinning occurs and 0 if it does not occur. If thinning occurs after a certain growth interval, for example, and $X_k = 1$, constraint set (15) results in a thinning volume greater than or equal to H_{2k} . If thinning does not take place, however, and $X_k = 0$, the right side of the relevant inequality is insignificant. To ensure that X_k is unity if thinning occurs after period k , the right hand side of constraint set (14) is changed to $H_{1k} X_k$, as presented in the complete model statement of Appendix A.

The variable X_k represents the binary choice of thinning versus not thinning after period k . Allowing X_k to range between 0 and 1, however, avoids the differentiability and combinatorial problems associated with incorporating

discrete 0-1 variables. The X_k variables may be permitted to vary continuously between 0 and 1, with extreme discrete values being forced by suitably adjusting the objective function. That is, selecting M as a large positive constant, one may add the objective function terms presented in relation (17).

$$-MX_k(1-X_k) \quad (k=0, \dots, G) \quad (17)$$

These terms penalize values of X_k different from either 0 or 1. Provided M is large enough to offset any potential gains from non-binary X_k values, optimal values close to either 0 or 1 will result.

The relation presented in (17) is convex, yet the objective is to maximize present value. The term therefore results in a nonconvex relationship. The nonlinear programming problem is already nonconvex, however, as will be demonstrated subsequently. Specifying appropriate values for M will be considered in demonstrating the formulated thinning model. An alternative to the preceding technique would be to solve the problem for fixed (0,1) values of the X_k variables, comparing the optimum objective values obtained in each case.

Constraint sets (15) and (16), and the objective function terms in (17) provide a means of modeling thinning volumes considered minimum for marketing reasons. Certain harvest levels may also be required to recover the fixed

costs associated with thinning. This aspect of the model formulation, however, will be discussed in association with the objective function.

Objective Function. Maximizing the present value of both land and timber was specified as the economic objective for the hardwood thinning model. The objective function was formulated as the present value of all timber harvested, plus the present value of selling the land after final harvest. While owners of hardwood timberland may or may not wish to sell their land after final harvest, representing the possible value is necessary to determine the final harvest age which maximizes the present value of both land and timber. The land sale value assumed therefore replaces SEV in the numerator of equation (5). The value assumed for land sale may be higher than the SEV, if alternative uses for the land are considered.

Decision variables for the hardwood thinning model were specified as the number of trees to cut from each species/diameter class combination, after each growth period. The important elements for determining the present value of timber harvests are therefore available. That is, size and species should adequately reflect per unit timber values, and the relevant growth periods define the future points in time when harvest incomes occur. Equation (18) represents the objective function in present value terms,

assuming constant per unit prices.

$$\text{Maximize: PV} = \left\{ \sum_{k=0}^G \left[\sum_{i=1}^s \sum_{j=1}^{n_i+k} [P_{ij} / (1+r)^{kt}] N_{ijk} \right] - MX_k(1-X_k) \right\} + [L / (1+r)^{Gt}] \quad (18)$$

where:

PV=present value of land and timber,

P_{ij} =stumpage value per tree for species i ,

diameter class j , calculated as the price per unit of volume times the average volume per tree,

r =real alternative rate of return,

t =number of years per growth period,

L =land sale value, and

Other variables are as previously defined.

The objective function should adequately reflect differences in value due to quality, as prices are input by size class and species. The thinning model is intended for guidance in making stand-level decisions. For a given stand, such quality variables as proportion of clear bole, limb size, etc., should be closely related to diameter class and species. The per unit prices assumed for a given stand should therefore reflect distinctions between products such as pulpwood and sawtimber, as well as any quality distinctions which may be associated with the larger size classes in the stand. As previously discussed, the thinning

model may also include constraints to ensure that quality is not adversely affected by thinning, thereby maintaining the initially implied price/quality relationships for the stand.

As seen in equation (18), a present value for land sale after period G is added to the present value of timber from thinnings and final harvest. This term is a constant in deriving optimal thinning schedules for a given rotation age, but will affect the determination of which rotation age is optimal. That is, optimal rotation length may be derived by solving the thinning problem for one growth period, two growth periods, etc., and examining the resulting present values of land and timber. Optimal thinning and rotation are thus simultaneously derived, comparing the present values from solving the thinning model for increasing numbers of growth periods.

Harvesting costs were the final aspect of hardwood thinning modeled in the objective function. A theoretically complete thinning model must allow prices received to reflect the costs of thinning. Per unit prices may, for example, be modeled in relation to the proportion of the stand harvested. Incorporating an assumed relationship between stumpage prices and the stand proportion harvested was considered, as total volume cut and total stand volume may be derived from the variables N_{ijk}^C and N_{ijk}^R . Such relationships, however, result in a fractional objective

function, an undesirable property in programs with nonlinear constraints.

In a Douglas-fir thinning model, Brodie and Kao (1979) modeled stumpage prices and variable logging costs in relation to the quadratic mean diameter of trees removed. A fixed entry cost for thinning was also subtracted from the value function. In another dynamic programming application, Riitters et al. (1982) modeled the contribution of timber harvests to the return function as the present value of the difference between total harvest value and a fixed thinning entry cost. Total harvest value for a particular thinning was calculated as the sum over all diameter classes, of the number of trees harvested multiplied by a constant stumpage price for each class. As each diameter class is explicitly recognized in the function, variable costs are reflected by the per unit stumpage prices assumed for each class.

The approach used in the present study for incorporating harvesting costs in the hardwood thinning model is similar to that of Riitters et al. (1982). That is, variable costs of thinning should be reflected by the per unit stumpage prices assumed for each diameter class, yet fixed entry costs will be subtracted for each harvest which occurs. The approach used to incorporate such fixed costs involves using the X_k variable created to reflect when thinning does and does not occur. Letting FC represent a

fixed thinning entry cost, the following terms are added to the objective function:

$$-X_k(FC)/(1+r)^{kt} \quad (k=0, \dots, G-1) \quad (19)$$

Fixed entry costs are therefore only incurred when thinning takes place, i.e., when X_k approaches 1. Also, as fixed costs are necessary after final harvest, X_G in the objective function is defined equal to 1. The final form of the objective function is presented in the complete model statement of Appendix A.

Convexity

Problem convexity is an important property in nonlinear programming as the absence of locally optimal solutions which are not globally optimal is assured for convex programs. Hence, if a solution cannot be improved by a local perturbation, it may be declared globally optimal. For convex programs, therefore, the first-order Kuhn-Tucker local optimality conditions are necessary (under certain constraint qualifications) and sufficient to characterize a global optimum. For non-convex programs, however, the Kuhn-Tucker conditions are not sufficient and solutions meeting these conditions may not even represent local optima. The hardwood thinning model formulated in this study is non-convex. The residual defining constraints represent non-convex relations, as demonstrated in Appendix B, and the binary relationships result in non-convex terms in the

objective function.

Various techniques have been used to deal with obtaining optimal solutions to nonconvex programs. These techniques will be considered in solving for optimal thinning and rotation in a demonstration of the mixed-hardwood model.

Program Size

Evaluating program size is often necessary in nonlinear programming as solution algorithms may specify maximum numbers of variables and constraints. The gradient projection algorithm used by Adams and Ek (1974), for example, allowed a maximum of 40 variables and 80 constraints. Of the currently available nonlinear programming codes listed by Waren and Lasdon (1979), nine had fixed limits on both variables and constraints. Program size in the present study was evaluated by developing equations predicting the numbers of variables and constraints, based on the number of species, diameter classes, and growth periods projected. Reference will be made to equation sets in the complete model statement of Appendix A. Variables used have been previously defined.

Number of Variables. The residual defining constraint sets, (A2) through (A5), require two sets of variables, numbers of trees cut and residual numbers of trees for each species/diameter class combination, after each growth

period. Residual numbers of trees variables are not required after period G, however, as final harvest occurs. From constraint set (A2), two sets of variables are required for each species/diameter class combination. Hence, the number of variables required for constraint set (A2) is given by:

$$2\left(\sum_{i=1}^S n_i\right) . \quad (20)$$

The number of variables required for constraint sets (A3) and (A4) may be represented as a total count minus the number of residual variables counted after period G. The number of such variables is:

$$2\left[\sum_{k=1}^G \sum_{i=1}^S (n_i+k-1)\right] - \left[\sum_{i=1}^S (n_i+G-1)\right] . \quad (21)$$

The number of variables represented by constraint set (A5) is determined similarly as:

$$2(G*S) - S . \quad (22)$$

One other variable, X, is used in the model statement of Appendix A, required after periods 0 through G-1. Adding G to the sum of (20), (21), and (22), and simplifying yields the total number of thinning model variables:

$$G(S+1) + \left(\sum_{i=1}^S n_i\right) + 2\left(\sum_{k=1}^G \sum_{i=1}^S [n_i+k-1]\right) . \quad (23)$$

Number of Constraints. The number of constraints in constraints sets (A2) through (A5) in the thinning model

formulation, beginning with the residual defining constraints, are given below in relations (24) through (27).

$$\sum_{i=1}^s n_i \quad (24)$$

$$S * G \quad (25)$$

$$G \left(\sum_{i=1}^s (n_i) - 2S \right) + S \left(\sum_{k=1}^G k \right) \quad (26)$$

$$S * G \quad (27)$$

Equation sets (A6) through (A8), representing (3*G) constraints, must also be included. Non-negativity restrictions are not included in the constraint count, however. The total number of constraints in the thinning model formulation is therefore:

$$(G+1) \left(\sum_{i=1}^s n_i \right) + S \left(\sum_{k=1}^G k \right) + (3 * G) \quad (28)$$

The numbers of variables and constraints in the hardwood thinning model may be predicted with equations (23) and (28), respectively. The effects of program size on the choice of a solution algorithm will be discussed in a demonstration of the model.

Discussion

Several aspects of the hardwood thinning model formulated in the present study warrant further discussion. One area is the discretization of the thinning interval and rotation age. While numbers of trees to cut are continuous

variables in the formulation, thinnings are only allowed now and after a discrete number of growth intervals, each of fixed length. Also, the above model limits possible rotations to multiples of the growth period. For even-aged upland hardwoods, however, discretizing the timing of harvests should not affect the usefulness of model results. Stands with relatively slow growth rates may not be thinned as frequently as stands of faster growing species. Also, rotation lengths for such stands are commonly specified in multiples of 5 or 10 years.

While the timing of harvests is discrete in the model formulated, the harvest intensity for each species/diameter class combination is a continuous variable. Number of trees, however, is inherently integer valued. This problem would not be avoided by choosing volume as the decision variable, as harvest volumes specified by diameter class must eventually be related to an integer number of trees. Continuous solutions in the thinning model demonstration will be rounded to the nearest integer solution. According to the classification presented by Taha (1975), the thinning model formulation is a direct integer problem. This class of integer problems is the only one for which rounding should be considered. As discussed by Taha (1975), however, a solution obtained by rounding optimal continuous values may not be an integer optimum, although it is likely to be

near the optimum.

Applying the formulated thinning model to young stands is another area for discussion. Stands too young for commercial thinning may be projected to thinning age within the optimization model. This may be accomplished by specifying no harvesting until after a sufficient number of growth periods, or by specifying zero prices for the appropriate growth periods. A more efficient approach, however, is to project young stands to thinning age prior to applying the optimization model. This approach avoids the additional variables and constraints necessary for incorporating initial growth periods where thinning is not an option.

Further consideration should also be given to certain thinning model constraint sets. For example, the possibility of setting minimum thinning levels was modeled such that the constraints applied only if thinning occurred. Maximum levels for thinning volumes were incorporated, however, without determining whether cuttings represented thinnings or final harvest. This determination was not necessary, as final harvest is assumed after the last growth period modeled. All other harvests may therefore be subject to maximum thinning volumes.

Also regarding the constraints, setting minimum volume levels for thinnings may not be required. Fixed costs were

incorporated in the objective function, but are incurred only if harvesting occurs. To realize a net gain from harvesting, sufficient volume must be removed to recover the fixed costs. The thinning model may therefore be solved without minimum harvest volumes, adding such constraints if the volumes specified are still considered inadequate for marketing or other reasons.

Other types of constraints may also be included in the nonlinear programming thinning formulation. For example, non-timber considerations involving wildlife, recreation, watershed, etc., may be incorporated. Such relationships, however, must be expressed as functions of volumes cut and residual volumes, either total or by diameter class and/or species, after each growth period. Rather than using constraints, nontimber values might also be included as either constant or varying (with density) values, added to the objective function depending on whether or not final harvest has occurred, i.e., whether or not standing timber is present. The ability to reflect non-timber considerations can be an important aspect in modeling upland hardwoods, as both public and private landowners frequently consider such factors in their harvest decisions.

The thinning model was formulated with decision variables specifying the number of trees to cut from each species/diameter class combination. Aggregating numbers of

trees into small groups may be considered if solutions to the thinning model formulation cannot be obtained at the level specifying exact numbers of trees. Thinning schedules from such a formulation would prescribe numbers of tree groups of 2, 3, 4, etc., to be harvested from each species/diameter class combination.

Finally, applying thinning model prescriptions in the field may require adjustments and managerial judgement. This is true in implementing results from any such model. In general, the stand should be defined small enough that the thinning formulation accurately represents the real system being modeled. The accuracy with which model results can be applied is directly related to how closely the input data represents the stand to be thinned. The thinning model may be used to develop prescriptions for wide application to frequently occurring stand types, or to derive thinning policies for individual stands.

THINNING MODEL DEMONSTRATION

The thinning model developed in the preceding chapter is based on a growth model tentatively specified for stand-table projection of mixed-species hardwoods. Although data were not available for estimation of the growth model parameters, the thinning model will be demonstrated using assumed parameter values. Specification of the growth model parameters will be discussed, followed by thinning model formulations for two problem cases. To complete the thinning model demonstration, three techniques will be evaluated for solving the nonlinear programming formulations.

Growth Model Parameter Specification

Statistical estimation of the growth model parameters requires remeasurement data for the upgrowth and mortality parameters, and the potential proportions of upgrowth. The optimization aspects of the thinning model were investigated in the absence of such data by specifying a hypothetical, mixed-species stand, and assigning parameter values for projecting the stand. Growth model parameters were specified for an assumed stand of age 30, to be projected with 5-year growth intervals, with or without thinning, to age 45. The stand assumed for demonstration is comprised of

two species groups, a faster growing, higher valued group such as yellow-poplar, and a slower growing, lower valued group such as mixed-oaks. These groups will be referred to as species groups 1 and 2, respectively. The initial distribution of trees by diameter class and the average merchantable volumes per tree used in the demonstration are presented in Table 1. The distribution of the total number of trees by diameter class was compared to the even-aged upland hardwood distributions presented by Gingrich (1967). Height-diameter relationships were assumed for each species and merchantable volumes were obtained through linear interpolation of volumes presented by Schnur (1937). Volumes presented for species 1 correspond to yellow-poplar while those for species 2 correspond to white-oak.

To specify growth model parameters which would adequately project the initial stand, broad biological considerations were made. These considerations will be discussed, followed by the final parameter values used in the demonstration.

Biological Considerations

As species 1 was considered to be the faster growing species, growth model parameters were specified to yield relatively higher upgrowth proportions for this group. Also, as faster growing species are often less tolerant of competition, parameters for species 1 were specified to

Table 1. Initial stand-table and average volumes per tree assumed for the thinning model demonstration.

Diameter Class (in.)	Species 1		Species 2	
	# Trees	Vol./Tree*	# Trees	Vol./Tree*
2-3.9	14	0.00	60	0.00
4-5.9	55	1.95	75	1.40
6-7.9	79	6.61	52	5.04
8-9.9	45	12.80	10	10.55
10-11.9	--	19.94	--	17.86
12-13.9	--	28.62	--	26.44
14-15.9	--	39.38	--	36.30
Total Trees	193		197	
Total Volume	1205.44		472.58	

*Cubic-foot volume to a 4" top (o.b.), from Schnur's (1937) yellow-poplar and white-oak volume tables, for assumed height/diameter relationships.

result in higher mortality than species 2, under similar conditions. Species 1 mortality was also modeled as being more sensitive to stand volumes in greater diameter classes. For both species, mortality was modeled such that larger diameter classes experienced lower proportions dying. Also for both species, the relative effects of competition from smaller diameter classes, or understory, were modeled as diminishing as diameter increases.

A major assumption in the growth model parameter specification was that for both upgrowth and mortality, the effects on the residual stand of cutting either species would be the same. That is, $b_{2}^{ij} = b_{3}^{ij}$ and $b_{5}^{ij} = b_{6}^{ij}$ in relations (11) and (12), respectively, for all individual combinations of i and j . This property may or may not hold for actual mixed-species stands. For the present analysis, however, the assumption expedited the specification of parameters without detracting from the usefulness of the demonstration.

Parameter Values

Biological considerations assisted in defining several general relationships between growth model parameters and predicted results. Constrained by these considerations, parameter values were assigned such that realistic upgrowth and mortality proportions were predicted by the growth model. Parameter values, including the potential

proportions of upgrowth, were therefore adjusted until reasonable upgrowth and mortality estimates were generated. Final parameter values used in the thinning model demonstration are presented in Tables 2, 3, and 4. A total of 112 values were assigned.

For the assignment and adjustment process, growth model projections were made for the original stand (Table 1) for 1, 2, and 3 growth periods of 5 years each, corresponding to stand development from age 30 to 45. Growth model results for upgrowth and mortality for all species/diameter class combinations, as well as aggregate stand volume projected, were examined for thinning intensities ranging from no thinning to removal of over half the stand. Parameter values were adjusted until growth model projections for up to 3 periods were comparable to the even-aged hardwood results presented by Dale (1972), Gingrich (1971), and Schnur (1937). Projections beyond age 45 were not of interest in the present study, as the thinning model demonstration will be limited to 3 growth periods.

Thinning Model Examples

Thinning model formulations were developed and solved for two examples. A relatively small problem, Case I, was studied to provide insight into the structure and solution of the more complete formulation, Case II, of the thinning model for the initial stand. Case II is further divided

Table 2. Potential proportions of upgrowth assumed for the thinning model demonstration (relation (11)).

Diameter Class (in.)	Growth Period 1		Growth Period 2		Growth Period 3	
	Species 1	Species 2	Species 1	Species 2	Species 1	Species 2
2-3.9	.200	.150	.150	.100	.100	.050
4-5.9	.450	.250	.350	.200	.300	.150
6-7.9	.575	.350	.550	.300	.500	.250
8-9.9	.700	.450	.650	.400	.600	.350
10-11.9	---	---	.750	.500	.700	.450
12-13.9	---	---	---	---	.850	.550

Table 3. Growth model upgrowth (b_1 , b_2 , b_3) and mortality (b_4 , b_5 , b_6) parameters assumed for species 1 for the thinning model demonstration (relations (11) and (12)).

Diameter Class (in.)	Parameter					
	b_1	b_2	b_3	b_4	b_5	b_6
2-3.9	-.0006813	-.0002524	-.0002524	-.0000908	-.0000252	-.0000252
4-5.9	-.0003668	-.0002494	-.0002494	-.0000227	-.0000083	-.0000083
6-7.9	-.0003659	-.0001990	-.0001990	-.0000076	-.0000059	-.0000059
8-9.9	-.0003028	-.0002497	-.0002497	-.0000038	-.0000038	-.0000038
10-11.9	-.0002300	-.0002500	-.0002500	-.0000030	-.0000027	-.0000027
12-13.9	-.0001700	-.0003000	-.0003000	-.0000020	-.0000018	-.0000018

Table 4. Growth model upgrowth (b_1, b_2, b_3) and mortality (b_4, b_5, b_6) parameters assumed for species 2 for the thinning model demonstration (relations (11) and (12)).

Diameter Class (in.)	Parameter					
	b_1	b_2	b_3	b_4	b_5	b_6
2-3.9	-.0006056	-.0002271	-.0002271	-.0000379	-.0000076	-.0000076
4-5.9	-.0004164	-.0002079	-.0002079	-.0000088	-.0000041	-.0000041
6-7.9	-.0003280	-.0001621	-.0001621	-.0000038	-.0000022	-.0000022
8-9.9	-.0002649	-.0002123	-.0002123	-.0000012	-.0000006	-.0000006
10-11.9	-.0002000	-.0003000	-.0003000	-.0000009	-.0000002	-.0000002
12-13.9	-.0001200	-.0004000	-.0004000	-.0000005	-.0000001	-.0000001

into Cases IIa, IIb, and IIc, representing formulations for 1, 2, and 3 growth periods, respectively. The initial assumptions used in the models developed for both examples will be discussed, followed by the explicit formulations to be solved.

Input Assumptions

Assumptions regarding land sale value, fixed costs, interest rates, and per unit prices were necessary to define the objective function coefficients for the example problems. The same values were assumed for these inputs for both cases formulated. Certain input assumptions were relaxed in a limited sensitivity analysis, to be discussed following the problem formulations and solution analysis. Input values initially assumed are summarized in Table 5.

A constant land sale value of \$300 was assumed for the example problems. No attempt was made to establish actual post-clearcut land values or land appreciation rates for a particular region. Realistic estimates of land sale value over time should not be difficult to obtain, however, for applications of the model to actual stands in a given locality. The market value for bare land represents the value of land in its highest and best use and therefore represents an upper bound on the SEV determined considering forestry uses.

Fixed costs of \$4 per acre were used in the thinning

Table 5. Input values initially assumed for determining present values in the thinning model demonstration.

Land Sale Value.....	L=\$300/acre
Fixed Thinning Costs.....	FC=\$4/acre
Real Rate of Return (decimal percent).....	r=.08
Stumpage Prices:	
Species 1, 10+ inches.....	P =\$0.233870/cu.ft.
Species 1, <10 inches.....	P =\$0.050828/cu.ft.
Species 2, 10+ inches.....	P =\$0.204980/cu.ft.
Species 2, <10 inches.....	P =\$0.042890/cu.ft.

model examples. These costs are associated with marking and sale administration, and are included in the model in lieu of fixed logging costs, as data were not available to establish per unit stumpage prices net of such costs. Administrative costs are fixed, however, and for purposes of model demonstration will represent the cost variable (FC) defined in the theoretical formulation and discussion. Fixed costs were applied to thinnings and final harvest.

A real discount rate of 8 percent was assumed for determining present values in the example problems. Some thinning studies, e.g., Riitters et al. (1982), have used rates as low as 3 percent. For the present demonstration, however, private ownership is assumed and the rate represents a before-tax, real alternative rate of return. The assumed rate was reduced to 5 and 3 percent in subsequent analyses.

In both examples, it is assumed that all material harvested can be sold at the stumpage prices assigned. Per unit stumpage prices for the model demonstration were obtained by averaging monthly prices reported for the Southeast in Timber Mart-South¹ for January through August, 1982. Random-length log prices were applied for trees in diameter classes 10 inches and over, while roundwood prices

¹Monthly report of Timber Mart-South, Inc., published by F.W. Norris, Highlands, N.C.

were used for trees under this limit. Yellow-poplar and mixed-hardwood prices were used for logs of species groups 1 and 2, respectively. For roundwood diameters, soft-hardwood prices were used for species 1, prices for chemically processed hardwoods were used for species 2. Prices per thousand board feet (Doyle) and per standard cord were converted to values per cubic foot using average conversion factors, also published in Timber Mart-South. Sawtimber price differentiation for quality was not included in the initial analysis. The initial values assumed for the thinning model demonstration are presented in Table 5. Finally, a real stumpage price increase of 2 percent per year was assumed for sawlog diameters of both species. Although real increases in stumpage value are not expected for lower quality hardwoods in the immediate future, the U.S. Forest Service (1982) has projected price increases beyond the next few decades.

Case I

Two examples of the thinning model were formulated for demonstration. Case I is formulated for a stand of very simple structure, while Case II represents the thinning model for the stand used to assign parameter values, summarized in Table 1. Case I is formulated for a stand of age 40, which on a per acre basis has 49 trees of species 1 in diameter class 8-9.9, and 39 trees of species 1 in the

10-11.9 inch class. For species 2, the stand has 49 trees in diameter class 6-7.9, and 19 trees in the 8-9.9 inch class. The stand is therefore comprised of 156 trees, with a total merchantable volume of 1852 cubic feet. The growth model parameters used for Case I are the appropriate values from Tables 2, 3, and 4. The previously discussed input assumptions are the same for both examples.

For the Case I problem, the stand will be projected for a single 5-year growth period. From equations (23) and (28), the thinning model formulation involves 15 variables and 13 constraints. The purpose of the formulation is to determine the thinning policy, applied now, which maximizes the present value of land and timber over the next 5 years. The stand may be clearcut now, thinned now and clearcut in 5 years, or left unthinned and clearcut in 5 years. It is assumed that if thinning occurs, volume removed must range between 30 and 50 percent of the pre-thinning stand volume. The initial stand-table for Case I was specified so that the optimal thinning policy could be derived through an exhaustive search of all possible thinning regimes. Case I will be used to evaluate solution techniques and provide insight into the structure of the second example, where the optimal solution is unknown.

The thinning model was formulated for Case I following the equation sets presented in Appendix A. This formulation

is presented in Table 6, where for ease of presentation, the following substitutions have been made for the variables used in the Appendix.

$$\begin{array}{ll}
 X(1) = N_{120}^R & X(2) = N_{120}^C \\
 X(3) = N_{130}^R & X(4) = N_{130}^C \\
 X(5) = N_{210}^R & X(6) = N_{210}^C \\
 X(7) = N_{220}^R & X(8) = N_{220}^C \\
 X(9) = N_{121}^R & X(10) = N_{131}^R \\
 X(12) = N_{141}^R & X(12) = N_{211}^R \\
 X(13) = N_{221}^R & X(14) = N_{231}^R
 \end{array}$$

The present value equation in Table 6 represents the sum of discounted land sale value and discounted values per tree multiplied by numbers of trees cut. The growth model coefficients in the constraints are expressed in terms of numbers of trees, i.e., the original coefficients are multiplied by average volumes per tree and aggregated. The Case I formulation presented in Table 6 may be simplified through substitution. That is, an equivalent formulation may be obtained by:

- (1) substituting $X(1)=49-X(2)$, $X(3)=39-X(4)$, $X(5)=49-X(6)$, and $X(7)=19-X(8)$ into the residual-defining constraints for period 1,
- (2) adding constraints $X(2) \leq 49$, $X(4) \leq 39$, $X(6) \leq 49$, and $X(8) \leq 19$,
- (3) replacing $X(9)$ through $X(14)$ in the objective function with the expressions defined by the remaining equality constraints, and
- (4) simplifying and combining terms.

Table 6. Case I thinning model formulation, following the equation sets presented in Appendix A.

Maximize: PV = \$201.45+0.65X(2)+4.66X(4)+0.22X(6)+0.45X(8)+0.44X(9)+3.50X(10)
 +5.03X(11)+0.15X(12)+0.31X(13)+2.75X(14)-4X(0)-MX(0)(1-X(0))

Subject to:

$$\begin{array}{ll} X(1) + X(2) = 49 & X(3) + X(4) = 39 \\ X(5) + X(6) = 49 & X(7) + X(8) = 19 \end{array}$$

$$X(9) - X(1) (\text{EXP}(-.0000973X(1) - .0001515X(3) - .0000192X(5) - .0000802X(7)) - .6\text{EXP}(-.007072X(1) - .0110169X(3) - .0015261X(5) - .0058289X(7))) = 0$$

$$X(12) - X(5) (\text{EXP}(-.0000768X(1) - .0001196X(3) - .0000302X(5) - .0000633X(7)) - .25\text{EXP}(-.0062733X(1) - .0097726X(3) - .0024701X(5) - .0051706X(7))) = 0$$

$$X(10) - X(3) (\text{EXP}(-.0000384X(1) - .0001137X(3) - .0000151X(5) - .0000317X(7)) - .7\text{EXP}(-.002944X(1) - .0095712X(3) - .0011592X(5) - .0024265X(7))) - .6X(1)\text{EXP}(-.007072X(1) - .00110169X(3) - .0015261X(5) - .0058289X(7)) = 0$$

$$X(13) - X(7) (\text{EXP}(-.000023X(1) - .0000359X(3) - .0000091X(5) - .000019X(7)) - .35\text{EXP}(-.0061082X(1) - .0095154X(3) - .0024051X(5) - .0050345X(7))) - .25X(5)\text{EXP}(-.0062733X(1) - .0097726X(3) - .0024701X(5) - .0051706X(7)) = 0$$

$$X(11) - .70X(3)\text{EXP}(-.002944X(1) - .0095712X(3) - .0011592X(5) - .0024265X(7)) = 0$$

$$X(14) - .35X(7)\text{EXP}(-.006108X(1) - .0095154X(3) - .0024051X(5) - .0050345X(7)) = 0$$

$$\begin{array}{l} 12.8X(2) + 19.94X(4) + 5.04X(6) + 10.55X(8) \leq 926X(0) \\ 12.8X(2) + 19.94X(4) + 5.04X(6) + 10.55X(8) \geq 370X(0) \end{array}$$

$$X(0) \leq 1, \quad X(i) \geq 0 \quad (i=0,1,\dots,14)$$

The Case I formulation obtained with the above steps is presented in Table 7. The Table 7 formulation has 5 variables and 7 constraints, compared to 15 variables and 13 constraints in Table 6. The problem is now comprised of a nonlinear objective function, constrained by a small set of linear inequalities (5 of which are merely upper bounds). Following substitution, the nonlinear program is written entirely in terms of the true decision variables, the number of trees to cut from each species/diameter class combination. Non-negativity expressions are not required (in Table 7) for variables X(9) through X(14). The residual-defining equations in Table 6 represent proportions living minus proportions of upgrowth. Logically, the upgrowth proportion in a given diameter class cannot exceed the proportion of trees living in that class, following a growth period. The result can also be shown algebraically, however, based on the relative magnitudes of the exponential coefficients in Table 6.

Case II

The second thinning model example is comprised of three problems. Cases IIa, IIb, and IIc correspond to thinning model formulations for the stand initially assumed for projection (Table 1), for 1, 2, and 3 growth periods, respectively. Formulations for Case IIa are presented using vector notation in Tables 8 and 9. Vectors used in these

Table 7. Case I thinning model formulation, following substitution and simplification.

$$\begin{aligned}
 \text{Maximize: } PV = & \$201.45 + 0.65X(2) + 4.66X(4) + 0.22X(6) + 0.45X(8) - 4X(0) - MX(0)(1-X(0)) \\
 & + (49-X(2)) (.44\text{EXP}(-.01314 + .0000973X(2) + .0001515X(4) + .0000192X(6) + .0000802X(8)) \\
 & \quad + 1.836\text{EXP}(-.9617151 + .007072X(2) + .0110169X(4) + .0015261X(6) + .0058289X(8))) \\
 & + (39-X(4)) (3.5\text{EXP}(-.00766 + .0000384X(2) + .0001137X(4) + .0000151X(6) + .0000317X(8)) \\
 & \quad + 1.071\text{EXP}(-.6204371 + .002944X(2) + .0095712X(4) + .0011592X(6) + .0024265X(8))) \\
 & + (49-x(6)) (.31\text{EXP}(-.01111 + .0000768X(2) + .0001196X(4) + .0001196X(6) + .0000633X(8)) \\
 & \quad + .04\text{EXP}(-.9077994 + .0062733X(2) + .0097726X(3) + .0024701X(6) + .0051706X(8))) \\
 & + (19-X(8)) (.31\text{EXP}(-.003334 + .000023X(2) + .0000359X(4) + .0000091X(6) + .000019X(8)) \\
 & \quad + .854\text{EXP}(-.8838532 + .0061082X(2) + .0095154X(4) + .0024051X(6) + .0050345X(8)))
 \end{aligned}$$

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Subject to:

$$\begin{aligned}
 X(2) & \leq 49 \\
 X(4) & \leq 39 \\
 X(6) & \leq 49 \\
 X(8) & \leq 19
 \end{aligned}$$

$$\begin{aligned}
 12.8X(2) + 19.94X(4) + 5.04X(6) + 10.55X(8) & \leq 926X(0) \\
 12.8X(2) + 19.94X(4) + 5.04X(6) + 10.55X(8) & \geq 370X(0)
 \end{aligned}$$

$$\begin{aligned}
 X(0) & \leq 1 \\
 X(i) & \geq 0 \quad (i=0,1,2,3,4)
 \end{aligned}$$

Table 8. Case IIa thinning model formulation, following the equation sets in Appendix A (vectors are defined in Table 10).

$$\text{Maximize: } PV = \$201.45 + P_1^T N_1 - 4X_0 - MX_0(1-X_0).$$

Subject to:

$$\begin{aligned} N_{110}^R + N_{110}^C &= 14, & N_{210}^R + N_{210}^C &= 60 \\ N_{120}^R + N_{120}^C &= 55, & N_{220}^R + N_{220}^C &= 75 \\ N_{130}^R + N_{130}^C &= 79, & N_{230}^R + N_{230}^C &= 52 \\ N_{140}^R + N_{140}^C &= 45, & N_{240}^R + N_{240}^C &= 10 \\ N_{111}^R - N_{110}^R (\text{EXP}(B_1^T N_2) - .200\text{EXP}(B_2^T N_2)) &= 0 \\ N_{211}^R - N_{210}^R (\text{EXP}(B_3^T N_2) - .150\text{EXP}(B_4^T N_2)) &= 0 \\ N_{121}^R - N_{120}^R (\text{EXP}(B_5^T N_2) - .450\text{EXP}(B_6^T N_2)) - N_{110}^R .200\text{EXP}(B_2^T N_2) &= 0 \\ N_{131}^R - N_{130}^R (\text{EXP}(B_7^T N_2) - .575\text{EXP}(B_8^T N_2)) - N_{120}^R .450\text{EXP}(B_6^T N_2) &= 0 \\ N_{141}^R - N_{140}^R (\text{EXP}(B_9^T N_2) - .700\text{EXP}(B_{10}^T N_2)) - N_{130}^R .575\text{EXP}(B_8^T N_2) &= 0 \\ N_{221}^R - N_{220}^R (\text{EXP}(B_{11}^T N_2) - .250\text{EXP}(B_{12}^T N_2)) - N_{210}^R .150\text{EXP}(B_4^T N_2) &= 0 \\ N_{231}^R - N_{230}^R (\text{EXP}(B_{13}^T N_2) - .350\text{EXP}(B_{14}^T N_2)) - N_{220}^R .250\text{EXP}(B_{12}^T N_2) &= 0 \\ N_{241}^R - N_{240}^R (\text{EXP}(B_{15}^T N_2) - .450\text{EXP}(B_{16}^T N_2)) - N_{230}^R .350\text{EXP}(B_{14}^T N_2) &= 0 \\ N_{151}^R - N_{140}^R .700\text{EXP}(B_{10}^T N_2) &= 0 \\ N_{251}^R - N_{240}^R .450\text{EXP}(B_{16}^T N_2) &= 0 \\ 336X_0 &\leq V^T N_3 \leq 839X_0 \\ 0 &\leq X_0 \leq 1, \quad N_{ijk}^{R,C} \geq 0 \quad (i=1,2 \quad j=1,\dots,n_i+k \quad k=0,1) \end{aligned}$$

Table 9. Case IIA thinning model formulation, following substitution and simplification (vectors are defined in Table 10).

$$\begin{aligned}
 \text{Maximize: } PV = & \$201.45 + P_2^T N_3 - 4X_0 - MX_0(1-X_0) \\
 & +(14-N_{110}^C)(.0014\text{EXP}(-1.566703+B_2^T N_3)) \\
 & +(55-N_{120}^C)(.070\text{EXP}(-.052019+B_5^T N_3)+.072\text{EXP}(-1.0339986+B_6^T N_3)) \\
 & +(79-N_{130}^C)(.23\text{EXP}(-.0287178+B_7^T N_3)+.12075\text{EXP}(-.905679+B_8^T N_3)) \\
 & +(45-N_{140}^C)(.44\text{EXP}(-.0003563+B_9^T N_3)+2.142\text{EXP}(-.678275+B_{10}^T N_3)) \\
 & +(60-N_{210}^C)(.006\text{EXP}(-1.3972885+B_4^T N_3)) \\
 & +(75-N_{220}^C)(.04\text{EXP}(-.02165+B_{11}^T N_3)+.0275\text{EXP}(-1.047585+B_{12}^T N_3)) \\
 & +(52-N_{230}^C)(.15\text{EXP}(-.009600+B_{13}^T N_3)+.056\text{EXP}(-.7879961+B_{14}^T N_3)) \\
 & +(.10-N_{240}^C)(.31\text{EXP}(-.002415+B_{15}^T N_3)+1.098\text{EXP}(-.589199+B_{16}^T N_3))
 \end{aligned}$$

Subject to:

$$N_{110}^C \leq 14, \quad N_{120}^C \leq 55, \quad N_{130}^C \leq 79, \quad N_{140}^C \leq 45$$

$$N_{210}^C \leq 60, \quad N_{220}^C \leq 75, \quad N_{230}^C \leq 52, \quad N_{240}^C \leq 10$$

$$336X_0 \leq V^T N_3 \leq 839X_0$$

$$0 \leq X_0 \leq 1, \quad N_{ij0}^C \geq 0 \quad (i=1,2 \quad j=1,2,3,4)$$

problem statements are defined in Table 10. Vector notation was required to present the Case IIa formulations due to the size of the program.

Following the equation sets presented in Appendix A, the Case IIa formulation includes 27 variables and 21 constraints (Table 8). Substitutions corresponding to those outlined for Case I result in the Case IIa formulation in Table 9, with 9 variables and 11 constraints. Again the substitutions result in a nonlinear objective function, constrained by linear inequalities. Similar programs will result for any formulation for 1 growth period, as all exponential terms resulting from the growth model are transferred to the objective function.

Nonlinear programs were also defined for Cases IIb and IIc following Appendix A. As predicted by equations (23) and (28), Case IIb involved 50 variables and 36 constraints, while Case IIc had 77 variables and 53 constraints. Equivalent formulations through substitution were not developed for these examples. Redefining the thinning model formulations simply in terms of trees to cut after each period becomes increasingly difficult as the number of growth periods projected increases. Also, substitution will not replace all of the nonlinear constraints in models with more than one growth period. Nonlinear constraints corresponding to the inequalities added in step (2) of the

Table 10. Vectors used in the Case IIa thinning model formulations of Tables 8 and 9.

$$P_1 = (0.00, 0.10, 0.34, 0.65, 0.00, 0.06, 0.22, 0.45, 0.00, 0.0680683, 0.2393183, 0.4423791, 3.5050035, 0.00, 0.040835, 0.1497283, 0.3062624, 2.7495561)^T$$

$$P_2 = (0.00, 0.10, 0.34, 0.65, 0.00, 0.06, 0.22, 0.45)^T$$

$$N_1 = (N_{110}^C, N_{120}^C, N_{130}^C, N_{140}^C, N_{210}^C, N_{220}^C, N_{230}^C, N_{240}^C, N_{111}^R, N_{121}^R, N_{131}^R, N_{141}^R, N_{151}^R, N_{211}^R, N_{221}^R, N_{231}^R, N_{241}^R, N_{251}^R)^T$$

$$N_2 = (N_{110}^R, N_{120}^R, N_{130}^R, N_{140}^R, N_{210}^R, N_{220}^R, N_{230}^R, N_{240}^R)^T$$

$$N_3 = (N_{110}^C, N_{120}^C, N_{130}^C, N_{140}^C, N_{210}^C, N_{220}^C, N_{230}^C, N_{240}^C)^T$$

$$V = (0, 1.95, 6.61, 12.8, 0, 1.40, 5.04, 10.55)^T$$

$$B_1 = (0, -.0002262, -.0007668, -.0014848, 0, -.0001624, -.0005846, -.0012238)^T$$

$$B_2 = (0, -.0018207, -.0061718, -.0119514, 0, -.0013072, -.0047058, -.0098505)^T$$

$$B_3 = (0, -.0000887, -.0003008, -.0005824, 0, -.0000637, -.0022930, -.0004800)^T$$

$$B_4 = (0, -.0016238, -.0055041, -.0106586, 0, -.0011658, -.0041968, -.0087850)^T$$

$$B_5 = (0, -.0000605, -.0002049, -.0003968, 0, -.0000434, -.0001562, -.0003271)^T$$

$$B_6 = (0, -.0012016, -.0040731, -.0078874, 0, -.0008627, -.0031056, -.0065009)^T$$

$$B_7 = (0, -.0000148, -.0000892, -.0001728, 0, -.0000106, -.0000680, -.0001424)^T$$

$$B_8 = (0, -.0007135, -.0037340, -.0072307, 0, -.0005123, -.0002847, -.0059597)^T$$

$$B_9 = (0, -.0000074, -.0000251, -.0000393, 0, -.0000053, -.0000192, -.0000802)^T$$

$$B_{10} = (0, -.0005905, -.0020015, -.0070720, 0, -.0004239, -.0015261, -.0058289)^T$$

$$B_{11} = (0, -.0000252, -.0000853, -.0001651, 0, -.0000181, -.000065, -.00013610)^T$$

$$B_{12} = (0, -.0012174, -.0041266, -.0079910, 0, -.0008740, -.0031465, -.0065864)^T$$

$$B_{13} = (0, -.0000074, -.0000397, -.0000768, 0, -.0000053, -.0000302, -.0000633)^T$$

$$B_{14} = (0, -.0006396, -.0032396, -.0062733, 0, -.0004592, -.0024701, -.0051706)^T$$

$$B_{15} = (0, -.0000023, -.0000079, -.0000230, 0, -.0000017, -.0000060, -.0000190)^T$$

$$B_{16} = (0, -.0005166, -.0017510, -.0061082, 0, -.0003709, -.0013351, -.0050345)^T$$

substitutions for Case I remain in the formulation. All nonlinear equalities can be removed, however. If necessary, computer programs could be written to perform the substitutions for reformulating problems with more than one growth period. As will be discussed with the thinning model solutions, however, reformulating Cases IIb and IIc would not expedite the analysis in the present study.

Thinning Model Solution

Three techniques were considered for solving the thinning model examples. These techniques were Monte-Carlo Integer Programming (MCIP), Multistage Monte-Carlo Integer Programming (MS-MCIP), and a nonlinear programming subroutine titled VMCON. The Monte-Carlo or random search methods considered are heuristics, i.e., non-convergent iterative algorithms (Muller-Merbach 1981). Such algorithms are commonly used in estimating solutions to integer or combinatorial problems. Each of the three approaches considered in the present study will be described, with subsequent discussions concerning their application to solving Cases I and II. The relative advantages and disadvantages of each for solving thinning model formulations will be considered following the application.

Solution Techniques

Monte-Carlo Integer Programming. MCIP has been proposed by Conley (1980) for solving mathematical

programming problems and systems of equations. The approach is not new, however, and simply involves evaluating the objective function of a problem for randomly selected, feasible values of the decision variables. The best solution generated by the random sample of feasible points is used as the estimated optimum. The approach is an integer approach, as integer solutions are evaluated. Conley's title for the method is observed in the present study, rather than simple random sampling, because of his single statistical argument for the approach.

The basic argument presented by Conley (1980) in defense of MCIP involves examining the probability density function for objective function values to a particular programming problem. For combinatorial problems, the density is actually a discrete, bounded distribution, more properly termed a relative frequency or probability mass relation. Conley contends that the random search technique will yield estimates very close to the true optimum, for problems with distributions having light (i.e., non-extended) right-hand tails.

For a maximization problem, the optimal solution is that having the greatest objective value, i.e., the value at the extreme right of the distribution of objective function values. If this value is not isolated, or is not at the end of an extremely heavy right-hand tail, objective function

values within a very small upper region of the distribution will closely approximate the maximum. The probability (Pr) that at least one of (n) random solutions is within a given area (a) of the optimum is characterized by equation (29).

$$\text{Pr} = 1 - (1-a)^n \quad (29)$$

In this relation, (1-a) represents the probability that a given solution is within the area (1-a). The probability that all (n) solutions generated fall within this area is therefore $(1-a)^n$. The probability that all (n) did not fall within area (1-a), i.e., that at least one is in the upper (a) region, is $1-(1-a)^n$. The value approaches 1 with large random samples. For example, the probability that at least one of 10,000 random solutions is within the upper .001 region of the probability density function for a given problem is:

$$\text{Pr} = 1 - (1-.001)^{10000} = .9999548$$

For problems where the objective function values within the upper .001 region are near the true maximum, the random search technique should yield estimated solutions with values close to the optimum. The usefulness of the approach for a particular problem therefore depends on the shape of the right-hand tail of the probability density function of objective function values. These distributions will be considered for the thinning model examples to be solved.

Relation (29) may also be solved to determine the

number of samples required for certain probabilities and areas, i.e., the number necessary to state that the probability is (Pr) that at least one solution is within the upper (a) region. This relation is presented in equation (30).

$$n = \ln(1-Pr)/\ln(1-a) \quad (30)$$

Although Conley (1980) does not refer to previous studies, equations (29) and (30) were presented much earlier by Brooks (1958). Brooks proposed the use of simple random search in estimating optimal factor combinations in experimental design. Examining the probability distribution for objective function values was not fully developed by Brooks. Recommendations were made, however, for using relatively small values of (a) in problems where only a small portion of the experimental region is expected to yield high response values.

To implement the MCIP approach for a given problem, a computer program is written to select and evaluate the chosen number of feasible solutions. Random solutions are obtained using a psuedo-random number generator, i.e., identical sequences of random numbers are produced each time the same initial seed number is used. Programs used in the present study will be described in the application of MCIP to Cases I and II.

Multistage Monte-Carlo Integer Programming. MS-MCIP is a modification of MCIP where multiple sets of random samples are evaluated. Conley (1981) proposes MS-MCIP as a method of directing the random search toward the optimal solution. In the multistage approach, sets of random solutions are generated, with the range of possible values for each variable reduced after each set of (n) has been evaluated. Similar concepts were advanced over twenty years ago by McArthur (1961) and Karnopp (1963). In the present analysis, sufficient sets were considered to ensure that possible ranges for decision variable values were very small in the final (n) evaluations. Each set of random evaluations represents a separate stage in the multistage method.

The possible range of values for each variable is based on the value of that variable in the best solution generated thus far. Each time a solution is found with an objective value greater than the highest obtained thus far, the new solution is stored and the possible ranges of variable values are shifted, being formed around the decision variable values in the new solution. The possible ranges are reduced only after each set of (n) solutions has been generated. The positions of these ranges are adjusted, however, each time a solution is obtained with a greater objective function value.

Ranges for variable values are referred to as possible ranges as they represent the maximum possible range for each. If the solution stored as the current best has a variable with a value close to an upper or lower bound, for example, the range may be less than the current maximum possible. This results as the decision variable value is used as the center of the maximum range, with the actual range applied being reduced to reflect feasible values. For a non-negative variable whose current value is zero, for example, the actual range used will be the interval between zero and one half the current range possible.

Conley (1981) relates MS-MCIP to the argument for MCIP, stating that the first set of solutions generated should yield an objective value estimate in the upper (a) region, while the second set should yield at least one solution in an even smaller upper region, etc. In this manner, Conley argues that MS-MCIP will in many cases converge on the true optimum, although convergence is not shown. As will be shown for the thinning model examples, however, in some cases MS-MCIP yields solutions inferior to simple random sampling, where the same total number of solutions are evaluated with each method. The details of the MS-MCIP computer programs written for the thinning model examples will also be described in the application to Cases I and II.

Nonlinear Programming Subroutine VMCON. Methods for solving nonlinear programming problems may be classed as penalty function methods, generalized reduced gradient methods, augmented Lagrangian techniques, and methods based on solving quadratic subproblems. Subroutine VMCON is in the last category, implementing a variable metric method for constrained optimization proposed by Powell (1978a). The subroutine was developed at Argonne National Laboratory, Argonne, Illinois, by Crane et al. (1980). VMCON was used in the present study due to its immediate availability. The algorithm has no fixed limits on problem size, i.e., on the number of variables or constraints. A brief introduction to the basic algorithm used in VMCON will be followed by the input requirements necessary to use the subroutine.

The variable metric algorithm employed in VMCON is an iterative method designed to converge to a point satisfying the first-order Kuhn-Tucker conditions. The first step in the algorithm is to determine the search direction (d) which minimizes a quadratic approximation of the objective function, subject to linear approximations of the constraints. A one-dimensional search is then performed to determine the step length to be taken in the direction (d). The function minimized in this search is the objective function plus a weighted sum of constraint deviations. The weights are calculated using Lagrange multiplier estimates

obtained in the quadratic programming subproblem. The choice of weights for the line search is based on theoretical results on convergence derived by Han (1975), as well as numerical experiments reported by Powell (1978a,1978b). The one-dimensional minimization is designed to produce global convergence, i.e., to force convergence from poor starting estimates. For the line search problem, an approximate minimum is determined through an iterative procedure based on quadratic approximations.

After a search direction and step length are determined, the algorithm uses information based on differences between the previous and current values for the decision variables to update the estimated Hessian matrix for use in the next quadratic subproblem. A convergence test is performed on each iteration after the quadratic programming problem is solved. The algorithm stops if the predicted change in the value of the objective function, plus a measure of the complementarity error, is less than a user-specified tolerance. Output from VMCON can be specified for printing nearly all calculations made at each stage of the algorithm.

To use the VMCON subroutine, two programs are required. A main or calling program is needed, as well as a subroutine subprogram. The main program is changed very little when solving different problems. Calling program adjustments

involve changing the dimensions of various subscripted variables, based on formulas using the numbers of variables and constraints. The main program used in the present study was a modification of the calling program used by Crane et al. (1980), for solving an example in Bracken and McCormick (1968). The user-supplied subroutine, however, is fairly extensive. The subprogram must return the objective function value, the gradient of the objective function, and each constraint value and constraint gradient, given the decision variable values, and the number of variables and constraints. Other subroutines are also called by VMCON. These subprograms, however, have already been coded with VMCON, or may be called from standard subroutine libraries.

Case I Solution

As previously discussed, Case I was specified to aid in evaluating the 3 solution techniques used for the thinning model examples. The entire set of possible integer solutions to Case I was generated, and the optimum solution recorded. With the initial stand-table for Case I, allowing the option to cut 0 trees from any species/diameter class combination, $(49+1)*(39+1)*(49+1)*(19+1)=2,000,000$ possible ways exist of cutting the initial stand. Not all solutions are feasible, however, as it was assumed that if thinning occurred, the volume cut must be between 30 and 50 percent of the original stand volume.

A FORTRAN program was written to evaluate all 2,000,000 solutions to Case I. Of the possible solutions, 930,029 were feasible considering the restrictions on volume removed in thinning. For thinning regimes which were feasible, the residual stand was projected from age 40 to 45, where final harvest occurs. Present values were computed for each of these solutions and the maximum recorded as \$485.76. The optimal integer thinning solution to Case I is to cut (now) 38 trees of species 1 from the 10-11.9 inch diameter class, 27 trees of species 2 from the 6-7.9 inch diameter class, and 3 trees of species 2 from the 8-9.9 inch class. Again, final harvest of the residual stand is assumed at age 45. With the same input assumptions, clearcutting the stand before the first growth period yields a present value per acre of land and timber of \$528.92. Assuming an alternative rate of return of 8 percent, therefore, it would be preferable from a present value standpoint to sell all the timber now. Maximum present values when lower rates were assumed will be presented in the sensitivity analysis.

Knowing the optimal solution to Case I assisted in evaluating the performance of the 3 solution techniques. Solutions to Case I using the two random search methods will be considered, followed by results from applying VMCON.

Random Search Methods. The first step in evaluating the usefulness of the MCIP and MS-MCIP approaches for

solving Case I was to examine the probability density function for objective function values. Computer programs presented by Conley (1980) were used as models in writing a FORTRAN program to determine the points for plotting the desired distribution. The process involves evaluating all solutions, recording the minimum and maximum objective values, and dividing the difference into histogram intervals. The total number of objective function values occurring within each interval is then determined, and each is divided by the total number of solutions evaluated, thus obtaining the probabilities associated with each interval.

The distribution resulting from this process, for all feasible solutions to Case I, is presented in Figure 1. As previously discussed, the most important property for such distributions is that within small upper regions, the range of possible objective function values is small. This property is reflected for Case I by the light right-hand tail of Figure 1. The distribution therefore indicates that the random search approaches should yield estimated optimal solutions to Case I with objective function values close to the true optimum of \$485.76.

FORTRAN programs were written to generate random solutions to the Case I thinning model. General diagrams of the steps involved in solving thinning model formulations with the MCIP and MS-MCIP approaches are presented in

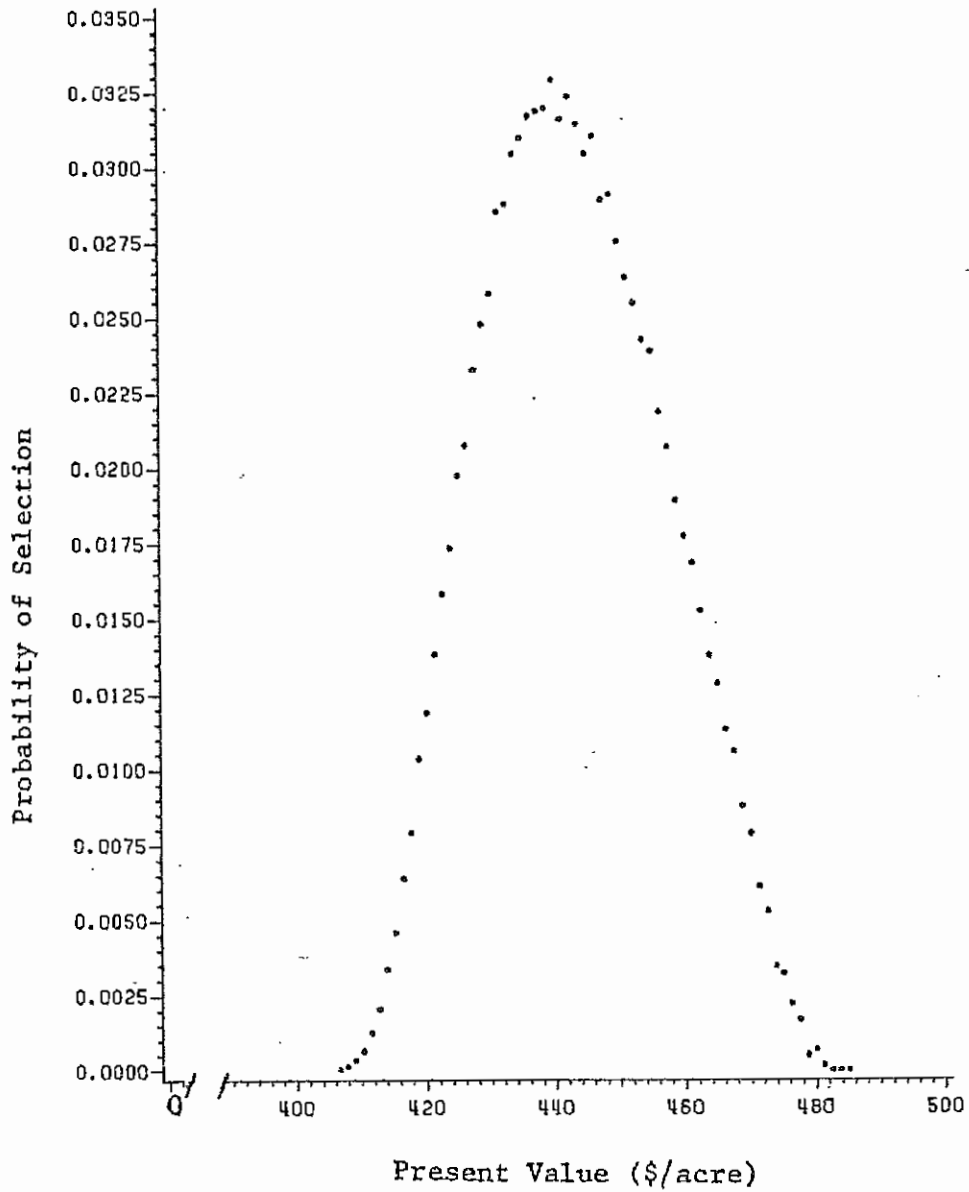


Figure 1. Probability density of Objective function values for all feasible solutions to Case I.

Figures 2 and 3, respectively. For both random search approaches, a pseudo-random number generator was coded as a function subprogram, requiring the specification of an initial seed number. The MCIP program generates random values for the numbers of trees to cut from each species/diameter class combination. For each feasible solution, the present value is determined over the 5-year growth interval, and compared to the current maximum value. The process is repeated until the required number of feasible solutions have been evaluated.

The MS-MCIP program for Case I was designed to evaluate 6 sets of random solutions, i.e., 6 stages were used in the multistage analysis. After each set the maximum range was reduced for each decision variable. The maximum ranges used for Case I were 100, 50, 30, 20, 10, and 4 trees per acre. In the first stage of the MS-MCIP program, the maximum range is 100. The value was chosen large enough that the initial range, for each species/diameter class combination, includes all possibilities, regardless of the current values of the decision variables. In this manner, the first stage of the MS-MCIP approach is equivalent to the MCIP program. That is, the first stage merely evaluates random solutions, with no narrowing of the variable ranges. Using the same initial seed number, output from the first stage of the MS-MCIP program should therefore correspond exactly to the results

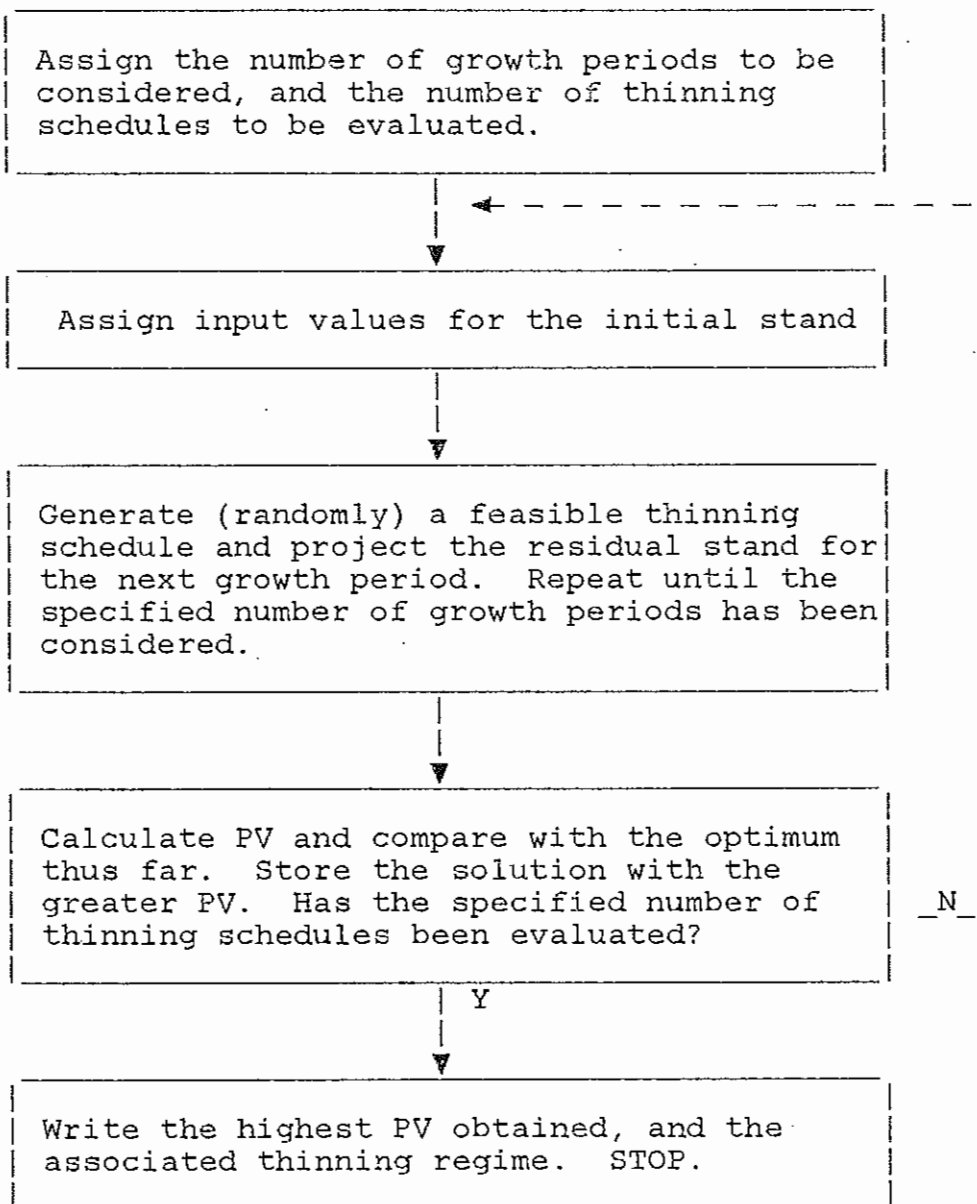


Figure 2. Diagram of the major steps involved in solving thinning model formulations with MCIP.

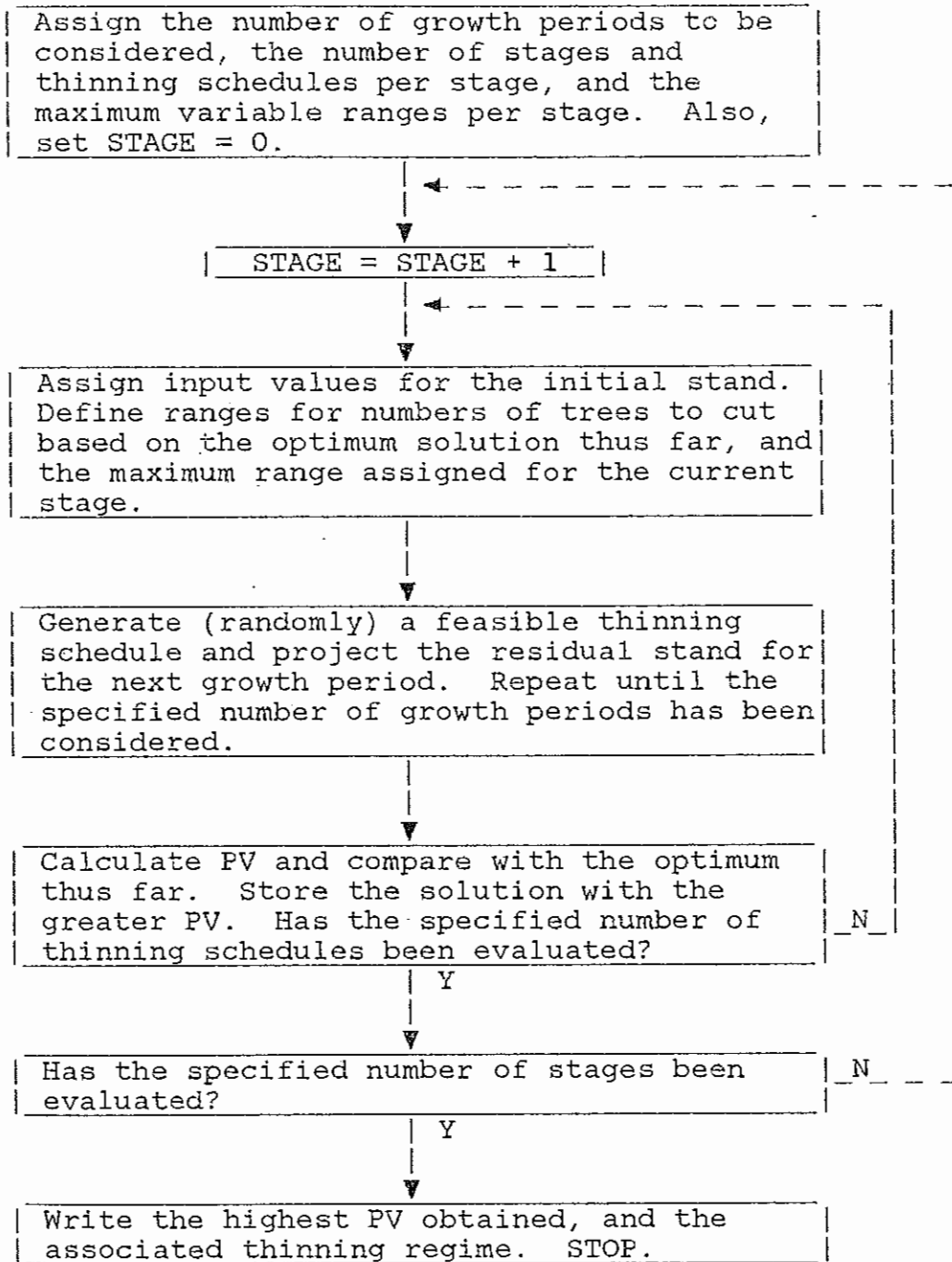


Figure 3. Diagram of the major steps involved in solving thinning model formulations with MS-MCIP.

using MCIP, when the same number of solutions are generated.

After the initial (n) feasible solutions have been evaluated with the MS-MCIP program, the maximum range of possible variable values is reduced to 50 trees and the second set of solutions is considered. This process continues until the final stage when the maximum range for trees to cut from each diameter class is reduced to 4. As previously discussed, however, the actual range implemented with the MS-MCIP program may change each time a solution is generated with a present value greater than the previous maximum.

Optimal solution estimates for Case I were obtained using MCIP and MS-MCIP, with the same initial seed number. Results for the two approaches, where 1,000 random solutions were evaluated for each stage of the MS-MCIP program are presented in Table 11. At each line of Table 11, the same number of solutions have been considered with each technique. The objective values are the same after the first stage of MS-MCIP, as the programs are equivalent until reduction in the decision variable ranges occurs. The MS-MCIP method results in higher present values than MCIP at each line of Table 11. The approach generated the true optimum during the final stage, evaluating only 6,000 solutions from a possible 930,029. The MCIP program produced an objective value of \$481.25 after 6,000

Table 11. Objective function values for solutions to Case I, with 1,000 random samples for each stage of the MS-MCIP approach (initial seed number = 39873).

Stage No. (for MS-MCIP)	Total No. of Samples	Present Values	
		MCIP	MS-MCIP
		-----\$/acre-----	
1	1,000	479.37	479.37
2	2,000	479.83	481.06
3	3,000	480.42	482.77
4	4,000	480.42	482.77
5	5,000	480.42	482.94
6	6,000	481.25	485.76*

*Optimal Value for Case I

solutions, \$4.51 below the optimum.

The MCIP and MS-MCIP programs were then used to solve Case I with a different initial seed number. Again, 1,000 feasible solutions were evaluated at each stage. Present values for these solutions are presented in Table 12. The MS-MCIP approach again generated the optimal solution with a total of 6,000 evaluations. Note, however, that the MS-MCIP present values are not higher than MCIP values after every stage. Using the MCIP technique, variables are allowed to assume any value within their initial ranges. The simple random search method therefore outperforms the reduced-range method in some instances.

Tables 11 and 12 present objective function values obtained for Case I with 1,000 evaluations for each stage of the MS-MCIP program. Tables 13 and 14 present the objective values obtained with 10,000 evaluations at each stage. The initial seed numbers used for Tables 13 and 14 correspond to those for Tables 11 and 12, respectively. Using 10,000 evaluations, neither approach generated the optimal solution to Case I. Although improved solutions are obtained after the first 10,000 evaluations, the solution used to begin the second stage of the MS-MCIP approach did not lead to the optimum. This result would not be expected in general, however, as using a small number of evaluations in the initial stage of the MS-MCIP approach may narrow variable

Table 12. Objective function values for solutions to Case I, with 1,000 random samples for each stage of the MS-MCIP approach (initial seed number = 42441).

Stage No. (for MS-MCIP)	Total No. of Samples	Present Values	
		MCIP	MS-MCIP
		-----\$/acre-----	
1	1,000	480.77	480.77
2	2,000	480.77	481.68
3	3,000	480.77	481.77
4	4,000	482.81	481.93
5	5,000	482.81	485.68
6	6,000	482.81	485.76 *

*Optimal Value for Case I

Table 13. Objective function values for solutions to Case I, with 10,000 random samples for each stage of the MS-MCIP approach (initial seed number = 39873).

Stage No. (for MS-MCIP)	Total No. of Samples	Present Values	
		MCIP	MS-MCIP
		-----\$/acre-----	
1	10,000	481.25	481.25
2	20,000	481.35	483.03
3	30,000	482.06	483.13
4	40,000	482.19	483.15
5	50,000	482.19	483.46
6	60,000	482.22	483.46

Table 14. Objective function values for solutions to Case I, with 10,000 random samples for each stage of the MS-MCIP approach (initial seed number = 42441).

Stage No. (for MS-MCIP)	Total No. of Samples	Present Values	
		MCIP	MS-MCIP
		-----\$/acre-----	
1	10,000	482.81	482.81
2	20,000	482.81	483.02
3	30,000	482.81	483.02
4	40,000	482.81	483.46
5	50,000	482.85	483.46
6	60,000	482.90	483.46

ranges too quickly, resulting in inferior final solution estimates.

To further evaluate the effects of sample size on the final estimates generated with MS-MCIP, 10 different initial seed numbers were used to generate solutions to Case I. The final solutions, for samples sizes of 1,000 and 10,000 per stage of the MS-MCIP approach, are presented in Table 15. The first two lines in Table 15 are the MS-MCIP results from Tables 11 through 14. As seen in Table 15, the optimal solution was generated 3 times using 1,000 samples per stage and only once with 10,000 per stage. Nine of the MS-MCIP solutions using 10,000 evaluations per stage had a final present value of \$483.46. The decision variable values for species 2 at this solution are to cut 10 trees from the 6-7.9 inch diameter class and 8 from the 8-9.9 inch class. In all solutions summarized in Table 15, 38 trees of species 1 are removed from the 10-11.9 inch diameter class.

A major problem with the MS-MCIP approach can be observed from the Case I solutions presented in Table 15. As values for species 1 are the same for all solutions, the values for trees to cut from species 2 result in the present value differences between solutions. The values for species 2 in the optimal solution are 27 and 3 (trees cut by diameter class). Species 2 values for the solution with objective value \$483.28 are 16 and 8 trees. For a slightly

Table 15. Objective function and decision variable values for solutions to Case I, with random samples of 1,000 and 10,000 for each stage of the MS-MCIP approach (input assumptions from Table 5).

Initial Seed No.	1,000 Samples/Stage PV (\$/acre)	No. Trees Cut*	10,000 Samples/Stage PV (\$/acre)	No. Trees Cut*
39873	485.76**	(0,38,27,3)	483.46	(0,38,10,8)
42441	485.76**	(0,38,27,3)	483.46	(0,38,10,8)
67815	483.29	(0,38,14,7)	483.46	(0,38,10,8)
98779	483.29	(0,38,14,7)	483.46	(0,38,10,8)
13591	485.76**	(0,38,27,3)	483.46	(0,38,10,8)
56783	483.28	(0,38,16,8)	483.46	(0,38,10,8)
45987	483.29	(0,38,14,7)	483.46	(0,38,10,8)
12125	483.29	(0,38,14,7)	485.76**	(0,38,27,3)
76533	483.46	(0,38,10,8)	483.46	(0,38,10,8)
98469	483.17	(0,38,23,4)	483.46	(0,38,10,8)

*Trees cut from (species 1, diameters 8-9.9 and 10-11.9, and species 2, diameters 6-7.9 and 8-9.9) at age 40, final harvest assumed at age 45.

**Optimal Solution for Case I

higher objective value, \$483.29, the species 2 values are 14 and 7. Also, for the most frequent solution, \$483.46, the values are 10 and 8. In each solution, the objective value increases slightly as the species 2 value for diameter class 6-7.9 decreases, from 16 to 14 to 10. In the optimal solution, however, the value is 27.

In the later stages of the MS-MCIP program, the possible ranges for variable values are reduced. For a decision variable such as the number of trees to cut from species 2, diameter class 6-7.9, to increase from 10 to 27, objective function values must show improvement for small changes in the decision variables. In this manner, the variable ranges can move toward a point where 27 is a possible value for trees to cut from the relevant species/diameter class combination. The number of trees cut from the smallest diameter class cannot approach 27 in the solution with objective value \$483.46, however, as the range of values in the final stage is from 8 to 12 trees, and small increases from 10 result in objective function decreases. To show improvement over the \$483.46 solution, a large change in the species 2 value for diameter class 6-7.9 is required. The MS-MCIP approach may therefore result in local optima. This property was recognized by Karnopp (1963) for similar multistage random search methods.

Increasing the number of samples evaluated at each

stage of the MS-MCIP approach does not necessarily improve the final solution estimate. For the MCIP method, however, increasing the total number of samples cannot lower the objective value, as the value is simply the greatest from a larger set of solutions. The estimated optimum for Case I using the simple random search method is \$479.37, after 1,000 solutions were evaluated (Table 11). For the same initial seed number, the estimated optimum is \$481.25 with 10,000 evaluations (Table 13). From equation (29), the probabilities that the above solutions are within the upper .001 region of Figure 1 are:

$$1 - (1-.001)^{1000} = 0.6323046, \text{ and}$$

$$1 - (1-.001)^{10000} = 0.9999548 .$$

Actual areas under the probability density function represented by Figure 1 were determined by recording the number of solutions greater than the estimated optima, and dividing by the total number of possible solutions. A total of 1226 solutions were recorded with present values greater than \$479.37, while only 429 had values greater than \$481.25. The actual areas to the right of these values are:

$$\frac{1,226}{930,029} = .0013168, \text{ and } \frac{429}{930,029} = .0004608 .$$

For the MCIP program with 1,000 evaluations, the estimated optimum is not within the upper .001 region, although the probability that the estimate would be was 0.6323046 . With 10,000 evaluations, however, the estimated

optimum is well within the upper .001 region. For the MCIP approach, as many evaluations should be performed as practical for a particular problem. Equations (29) and (30) may be of help, however, for problems where functional evaluations are particularly difficult or expensive.

VMCON. Subroutine VMCON was used in trying to solve the Case I thinning model formulations of Tables 6 and 7. The only change in the formulations actually implemented in the solution attempts was that thinning was assumed to occur. That is, $X(0)$ was defined equal to 1. This assumption simplified the coding of the user-supplied subroutine for VMCON, avoiding the problem of specifying an exact value for the constant M in the initial trials. Appropriate values for M may have to be determined through trial and error, as simply specifying a very large number may result in ill-conditioning of the problem. Another approach would be to solve the problem for both values of $X(0)$, i.e., $X(0)=1$ and $X(0)=0$. This alternative is only viable, however, in problems where the number of growth periods projected, and thus the number of binary choice combinations, is relatively small.

As previously noted, the VMCON calling program presented by Crane et al. (1980) was modified for use with the Case I formulations. The convergence tolerance level was specified as 10^{-9} . The necessary user-supplied

subroutines were coded for both formulations of Case I, in hopes of evaluating the gains from specifying the model entirely in terms of variables for trees to cut. For the formulation presented in Table 6, however, solutions were not obtained. For all starting solutions attempted, the number of functional evaluations for the initial line search exceeded the internal maximum for VMCON.

Solutions were obtained, however, for the formulation presented in Table 7, although problems were encountered. Many starting solutions were tried for the substituted formulation of Table 7, yet convergence was obtained for only two. Other starting points either resulted in exceeding the maximum evaluations for the line search, or resulted in FORTRAN errors for internal arithmetic overflows. In some cases, scaling techniques may be used to resolve overflow problems with nonlinear programming algorithms (Balachandran and Frair 1982). The objective function and objective function gradient for Case I, Table 7, were therefore divided by a constant to reflect values near unity. The scaling did not result in improved solutions with the VMCON subroutine, however.

Both solutions obtained with VMCON for Case I resulted in objective function values of \$478.25. Variables X(2), X(4), X(6), and X(8) in Table 7 correspond to numbers of trees to cut from species 1, diameters 8-9.9 and 10-11.9

inches, and species 2, 6-7.9 and 8-9.9 inches, respectively. The two initial starting points which generated the solution for \$478.25 were (1,1,1,1) and (25,25,25,25). As the global optimum for Case I with continuous values should be at least \$485.76, convergence to a common solution from different starting points is not necessarily reliable for obtaining global optima in non-convex problems. The final decision variable estimates from VMCON were $(-3 \times 10^{-33}, 36.387, 0.000, 19.000)$. The objective value for cutting 36 trees of species 1 in the 10-11.9 inch class, and 19 trees of species 2 in the 8-9.9 inch class, was determined using the MCIP program, specifying the above values. The integer solution yields a present value of \$480.32 .

One of the goals in using the VMCON program for Case I was to use the estimated optimal solutions from the MCIP and MS-MCIP approaches as starting estimates, observing the degree of improvement obtained. In each case where random search solutions were used as starting estimates, no improvements were made. Due to the problems encountered with obtaining solutions to Case I with VMCON, further efforts to produce the global optimum were not pursued. Such efforts might have included an analysis of the solution results from partitioning the set of possible starting values.

Case II Solution

The random search methods used for Case I were also used in estimating solutions to the three subproblems of Case II. The results obtained with the MCIP and MS-MCIP approaches will be presented, followed by the application of subroutine VMCON to the Table 9 formulation of Case IIa.

Random Search Methods. Exhaustive search could not be used to determine global optima for the Case II problems. For Case IIa, for example, there are $8.3548583 \cdot 10^{12}$ possible ways to thin the stand. The global optima for the Case II problems are therefore unknown. The exact shapes of the probability density functions for objective values are also unknown.

The distribution of present values for all solutions to Case I was presented in Figure 1. The distribution resulting from 10,000 random solutions to Case I is presented in Figure 4. The relationship plotted for the large random sample of solutions corresponds to the general shape of the distribution for all feasible solutions to Case I (Figure 1). For the Case II problems, therefore, the distributions resulting from 10,000 random solutions to each problem were plotted. These relationships are presented in Figures 5, 6, and 7 for Cases IIa, IIb, and IIc, respectively.

Figures 5, 6, and 7 do not provide conclusive evidence

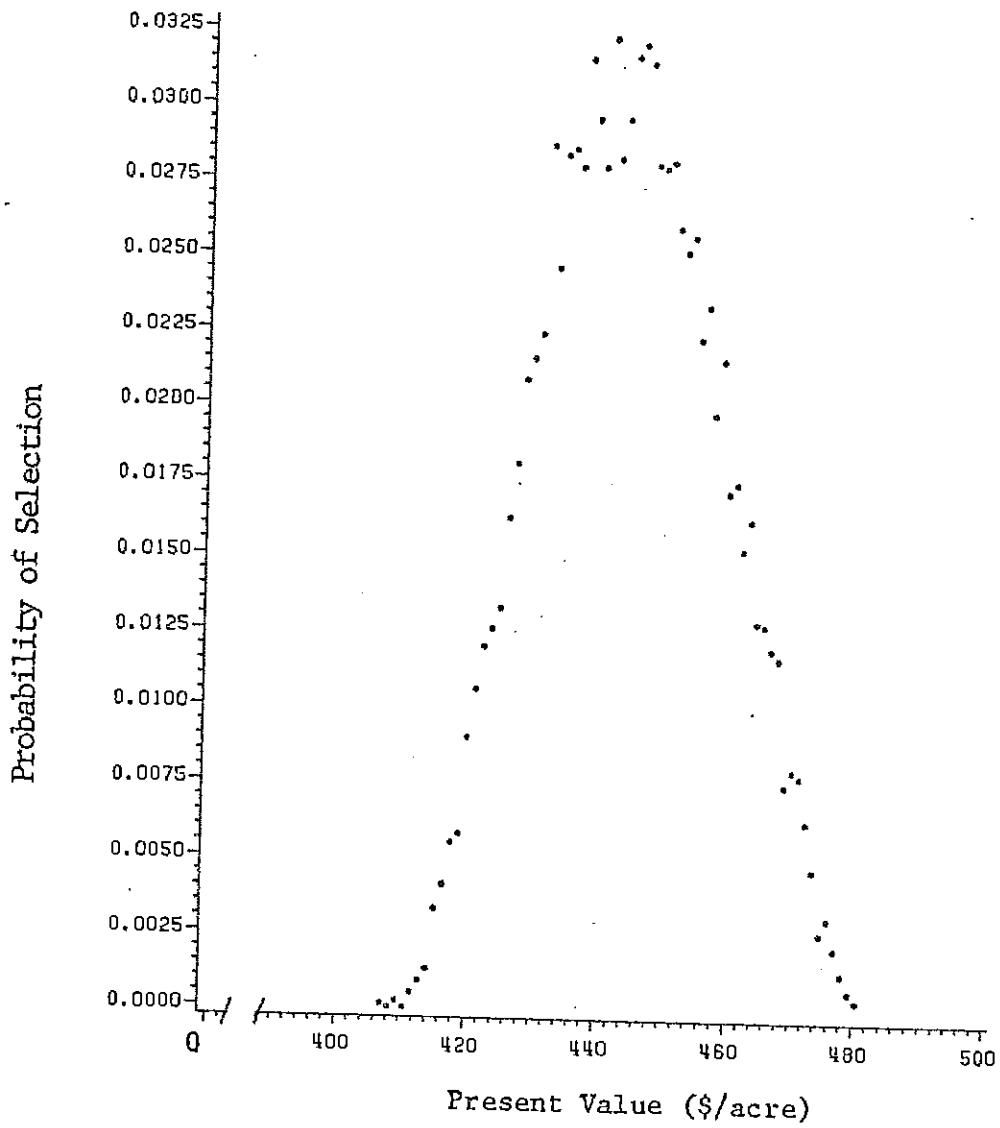


Figure 4. Probability density of objective function values resulting from 10,000 random solutions to Case I.

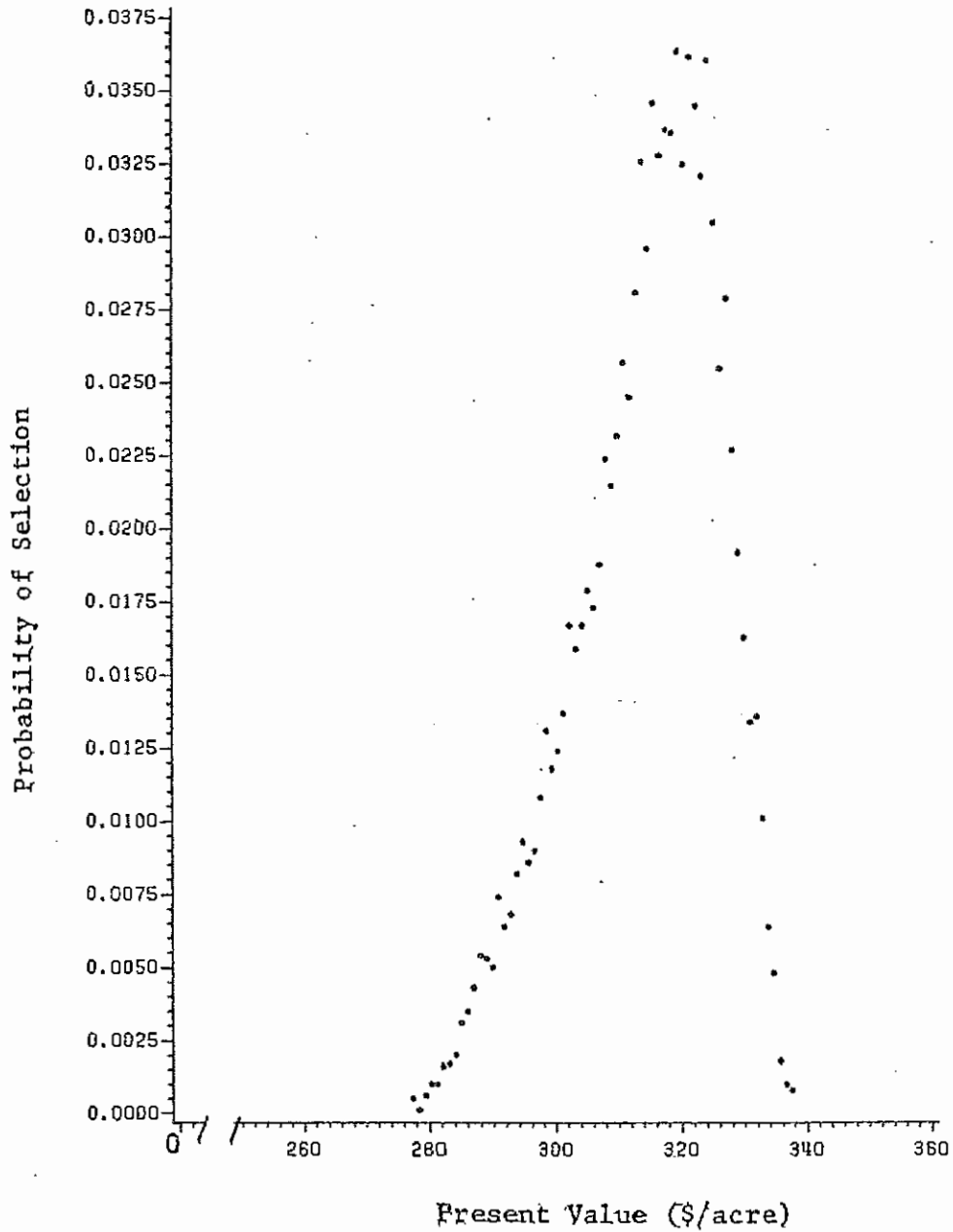


Figure 5. Probability density of objective function values resulting from 10,000 random solutions to Case IIa.

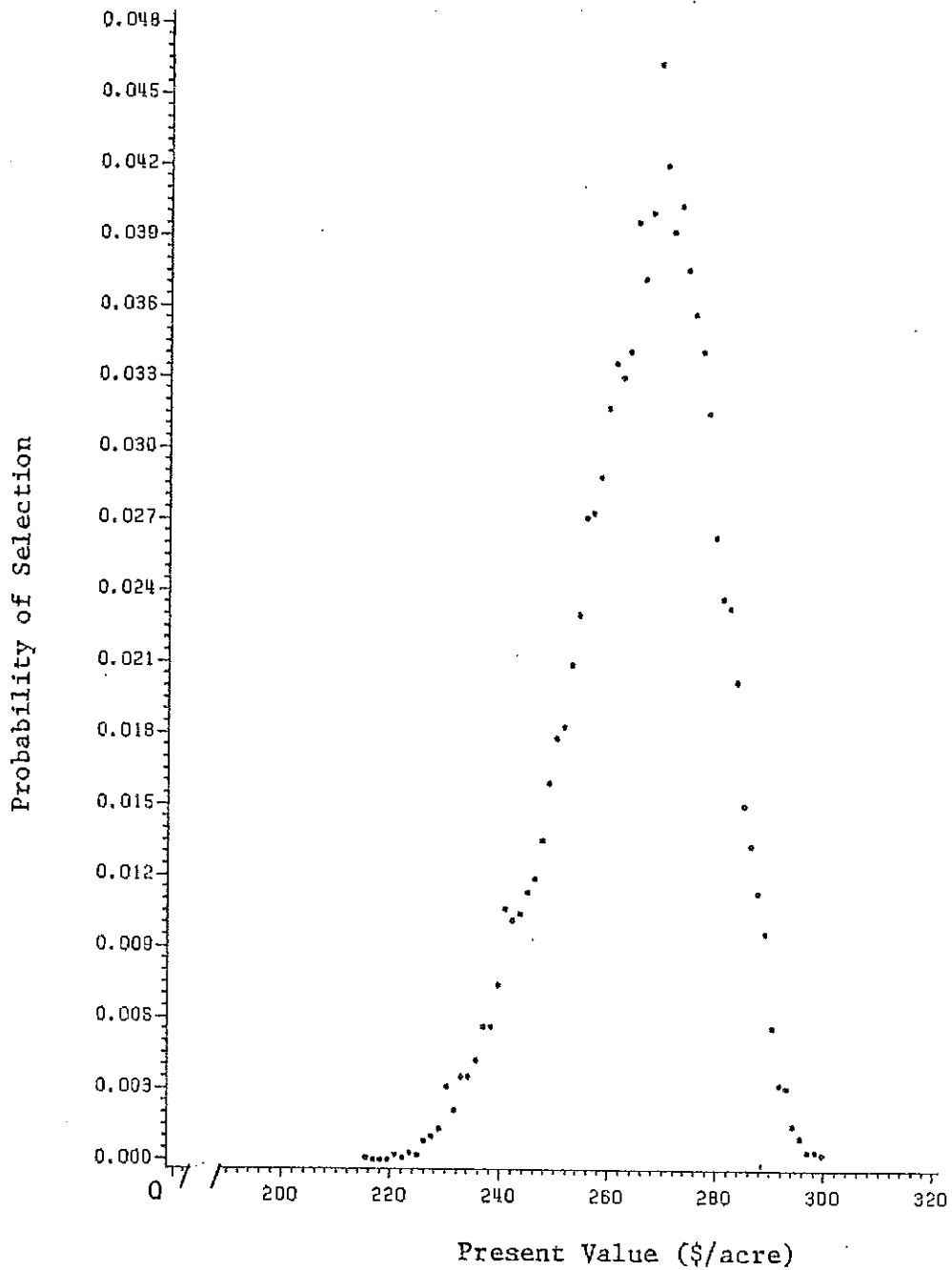


Figure 6. Probability density of objective function values resulting from 10,000 random solutions to Case IIb.

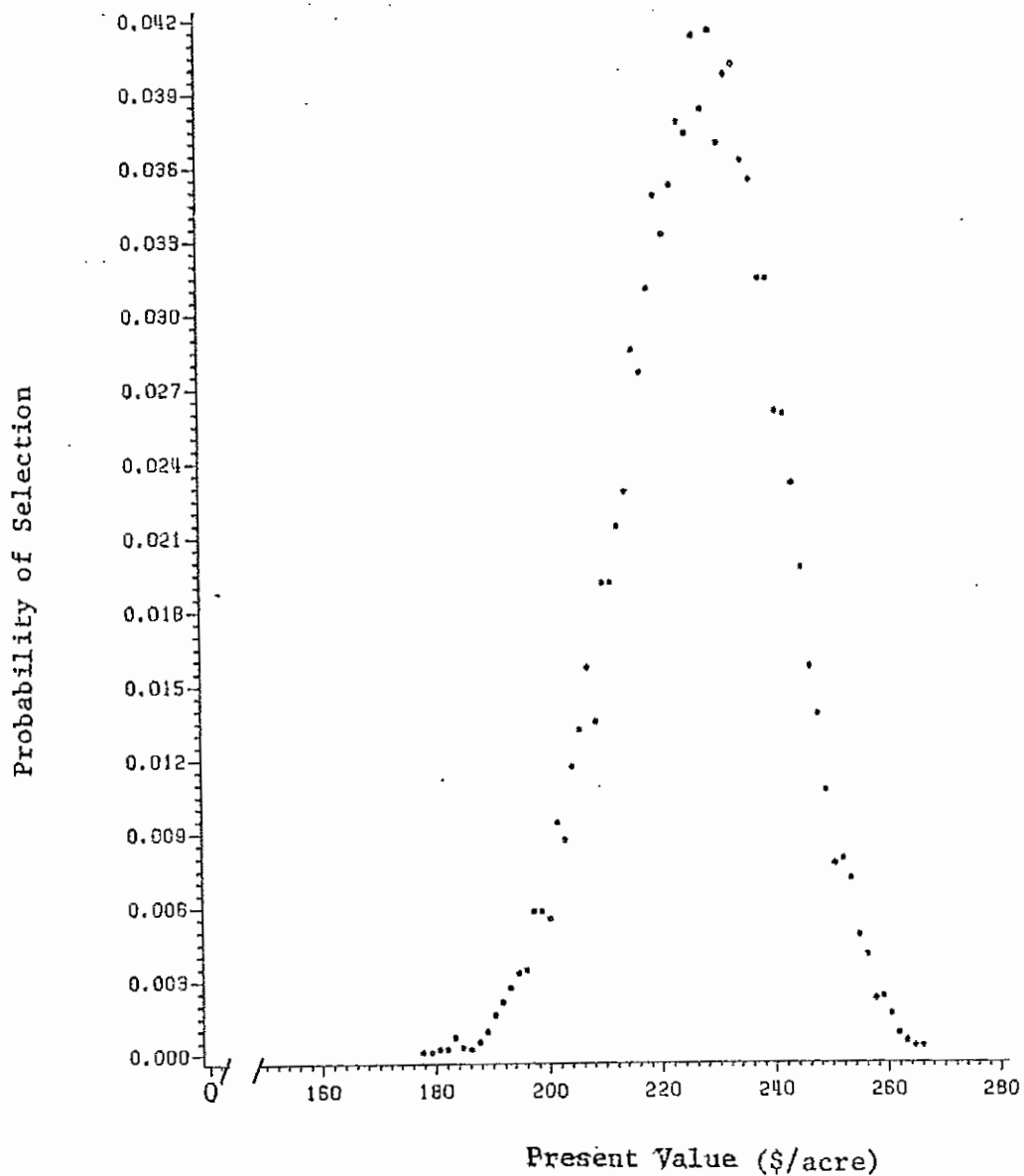


Figure 7. Probability density of objective function values resulting from 10,000 random solutions to Case IIc.

of the exact shape of the right-hand tails of the unknown distributions for Case II. A nonexhaustive sample of feasible solutions is unlikely to reveal an extended right-hand tail. Such a tail would be indicated, however, if a few solutions were obtained with objective values very much greater than the majority evaluated. Figures 5, 6, and 7 do not, however, indicate isolated values. If these distributions correspond to those for the entire sets of feasible solutions, as resulted for Case I, the random search methods should provide solution estimates near the true optima.

The MCIP program for Case II was developed in three segments, corresponding to 3 growth periods. In the first section, a feasible thinning schedule is generated randomly and the residual stand projected to age 35. For Case IIa, present values are calculated and compared at this point. For Cases IIb and IIc, however, another feasible thinning schedule is generated and the residual stand projected to age 40. For Case IIc solutions, a third thinning schedule is generated and the stand projected to age 45. A single feasible solution for the 3-growth period thinning model therefore involves 3 thinning schedules, with values for a total of 30 decision variables. Two thinning plans, with 18 variables, are required for each solution to the 2-period model. Other details of the MCIP program were similar to

the program for Case I.

The MS-MCIP program for Case II was also developed in three segments. Seven stages were used for the MS-MCIP approach, with maximum variable ranges of 300, 100, 50, 30, 20, 10, and 4 trees per acre, for stages 1 through 7, respectively. The MS-MCIP program for Case IIa, 1-growth period, corresponds to the program discussed for Case I. For the 2 and 3-period thinning models, however, a more detailed procedure was used to establish actual ranges for possible numbers of trees to cut after periods 1 and 2.

The MS-MCIP program randomly selects the numbers of trees to cut from each species/diameter class combination prior to growth period 1. The residual stand is then projected to the end of period 1, where values are selected for trees to cut before growth period 2. The number of trees available for cutting cannot exceed the number projected after growth period 1. Therefore, the range of possible values for trees to cut cannot simply be formed around the value of each variable in the optimal solution generated thus far. That is, in the optimum thus far, the number of trees cut after period 1 may be greater than the number of trees projected at the current solution, i.e., considering the thinning regime prescribed for period 1 at the present evaluation. If this occurs, or if the variable value in the optimum thus far, plus one half the current

maximum range, is greater than the projected number of trees, the MS-MCIP program uses the number projected as the upper bound on the range of possible values for the variable.

The lower bound for each variable range is defined as the value in the optimal solution thus far, minus one half the maximum range possible. If the lower bound is greater than the number of trees projected, the lower bound is redefined as the number projected minus one half the maximum range, or redefined as 0 if this quantity is negative. In the 3-period model, a similar procedure was coded for each species/diameter class combination, for choosing upper and lower bounds for possible trees to cut after period 2.

Similar to the solutions evaluated for Case I in Tables 11 through 14, Case II solutions are presented in Tables 16 through 19. Present values are presented in Tables 16 and 17 using 1,000 solutions per stage of the MS-MCIP program, for the initial seed numbers specified. The same seed numbers were used with 10,000 evaluations per stage, and the present values summarized in Tables 18 and 19.

For all solutions, the present values after the first stage of the MS-MCIP program correspond exactly to the MCIP solutions. This results as 300 was specified as the initial maximum range for trees cut per acre from each species/diameter class combination in the MS-MCIP program.

Table 16. Objective function values for solutions to Cases IIa, IIb, and IIc, with 1,000 random samples for each stage of the MS-MCIP approach (initial seed number = 39873).

Stage No. (for MS-MCIP)	Total No. of Samples	Case IIa		Present Values Case IIb		Case IIc	
		MCIP	MS-MCIP	MCIP	MS-MCIP	MCIP	MS-MCIP
-----\$/acre-----							
1	1,000	337.89	337.89	299.22	299.22	266.51	266.51
2	2,000	338.13	338.15	299.22	299.22	266.51	268.09
3	3,000	338.13	338.54	299.22	299.22	266.51	272.60
4	4,000	338.13	338.54	299.22	299.22	266.76	274.72
5	5,000	338.13	340.92	299.22	299.87	266.76	276.00
6	6,000	338.13	341.16	299.22	302.25	266.76	277.67
7	7,000	338.13	341.21	300.38	303.40	266.76	280.21

Table 17. Objective function values for solutions to Cases IIa, IIb, and IIc, with 1,000 random samples for each stage of the MS-MCIP approach (initial seed number = 42441).

Stage No. (for MS-MCIP)	Total No. of Samples	Present Values					
		Case IIa		Case IIb		Case IIc	
		MCIP	MS-MCIP	MCIP	MS-MCIP	MCIP	MS-MCIP
		-----\$/acre-----					
1	1,000	340.29	340.29	295.10	295.10	262.54	262.54
2	2,000	340.29	340.29	295.10	295.33	262.54	262.54
3	3,000	340.29	340.29	295.10	300.77	262.54	268.10
4	4,000	340.29	340.73	295.57	300.77	266.04	269.97
5	5,000	340.29	340.97	299.87	300.88	267.11	270.83
6	6,000	340.29	341.16	299.87	302.07	267.11	274.47
7	7,000	340.29	341.19	299.87	303.92	267.11	280.97

Table 18. Objective function values for solutions to Cases IIa, IIb, and IIc, with 10,000 random samples for each stage of the MS-MCIP approach (initial seed number = 39873).

Stage No. (for MS-MCIP)	Total No. of Samples	Present Values					
		Case IIa		Case IIb		Case IIc	
		MCIP	MS-MCIP	MCIP	MS-MCIP	MCIP	MS-MCIP
-----\$/acre-----							
1	10,000	338.13	338.13	300.38	300.38	266.76	266.76
2	20,000	338.22	340.06	302.99	302.69	275.30	271.66
3	30,000	338.22	340.43	302.99	302.69	275.30	272.74
4	40,000	340.37	341.27	304.25	302.75	275.30	272.74
5	50,000	340.43	341.59	304.25	305.41	275.30	275.24
6	60,000	340.43	341.61	304.25	306.85	275.30	278.89
7	70,000	340.43	341.65	304.25	308.52	275.30	280.93

Table 19. Objective function values for solutions to Cases IIa, IIb, and IIc, with 10,000 random samples for each stage of the MS-MCIP approach (initial seed number = 42441).

Stage No. (for MS-MCIP)	Total No. of Samples	Present Values					
		Case IIa		Case IIb		Case IIc	
		MCIP	MS-MCIP	MCIP	MS-MCIP	MCIP	MS-MCIP
		-----\$/acre-----					
1	10,000	340.29	340.29	299.87	299.87	267.11	267.11
2	20,000	340.29	340.29	299.87	299.87	269.23	271.23
3	30,000	340.29	340.91	302.04	304.58	271.70	271.94
4	40,000	340.29	341.24	303.60	304.58	273.17	275.22
5	50,000	340.29	341.54	303.60	304.58	273.17	279.19
6	60,000	340.29	341.60	303.60	306.67	274.91	284.13
7	70,000	340.29	341.60	303.60	308.37	274.91	287.99

This ensures that no reduction in the ranges of possible values for the decision variables occurs in the first stage.

A total of 12 MS-MCIP solutions are presented in Tables 16 through 19. In all solutions evaluated for Case II, the MS-MCIP program resulted in greater final present value estimates than were obtained with the MCIP method. No solutions were obtained with greater present values than the \$378.05 for clearcutting the stand now, however. Of the Case II solutions generated, the highest present values obtained were \$341.65, \$308.52, and \$287.99. The thinning regimes associated with these solutions are presented in Table 20.

One difference between the Case I and Case II solutions examined is that increasing the number of evaluations to 10,000 per stage resulted in greater present value estimates for Case II. The improvements are evident with both approaches, and are greatest for the Case IIb and IIc examples, problems with greater numbers of possible feasible solutions. These results, however, are due to the respective shapes of the previously discussed objective function distributions. The total number of possible solutions to a problem should have no bearing on the degree of objective function sensitivity to the fraction of the total evaluated.

Finally, the present value estimates for Case II

Table 20. Thinning schedules for solutions to Cases IIa, IIb, and IIc, with present values of \$341.65, \$308.52, and \$287.99, respectively.

	Diameter Class (in.) and Species (1,2)											
	2-3.9		4-5.9		6-6.9		8-9.9		10-11.9		12-13.9	
	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
	-----No. of Trees Harvested-----											
<u>Case IIa</u>												
Period 0:	8	12	0	57	70	51	2	1	-	-	-	-
<u>Case IIb</u>												
Period 0:	12	57	27	55	39	50	6	5	-	-	-	-
Period 1:	1	3	18	10	3	4	31	3	1	0	-	-
<u>Case IIc</u>												
Period 0:	10	54	11	66	19	10	18	9	-	-	-	-
Period 1:	0	2	20	4	14	31	3	6	1	0	-	-
Period 2:	1	1	9	4	20	1	13	1	3	0	4	0

decrease as the number of growth periods projected increases. This results for both random search methods, for each of the solutions generated. It should not be concluded, however, when present value decreases occur for a given problem, that decreases will continue as the number of periods considered is increased. The present value relationship for the thinning model is not necessarily concave with respect to the number of growth periods projected. The relationship depends on the input assumptions, as will be discussed in the sensitivity analysis. For a given problem, sufficient periods should be projected that all value increases assumed have been reflected.

VMCON. The nonlinear programming subroutine was used to solve the Table 9 formulation of Case IIa, under the assumption that thinning occurs, i.e., $X(0)=1$. The formulation presented in Table 8, developed from the equations in Appendix A, was not coded due to the lack of success in solving the Case I formulation with equality constraints. For the Table 9 formulation of Case IIa, arithmetic overflows resulted in premature termination of the VMCON algorithm for all starting solutions attempted. Due to these results, the subroutine was not applied to the much more involved 2 and 3-period formulations.

Although convergence was not attained, the VMCON

program did produce one continuous solution to Case IIa with a higher present value than any integer solution obtained with the random search methods. From a starting solution of cutting 1 tree from each species/diameter class combination, the algorithm produced the following continuous solution with an objective value of \$343.66 .

$$\begin{array}{rcl}
 N_{110}^C & = & 0.910 \\
 N_{120}^C & = & 19.720 \\
 N_{130}^C & = & 77.295 \\
 N_{140}^C & = & 0.000 \\
 N_{210}^C & = & 0.563 \\
 N_{220}^C & = & 19.676 \\
 N_{230}^C & = & 52.000 \\
 N_{240}^C & = & 0.000
 \end{array}$$

Rounding these values to integers, however, yields a present value of \$340.93 . An integer solution with objective value \$341.65 was obtained with random search. Had premature termination not occurred, however, the subroutine may have produced integer solutions to Case IIa superior to the random search results. All attempts to use random search solutions as starting points for the algorithm resulted in termination without changing the initial estimates.

Sensitivity Analysis

Thinning model results were presented for various solutions to Case I, and Cases IIa, IIb, and IIc. The sensitivity of these results to changes in certain input assumptions was examined. A limited number of changes were evaluated as the input parameters were not developed through

estimation. Results of this analysis are intended to emphasize general properties of the thinning model formulation. Most changes were evaluated only for Case I, as the global optimum for this problem could be determined.

As noted in the Case I solution, using an 8 percent discount rate the present value of land and timber if the stand were clearcut now is \$528.92 . If final harvest is postponed 5 years, the optimal policy includes thinning now, and results in a present value of \$485.76 . Assuming a discount rate of 5 percent, however, the present value of the thinning option is \$531.95, indicating the final harvest should be postponed. Further reduction to a rate of 3 percent results in a present value of \$567.43 . Optimal thinning schedules, however, did not change as the interest rate was varied.

The results from two changes in the original price assumptions were also determined for Case I. As presented in Table 5, random-length log prices were originally assumed for trees in diameter classes above 10 inches. Lowering this limit to 8 inches, and assuming 25 percent higher prices for trees above 10 inches, resulted in a present value from thinning of \$639.27, compared to a present value from clearcutting now of \$721.70 . Significant increases in present value are expected in cases where smaller diameters are used as logs rather than roundwood. Changes also occur

in the optimal thinning schedule under these assumptions. In the previous solution, 27 trees of species 2 were cut from the 6-7.9 inch diameter class, while only 3 were removed from the 8-9.9 inch class. Under the new assumptions, however, only 3 trees are removed from the smaller class while 12 are cut from the 8-9.9 inch class. Trees in the larger diameter class have a greater value than previously, and present value maximization requires they be harvested earlier than before. More of the 6-7.9 inch trees are allowed to grow into the higher valued diameter class before being harvested.

The second price assumption varied for Case I involves the difference between stumpage prices for thinned volume versus volume removed in a clearcut. Some researchers (e.g., Broderick et al. 1982) have modeled the effects of increased thinning costs by reducing per unit stumpage prices as a percentage of clearcut prices. Initial stumpage price assumptions for Case I were changed, with Table 5 prices representing thinning volumes, and assuming 25 percent higher prices for volume in the final harvest. Present values under this assumption were \$587.12 for clearcutting now, and \$512.09 for thinning now and clearcutting in 5 years. The optimal thinning schedule under this assumption included removing 26 trees of species 1 from diameter class 10-11.9, while for species 2, 48 trees

were cut from the 6-7.9 inch class and 3 were removed from the 8-9.9 inch class. As 25 percent higher prices are obtained at final harvest, more of the larger, species 1 trees are left in the residual stand. For the lowest diameter class of species 2, however, the 25 percent increase represents a much smaller gain. More of the smaller trees are used to comprise the necessary volume for the thinning to be feasible.

Finally, in determining the overall policy which maximizes present value, the number of growth periods considered should be sufficient to reflect all input assumptions for the stand. Final harvest age is sensitive to such factors as the interest rate and the product values assumed. For the formulations presented in the present study, the present value relationship is not necessarily concave with respect to the number of growth periods considered. To demonstrate this, consider Case II with a discount rate of 3 percent, and random-length log prices for trees in the 14-15.9 inch diameter class only. Estimated solutions to Cases IIa, IIb, and IIc were obtained with 1,000 evaluations per stage of the MS-MCIP program. The present value estimates for the 1, 2, and 3-growth period formulations were \$341.64, \$307.37, and \$328.58, respectively. It is also recognized that the solutions generated are merely estimated optima. The present value

differences are of sufficient size, however, to indicate that the true optima would follow a similar order of magnitude. Trees do not advance into diameter class 14-15.9 until the third projection period, resulting in a present value increase following the decrease for the 2-period model. Under these assumptions, sufficient growth periods would have to be considered to fully reflect future growth into the sawlog diameter classes.

The sensitivity of thinning model solutions to certain input assumptions was considered. The analysis did not reveal any unexpected relations, but demonstrated the need to consider the input assumptions in evaluating when final harvest should occur. Thinning model results are also related to the growth rates implied by the stand-table projection parameters. The parameters assumed in the present analysis were not varied in the sensitivity evaluation, however, since these values were assigned to achieve certain growth and yield results. Arbitrary changes in the parameter values assumed for the hypothetical stand may result in illogical growth model predictions.

Discussion

The thinning model formulated in the present study represents an entire class of problems. The model cannot be solved for a single set of inputs, and the solution universally applied. Optimal thinning schedules vary with

species composition, stand age and structure, site quality, and other biological and economic factors associated with mixed-species stands. The model must therefore be solved for every stand for which a thinning policy is considered, requiring an easily applied solution technique.

The nonlinear programming subroutine used in solving the thinning model examples is not easily applied, and adequate solutions were not obtained, even for an assumed stand of very simple structure. Respecifying the user-supplied subroutine for VMCON alone detracts from its use in solving repeated problems. Random search methods, however, are easily applied. Such techniques become competitive for solving optimization problems when function characteristics are difficult to calculate, when computer storage is limited, or when numerous local optima exist (Solis and Wets 1981).

Random search techniques for optimization are direct search methods, as function gradients are not considered. Many such approaches are dismissed as possible solution methods due to their lack of a theoretical basis and demonstrated inefficiencies for certain problems. These factors should not result in ignoring direct search methods for many applied problems, however (Swann 1974). The following discussion concerns the use of simple and multistage random search methods for solving thinning model

formulations.

Several reasons for using random search techniques in optimization were presented by Karnopp (1963). The advantages of using such techniques for solving the hardwood thinning model include the use of very little computer memory, and the possibility of designing a single program for use with input data from different stands. Such programs could be developed for microcomputers, expediting applications of the thinning model. The longest FORTRAN program coded for the previous examples was approximately 500 lines. Solutions generated in the present study required execution times from a few seconds to 3 minutes, on an IBM 3081 central processing unit. Another advantage in using random search techniques to solve thinning problems is that integer solutions are obtained, avoiding the problems involved with rounding continuous values. Also, if problems are encountered with generating feasible solutions, the random number sets resulting in infeasible answers may be modified to yield acceptable alternatives.

The random search approaches applied in the present study also have shortcomings, however. These methods are clearly not the most practical for many problems, and would be extremely inefficient in solving problems with certain structures, e.g., linear programs, problems which can be solved using calculus, etc.

Convergence is another issue with these techniques. The simple random search approach converges only as the number of solutions evaluated approaches infinity (Matyas 1965), while the multistage approach was shown to result in a local maximum in some solutions to Case I. Convergence results have been demonstrated for other random search algorithms by Solis and Wets (1981), although examples for constrained optimization were not presented. A method of searching for the global optimum using random search was presented by Anderssen (1972). The method involves testing the hypothesis that the decision variable values obtained are elements of a set containing the values in the globally optimal solution. Repeated sampling and refinement of the designated set is performed until the hypothesis is not rejected.

A serious criticism of the simple random search approach was presented by Golden and Assad (1981). These reviewers contended that Conley's (1980) argument in defense of MCIP is not the most appropriate. Conley's defense of simple random search is based on the probability of obtaining an objective function value within a certain fraction of the global optimum, considering all possible solutions. Golden and Assad propose the actual objective value as the most important consideration, and the most appropriate goal as obtaining at least one solution with an

objective value within a certain percentage of the optimal value. Such a goal requires a much larger sample size than the argument presented by Conley. Golden and Assad do not consider the shape of the distribution for objective function values, however. As previously discussed, if the distribution is characterized by a relatively light right-hand tail, objective values within a small upper region will be near (in actual value) the optimal solution, achieving the result specified by Golden and Assad. The MCIP solution of \$481.25 for Case I, for example, is within 99 percent of the optimal value of \$485.76 .

The greatest shortcoming of the MCIP technique is that for actual problems, the entire distribution of objective values, including the exact shape of the right-hand tail, is unknown. The general shape of the entire distribution for a problem may be inspected for large random samples of solutions. Such procedures may indicate problems for which random search methods should not be used, but cannot result in complete confidence in using the approaches for a particular problem.

Procedures for evaluating heuristic solutions to large combinatorial problems were investigated by Dannenbring (1973,1977). Two general approaches were considered for estimating optimum solution values. One set of procedures involves random sampling to obtain reduced-bias estimates of

the optimum value. The second method uses concepts developed in statistical extreme-value theory to derive best-fit estimates of parameters for the asymptotic distribution of extrema. One of the parameters obtained is an appropriate estimate of the optimum solution value. As previously discussed, the tail behavior of the objective value relative frequency distribution will affect the performance of random search algorithms. Dannenbring did, however, address tail behavior in considering procedures for evaluating the performance of such methods. A truncation point estimator was proposed as superior for the combinatorial problems used in his analysis, regardless of the objective value distribution. The statistical extreme-value approach was used by McRoberts (1971) in evaluating solutions obtained with a heuristic algorithm. In general, the estimated optimal objective value may be compared with estimates obtained with inexact algorithms, thereby evaluating the performance of such methods as random search for solving particular problems. Additional methods for evaluating the quality of heuristic algorithms in general were reported by Silver et al. (1980).

For the examples used in the present study, the MS-MCIP approach resulted in higher final present value estimates than MCIP, for the same number of solutions generated. The approach should not be considered superior to MCIP for all

thinning model problems, however, based on the solutions generated for the previous examples. Examining complete objective value probability densities for problems developed for actual stands, using estimated growth model parameters, is required before final conclusions can be made on the effectiveness of these techniques for estimating optimal hardwood thinning schedules. Of the two approaches, however, the multistage method appears to have more potential in yielding estimated optima for such problems. Based on results from the examples in the present study, further investigation of this technique should include varying the number of stages, the numbers of evaluations generated at each stage, and the reductions in the possible ranges used for decision variables.

Results from using random search heuristics in the thinning model demonstration were generally positive. Such methods should be given further consideration for solving this class of problems. MCIP and MS-MCIP are not the only random search possibilities, however. A random search method for constrained optimization was presented by Luus and Jaakola (1973), for example. The algorithm presented by Solis and Wets (1981) for unconstrained minimization is another method which might be adapted to the present problem. Further study of approaches for mixed-hardwood thinning formulations would benefit from final growth model

specification, with parameters estimated from remeasurement data.

V. SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

Summary And Conclusions

Upland hardwood stands of mixed-species are the most common forest types in the United States. Thinning such stands has not been widely practiced in the past, chiefly due to inadequate markets for lower quality hardwood raw materials. Markets for lower grade hardwoods are expanding, however, and increasing emphasis is being placed on hardwood management. The present study involves deriving optimal thinning and rotation for mixed-hardwood stands. A general formulation of the problem was developed and solution techniques were considered.

A means of projecting growth and yield for mixed-hardwood stands was required prior to formulating a thinning optimization model. The growth model must reflect both biological and economic effects from partial harvests, and therefore must predict stand volume over time by diameter class and species. A stand-table projection model was tentatively specified with upgrowth and mortality equations for each species/diameter class combination.

Upgrowth by species and diameter class was modeled by reducing an estimated upper potential to an actual upgrowth estimate, using stand volume measures to determine the proportion of potential realized. Thinning therefore

results in increased diameter growth rates for the residual stand, as measures of stand density are reduced. The mortality relation for each species/diameter class combination was specified with the same variables used in modeling upgrowth. Both equations included measures of stand volume for each species group recognized.

The stand-table projection model specified for mixed-hardwoods was used in formulating a thinning optimization model with nonlinear programming. The interface between growth model and thinning model was accomplished by specifying numbers of trees to cut from each species/diameter class combination as decision variables in the nonlinear program. Constraints were developed for defining the residual stand after each thinning. Optimal thinning schedules are derived for successive numbers of growth periods. The rotation with the greatest present value of land and timber is selected as optimal, among the set of growth periods projected.

The thinning model was formulated for stands which are presently of thinning age. Application to younger stands may be accomplished, however, by projecting such stands to thinning age prior to solving for optimal thinning schedules. The model has sufficient resolution to reflect mixed-hardwood factors such as interspecific growth rates, thinning effects, and value-by-size-class relationships.

Another property of the thinning model is that constraints may be added to represent wildlife, recreation, or other management objectives.

The thinning model was demonstrated for a hypothetical stand of two species. Growth model parameters were assigned for the demonstration for projecting the stand in 5-year intervals. A stand of very simple structure was also specified to aid in evaluating solution techniques. Two general approaches were used in solving thinning model formulations: a nonlinear programming algorithm, and heuristic algorithms involving random search. Both simple random search and a multistage random search approach were included in the evaluation.

Considering a single 5-year growth period, the optimal thinning policy for the simple stand, Case I, was determined through an exhaustive search of the entire feasible region. The problem had 2 million possible solutions, and was therefore large enough to evaluate both random search methods and the nonlinear programming algorithm. Problems were encountered in obtaining solutions with the nonlinear programming algorithm. Two solutions to Case I were obtained, however, from widely different initial estimates. The solutions obtained were identical but were not globally optimal. The example indicates the unreliability of estimating global optima to nonconvex problems based on

convergence to a common solution from different starting points.

Before applying the random search techniques to Case I, the probability density of objective function values for the problem was examined. Random search methods may be considered for problems where functional evaluations are relatively inexpensive, and the probability density of objective function values has a light right-hand tail. The distribution for Case I had the desired property. A disadvantage of using random search methods is that for problems of realistic size, the entire distribution cannot be examined. For Case I, simple random search provided solutions with objective values within 99 percent of the optimum using very little computer storage and execution time. The multistage random search method produced the global optimum in several trials. It was also demonstrated, however, that the multistage approach may result in local optima.

Case II was formulated for the stand assumed for growth model parameter assignment. The problem involved formulations for 1, 2, and 3 growth periods, corresponding to Cases IIa, IIb, and IIc, respectively. The nonlinear programming algorithm was applied to a simplified formulation of Case IIa with little success. One solution was obtained with a greater objective value than obtained

with random search. Rounding the solution to integer numbers of trees, however, resulted in a lower present value than obtained with the other methods. Due to the lack of success in solving the 1-growth period formulation, the algorithm was not applied to solving the more involved 2 and 3-period problems.

Random search solutions were generated for all Case II formulations. Probability densities resulting from 10,000 random solutions were examined for each subproblem. The distributions were characterized by light right-hand tails. In all solutions generated for the Case II problems, the multistage method resulted in greater present value estimates than simple random search. Final solutions to Cases IIb and IIc were more sensitive to the number of thinning schedules evaluated.

A limited analysis of thinning model sensitivity was performed for changes in several input assumptions. Although results from such changes were as expected, an important property of the model became evident during the analysis. The thinning model present value relationship is not necessarily concave with respect to the number of growth periods projected. In determining optimal final harvest age, therefore, a sufficient number of growth periods must be projected to ensure that all input assumptions concerning relative product values are fully reflected.

The thinning model developed for mixed-hardwoods represents an entire set of problems. Optimal thinning plans vary with species composition, stand age and structure, site quality, and other biological and economic factors associated with such stands. The model must therefore be solved for every stand for which thinning is considered, requiring a solution method that can be easily and inexpensively applied. The random search methods evaluated are viable alternatives for solving the thinning model. Although convergence to the global optimum is not guaranteed, procedures involving extreme-value estimation are available for evaluating the estimated results from such solution methods. In addition, the methods are easily used, and could be adapted for solution on microcomputers, expediting a wide and inexpensive application of the model for diverse stands.

Of the random search methods evaluated, the multistage approach appears to have the most potential for solving thinning model formulations. Other random search techniques should also be considered, however. Using growth model parameters estimated from remeasurement data would ensure future evaluations free of any artifacts which may have resulted from the parameter values assigned for the present demonstration. Also, the final growth model specification directly influences the exact thinning model formulation,

and thus the solution methods considered.

Recommendations For Further Research

Recommendations for further study are presented for both the growth model and the thinning model. Specification as well as actual implementation of the growth model are discussed. For the thinning model, recommendations for further study are presented for both formulating and solving the optimization problem.

Growth Model

Specification. The growth model specified for use in deriving optimal mixed-hardwood thinning schedules incorporates certain mixed-species modeling concepts in a stand-table projection framework. The method is an original synthesis of concepts in modeling growth and yield, and in the absence of data, only a tentative specification was proposed. Further study of this approach to modeling stand growth must include estimating potential proportions of upgrowth, as well as estimating the adjustment and mortality function parameters. Final specification of these relations must consider the ability of alternate forms to reflect remeasurement data.

In evaluating alternate specifications of the growth model, stand-table projections should be made as integer numbers of trees. The ultimate use of the growth model requires reliable integer projections, following the removal

of integer numbers of trees during thinning.

Further study of the growth model should also consider incorporating stand age in the mortality function. Age is represented in the upgrowth relation, as different potential proportions are specified after each growth period. The tentative form for the mortality relation, however, merely incorporates the diameter class and species, and a measure of the degree of competition experienced during a particular growth interval.

Implementation. The feasibility of developing parameter estimates for general use in implementing the thinning model should be investigated following the final specification. Potential proportions of upgrowth would be required by age, diameter class, and species group. For the adjustment and mortality functions, parameter estimates would be needed by diameter class, for species groups commonly associated in mixed-hardwood forest types. With tables of such parameter estimates, optimal thinning and rotation could be estimated for any mixed-hardwood stand, given the initial age and stand-table information.

Thinning Model

Formulation. The complexity of the thinning model formulation for mixed-hardwoods is directly related to the final form of the growth model. Further study of the formulation may therefore be required if significant changes

are necessary for the growth model to adequately reflect growth and yield data. The formulation is also related to the solution method used, however. If random search techniques are employed, for example, the thinning model may be much more detailed than if a nonlinear programming algorithm is applied. Developing a general model for deriving optimal thinning and rotation for mixed-hardwoods requires joint considerations in all phases of modeling the problem.

The thinning formulation presented for mixed-hardwoods is a stand-level model, as opposed to forest-level harvest scheduling models. Most even-aged, mixed-hardwoods are privately owned, relatively small, and have a common management history throughout the stand. The thinning and rotation problem was therefore approached from the beginning as a stand-level problem, i.e., in many cases for mixed-hardwoods, the stand and forest are synonymous. In other situations, however, stand treatments cannot be considered alone. Optimal forest-level policies may be derived by aggregating optimal stand treatments in the case of fully regulated forests, or if harvest-level constraints are unnecessary (Hann and Brodie 1980). For most applications, however, integrating stand-level optimization with forest-level harvest scheduling is required. Methods of accomplishing this have been presented by Nazareth (1973),

Williams (1976), and De Kluyver et al. (1980). Further study of the mixed-hardwood thinning model should include investigating means of formulating the problem as a forest-level harvest scheduling model.

Solution. Two random search methods were used in solving thinning model formulations. Further study of these and other random search approaches is recommended following growth model specification and parameter estimation. Further study of the multistage approach should include varying the numbers of stages and evaluations per stage, as well as the variable ranges used. The multistage approach presented by Luus and Jaakola (1973) should also be considered. The method involves evaluating relatively few random solutions at each of hundreds of stages. Initial variable ranges are reduced by a very small percentage after each stage. The algorithm presented by Solis and Wets (1981) for unconstrained minimization should also be considered for adaptation to solving thinning model formulations.

Further research concerning thinning model solutions should also include the method for obtaining global optima with random search presented by Anderssen (1972). Anderssen's refinement procedure involves hypothesis testing and requires that several parameter values be assigned. Evaluating the process for the mixed-hardwood formulation

may involve significant trial and error before suitable values for these parameters are established. Incorporating concepts presented by Dannenbring (1977) for evaluating heuristic solutions should also be investigated. Comparing estimated extreme (optimal) solution values with the values obtained with random search algorithms could be used in developing meaningful stopping criteria for random search methods.

The final stage of research for the mixed-hardwood thinning problem involves developing programs for implementing the model on microcomputers. A single program could be coded to estimate optimal thinning and rotation for various stands. Input to the program would include appropriate values for the economic parameters, stand age, and stand-table data. Using tables of growth model parameter estimates and an appropriate random search solution method, inexpensive estimates of optimal thinning and rotation would be readily available for wide application to mixed-hardwood forest types.

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APPENDICES

Appendix A. Nonlinear programming specification of the
hardwood thinning model.

Appendix A is a complete and general statement of the nonlinear programming formulation of the hardwood thinning problem, including a list of definitions for all variables used in the general statement.

Objective Function:

$$\text{Maximize: PV} = \left\{ \sum_{k=0}^G \left[\sum_{i=1}^S \sum_{j=1}^{n_i+k} \left[(P_{ij} / (1+r)^{kt}) N_{ijk}^C \right] \right. \right. \quad (A1)$$

$$\left. \left. - X_k FC / (1+r)^{kt} - MX_k (1 - X_k) \right] + [L / (1+r)^{Gt}] \right\}$$

Subject to:

Residual Defining Constraints

Initial residual trees (for all species/diameter class combinations):

$$N_{ijk}^R - N_{ijk}^I + N_{ijk}^C = 0 \quad (i=1, \dots, S \quad j=1, \dots, n_i \quad k=0) \quad (A2)$$

Residual trees in the smallest diameter classes (after each period):

$$N_{ijk}^R - (N_{ijk-1}^R) \left(\text{EXP} [b_{s+2}^{ij} (V_{T,k-1}^R) + \sum_{m=1}^s (b_{s+2+m}^{ij} (V_{m,>j,k-1}^R))] \right)$$

$$- (PP_{ijk}) \text{EXP} [b_1^{ij} (V_{T,k-1}^R) + \sum_{m=1}^s (b_{m+1}^{ij} (V_{m,>j,k-1}^R))] \quad (A3)$$

$$+ N_{ijk}^C = 0 \quad (i=1, \dots, S \quad j=1 \quad k=1, \dots, G)$$

Residual trees in the intermediate diameter classes
(after each period):

$$\begin{aligned}
 N_{ijk}^R - (N_{ijk-1}^R) & \left(\text{EXP} \left[b_{s+2}^{ij} (V_{T,k-1}^R) + \sum_{m=1}^s (b_{s+2+m}^{ij} (V_{m,\geq j,k-1}^R)) \right] \right. \\
 & \left. - (PP_{ijk}) \text{EXP} \left[b_1^{ij} (V_{T,k-1}^R) + \sum_{m=1}^s (V_{m,\geq j,k-1}^R) \right] \right) + N_{ijk}^C \quad (A4) \\
 & - (PP_{i,j-1,k}) \text{EXP} \left[b_1^{ij-1} (V_{T,k-1}^R) + \sum_{m=1}^s (b_{m+1}^{ij-1} (V_{m,\geq j-1,k-1}^R)) \right] \\
 (N_{i,j-1,k-1}^R) & = 0 \quad (i=1, \dots, S \quad j=2, \dots, n_1+k-1 \quad k=1, \dots, G)
 \end{aligned}$$

Residual trees in the largest diameter classes (after
each period):

$$\begin{aligned}
 N_{ijk}^R - (N_{i,j-1,k-1}^R) & \left((PP_{i,j-1,k}) \text{EXP} \left[b_1^{ij-1} (V_{T,k-1}^R) + \right. \right. \quad (A5) \\
 & \left. \left. \sum_{m=1}^s (b_{m+1}^{ij-1} (V_{m,\geq j-1,k-1}^R)) \right] \right) + N_{ijk}^C = 0 \\
 & (i=1, \dots, S \quad j=n_1+k \quad k=1, \dots, G)
 \end{aligned}$$

Maximum Harvest Volume Constraints:

$$\sum_{i=1}^s \sum_{j=1}^{n_1+k} (V_{ij} N_{ijk}^C) \leq H_{1k} X_k \quad (k=0, \dots, G-1) \quad (A6)$$

Minimum Harvest Volume Constraints (if harvesting occurs):

$$\sum_{i=1}^s \sum_{j=1}^{n_1+k} (V_{ij} N_{ijk}^C) \geq H_{2k} X_k \quad (k=0, \dots, G-1) \quad (A7)$$

Constraints Defining a Range for X_k :

$$0 \leq X_k \leq \quad (k=0, \dots, G-1) \quad (A8)$$

Non-negativity Restrictions:

$$N_{ijk}^{R,C} \geq 0 \quad (i=1, \dots, S \quad j=1, \dots, n_i+k \quad k=0, \dots, G) \quad (A9)$$

Variable Definitions:

In all cases, indexes used are: i for species, j for diameter class, and k for growth period. Indexes m and q are used for summation in the problem statement, in cases where i or j are held constant. All other variables used in the general problem statement as well as the text of the study are defined below in alphabetical order. Vectors used in the Case IIa formulations of Tables 8 and 9 are not included in the definitions. These vectors are defined in Table 10 and are not used elsewhere in the study.

A represents the number of age periods in a

discrete dynamic programming network,

a is an area in the right-hand tail of a

probability density of objective function values,

ADJ_{ijk} is an adjustment to the potential proportion

of upgrowth, representing the percentage of PP_{ijk}

realized,

$b_m^{ij} \leq 0, m=1, \dots, 2S+2$, growth model upgrowth

($m=1, \dots, S+1$) and mortality ($m=S+2, \dots, 2S+2$)

parameter estimates,

C is used as a superscript denoting numbers of

trees cut,

D is the number of diameter classes in a discrete dynamic programming network,

FC is a fixed thinning and final harvest cost,

G is the number of growth periods modeled,

H_{1k} is a maximum harvest volume after period k ,

H_{2k} is a minimum harvest volume after period k ,
observed only if a harvest occurs,

HV represents harvest value in dollars,

I is a superscript denoting initial numbers of trees,

L represents the land sale value assumed,

M is a superscript denoting numbers of trees dying (mortality),

n is the number of solutions evaluated using random search,

n_1 is the initial number of diameter classes (by species),

N_{ijk}^C represents the number of trees cut (by species, diameter, and growth period),

N_{ijk}^I represents the initial number of trees (by species, diameter, and growth period),

N_{ijk}^R represents the residual number of trees (by species, diameter, and growth period),

QTY_{ijk-1} represents a quantity at the beginning of growth period k (in units projected),

P_{ij} is a per tree stumpage value, calculated as the relevant price (per unit volume) times the appropriate average volume per tree,

PP_{ijk} represents the potential proportion of upgrowth (by species, diameter, and growth period),

Pr is the probability that at least one random search solution is obtained within area (a),

PV represents the present value of land and timber cut,

r is a real alternative rate of return,

R is a superscript denoting numbers of residual trees,

RL is rotation length in years,

S is the number of species groups represented,

SEV is soil expectation value,

t is the number of years per growth period,

TC is the number of classes used for numbers of trees in a discrete dynamic programming network,

U is a superscript for numbers of trees projected as upgrowth,

UPG_{ijk} is upgrowth (in units projected),

V_{ij} is an average volume per tree (by species and diameter),

$V_{m, \geq j, k}^R = \sum_{m=j}^{n_i+k} (V_{mq}^R N_{mqk}^R)$ represents residual

volume (by species and growth period) in diameter classes $\geq j$,

$$V_{T,k}^R = \sum_{i=1}^s \sum_{j=1}^{n_i+k} (V_{ij} N_{ijk}^R)$$

represents a total

residual volume (by growth period), and

X_k is an intermediate variable used to reflect whether or not harvesting occurs after period k .

Appendix B. Evaluation of constraint convexity.

Twice differentiable multivariate functions may be characterized as convex if and only if the Hessian matrix is positive-semidefinite. This procedure was not used to evaluate program convexity in the hardwood thinning model, however, due to the number of variables involved and the resulting dimensions of the Hessian. The structure of the residual-defining constraint sets ((A3) through (A5)) was examined, however, by simplifying the relation used for intermediate diameter classes.

Consider an equation from constraint set (A4) for residual volume after growth period 1, assuming $S=2$ and $2 \leq j \leq n$.

Let:

$$B_1 = b_1^{ij} \left[\sum_{i=1}^2 \sum_{j=1}^{n_1} (V_{ij} N_{ij0}^R) \right] + b_2^{ij} \left[\sum_{m=j}^{n_1} V_{1m} N_{1m0}^R \right] \quad (B1)$$

$$+ b_3^{ij} \left[\sum_{m=j}^{n_2} V_{2m} N_{2m0}^R \right],$$

$$B_2 = b_4^{ij} \left[\sum_{i=1}^2 \sum_{j=1}^{n_1} (V_{ij} N_{ij0}^R) \right] + b_5^{ij} \left[\sum_{m=j}^{n_1} V_{1m} N_{1m0}^R \right] \quad (B2)$$

$$+ b_6^{ij} \left[\sum_{m=j}^{n_2} V_{2m} N_{2m0}^R \right], \text{ and}$$

$$B_3 = b_1^{ij-1} \left[\sum_{i=1}^2 \sum_{j=1}^{n_1} (V_{ij} N_{ij0}^R) \right] + b_2^{ij-1} \left[\sum_{m=j-1}^{n_1} V_{1m} N_{1m0}^R \right] \quad (B3)$$

$$+ b_3^{ij-1} \left[\sum_{m=j-1}^{n_2} V_{2m} N_{2m0}^R \right].$$

Using relations (B1), (B2), and (B3), the residual defining constraint from set (A4) may be written:

$$N_{ij1}^R = N_{ij0}^R (e^{B_2} - (PP_{ij1}) e^{B_1}) + (PP_{i,j-1,1}) N_{i,j-1,0}^R e^{B_3} - N_{ij1}^C \quad (B4)$$

From constraint set (A2), however, $N_{ij0}^R = N_{ij0}^I - N_{ij0}^C$. Substituting this relation into equation (B4) and multiplying yields:

$$N_{ij1}^R = N_{ij0}^I e^{B_2 - PP_{ij1}} e^{B_1} N_{ij0}^I - N_{ij0}^C e^{B_2 + PP_{ij1}} e^{B_1} N_{ij0}^C + PP_{i,j-1,1} N_{i,j-1,0}^I e^{B_3 - PP_{i,j-1,1}} N_{i,j-1,0}^C e^{B_3 - N_{ij1}^C} \quad (B5)$$

The same expression may be substituted for N_{ij0}^R in relations (B1), (B2), and (B3). Considering (B1), for example:

$$B_1 = b_1^{ij} \sum_{i=1}^2 \sum_{j=1}^{n_1} V_{ij} (N_{ij0}^I - N_{ij0}^C) + b_2^{ij} \sum_{m=j}^{n_1} V_{1m} (N_{1m0}^I - N_{1m0}^C) + b_3^{ij} \sum_{m=j}^{n_2} V_{2m} (N_{2m0}^I - N_{2m0}^C) \quad (B6)$$

Expanding relation (B6) and collecting constants yields:

$$B_1 = b_1^{ij} \sum_{i=1}^2 \sum_{j=1}^{n_1} V_{ij} N_{ij0}^I + b_2^{ij} \sum_{m=j}^{n_1} V_{1m} N_{1m0}^I + b_3^{ij} \sum_{m=j}^{n_2} V_{2m} N_{2m0}^I - b_1^{ij} \sum_{i=1}^2 \sum_{j=1}^{n_1} V_{ij} N_{ij0}^C - b_2^{ij} \sum_{m=j}^{n_1} V_{1m} N_{1m0}^C - b_3^{ij} \sum_{m=j}^{n_2} V_{2m} N_{2m0}^C \quad (B7)$$

Relation (B7) may be expressed as:

$$B_1 = k_1 - b_{(1)}^T N \quad (B8)$$

Where k is a non-negative constant, $b_{(1)}$ is a vector of

regression constants from equation (B1), and N is a vector of N_{ij0}^C variables multiplied by appropriate V_{ij} constants. Similar expressions for equations (B2) and (B3) are:

$$B_2 = k_2 - b^{(2)T} N \quad (B9)$$

$$B_3 = k_3 - b^{(3)T} N \quad (B10)$$

Using results (B8), (B9), and (B10), and letting K 's also represent non-negative constants, equation (B5) may be expressed:

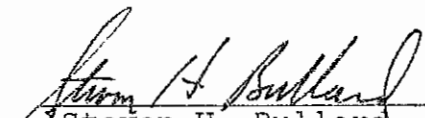
$$N_{ij1} = K_1 e^{-b^{(2)T} N} - K_2 e^{-b^{(1)T} N} - K_3 N_{ij0}^C e^{-b^{(2)T} N} + K_4 N_{ij0}^C e^{-b^{(1)T} N} \quad (B11)$$

$$+ K_5 e^{-b^{(3)T} N} - K_6 N_{i,j-1,0}^C e^{-b^{(3)T} N} - N_{ij1}^C$$

Equation (B11) is written entirely in terms of variables expressing numbers of trees to cut. As this expression is a nonlinear equality, it represents a nonconvex feasible region. In addition, equation B(11) lacks any convexity structure, as the right hand side includes sums of both convex and concave functions of the decision variables. The first-order Kuhn-Tucker local optimality conditions are therefore not sufficient to characterize a solution as globally optimal. Similar results could be shown for the other constraint sets defining residual numbers of trees ((A3) and (A5)).

VITA

The author, son of Henry B. and Margaret E. Bullard, was born April 8, 1955, in Camp LeJeune, North Carolina. Grade-schools were attended in Norfolk, Virginia; Rota, Spain; Memphis, Tennessee; and Woodland, Mississippi. The author graduated from Woodland Attendance Center in 1973 and entered the forestry program at Mississippi State University. On December 15, 1975, he married the former Gail "Susie" Partridge of Maben, Mississippi. The B.Sc. degree in forestry was conferred in 1977. After working one year as a research assistant in Forest Entomology at MSU, the author entered the forestry graduate program in 1978. On March 13, 1979, the author and his wife were blessed with a son, Jacob Henry. The M.Sc. degree in Forest Management-Economics was awarded in 1980. The author and family presently reside in Blacksburg, Virginia, where he is a Ph.D. candidate in Forest Management-Economics at Virginia Polytechnic Institute and State University. The author hopes to pursue a career in teaching/research at the university level. He is a member of the Society of American Foresters.


Steven H. Bullard

MIXED-HARDWOOD
THINNING OPTIMIZATION

by

Steven H. Bullard

(ABSTRACT)

Expanding markets are expected to create new opportunities for active forest management in upland hardwood stands. A procedure was developed for estimating economically optimal thinning policies for mixed-species hardwoods by interfacing a stand-table projection growth model with a nonlinear programming thinning model formulation. The thinning model provides information on numbers of trees to harvest over time by species and diameter class, and therefore has sufficient resolution to reflect interspecific growth rates and value-by-size-class relationships.

The diversity of biological and economic factors associated with mixed-hardwoods requires solution methods which can be easily and inexpensively applied to formulations for individual stands. A nonlinear programming algorithm and heuristic methods involving random search were evaluated as solution techniques in a demonstration of the thinning model. For the demonstration, growth model parameters were specified for a hypothetical stand. Both

simple random search and multistage random search methods appear promising for solving thinning model formulations for mixed-hardwoods. As formulated, thinning problems are combinatorial in nature, belonging to a class of problems for which heuristics are often used. Further study is needed, however, to evaluate such methods for solving mixed-hardwood thinning problems, using growth model parameters estimated from remeasurement data.