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Observations on Convexity

Chad A. Huckaby
Stephen F Austin State University, chad_huckaby2000@yahoo.com

OBSERVATIONS ON CONVEXITY

by

CHAD ALAN HUCKABY, B.S.

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CHAD ALAN HUCKABY, B.S.

APPROVED:

Roy Joseph Harris, Ph.D., Thesis Co-Director

Sarah Triana Stovall, Ph.D., Thesis Co-Director

Clint Richardson, Ph.D., Committee Member

Lee W. Payne, Ph.D., Committee Member

Dr. Richard A. Berry, D.M.A.,
Dean of the Graduate School

ABSTRACT

This thesis will explore convexity as it pertains to sets of complex-valued functions. These include preliminary looks at established linear and polynomially convex hulls, along with the development of new types of convex hulls. These types will include, but are not limited to the hulls determined by inversions, shift inversions, and Möbius transformations. A convex hull must be preceded by the set of functions involved. These hulls are the smallest convex sets that contain the original set. Justifications and precise definitions are included within the body of the work.

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“I can do all things through Christ who strengthens me”- Philipians 4:13

I believe all victories should end in the SFA Alma Mater:

“All Hail to SFA”
Oh future bright ’neath the
Purple and White
All hail to SFA.
’Mid Texas pines we have
Found peaceful shrines
Where ev’ry month is May.
Long live our Alma Mater,
Honor to thee for aye.
As years unfold, happy
Mem’ries we’ll hold,
All hail to SFA.

CONTENTS

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1 INTRODUCTION

A common topic in complex variables is the convex hull of a set. The goal of this thesis is to explore different definitions of the convex hull. The functional definition of a convex hull of a set is as follows: Let $f \in \mathcal{F}$ where $\mathcal{F}$ is a set of functions and Ω is any compact subset of $\mathbb{C}$. Then the $\mathcal{F}$-convex hull of a set Ω, denoted $\hat{\Omega}_{\mathcal{F}}$, is defined by

$$\hat{\Omega}_{\mathcal{F}} = \bigcap_{f \in \mathcal{F}} \left\{ w \in \mathbb{C} : |f(w)| \leq M_f \right\},$$

where $M_f = \max_{z \in \Omega} |f(z)|$. This definition requires further definitions to explain, all of which including modulus and compactness, are presented as the remainder of this chapter.

1.1 Results from Complex Variables

The usual first definition of convexity is the type called geometric convexity.

Definition 1.1. A set G is **geometrically convex** if given any two points a and b in G the line segment joining a and b, $\overline{ab}$, lies entirely in G.

This type of convexity may also be referred to as linear convexity.

Definition 1.2. The **modulus** of a complex number $z = x + iy$ is defined as

$$|z| = \sqrt{x^2 + y^2},$$

where $x, y \in \mathbb{R}$.

Definition 1.3. The **$\mathcal{F}$-convex hull of a set** is defined as follows: Let $\mathcal{F}$ be a set of functions and Ω any compact subset of $\mathbb{C}$. Then the $\mathcal{F}$-convex hull of a set Ω, denoted $\hat{\Omega}_{\mathcal{F}}$, is defined by

$$\hat{\Omega}_{\mathcal{F}} = \bigcap_{f \in \mathcal{F}} \{w \in \mathbb{C} : |f(w)| \leq M_f\},$$

where $M_f = \max_{z \in \Omega} |f(z)|$.

Remark 1.4. Notice if $z \in \Omega$, then $|f(z)| \leq M_f$ so that $\Omega \subseteq \{w \in \mathbb{C} : |f(w)| \leq M_f\}$. That is, $\Omega \subseteq \hat{\Omega}_{\mathcal{F}}$.

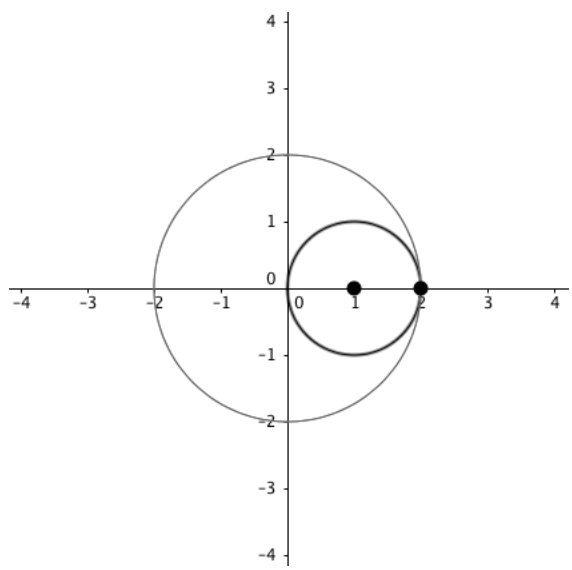

Figure 1.1: Example 1.5 Illustration

Example 1.5. Let $\mathcal{F} = \{z, z^2\}$ and $\Omega = \{z : |z-1| \leq 1\}$. Find $\hat{\Omega}_{\mathcal{F}}$.

First, it is necessary to find where each of the functions in $\mathcal{F}$ achieve their respective maxima. On Ω, both $f(z) = z$ and $g(z) = z^2$ achieve maxima at $z = 2$. Then the $\mathcal{F}$-convex hull is the intersection of the sets defined by two functions have modulus less than their maxima. For f, the set of all $w \in \mathbb{C}$ that causes $|f(w)| \leq 2$ is the disk centered at the origin of radius 2. For g, the set of all $w \in \mathbb{C}$ that causes $|f(w)| \leq 4$ is also the disk centered at the origin of radius 2. Thus the intersection of these sets is the disk centered at the origin of radius 2. Therefore, the $\mathcal{F}$-convex hull of Ω is $\hat{\Omega}_{\mathcal{F}} = \{z : |z| \leq 2\}$.

1.2 Results from Topology

Definition 1.6. Let $X \neq \emptyset$ be a set. Then a **topology** on X is a collection of subsets of X, denoted $\mathcal{T}$, obeying the following axioms:

(a) X and $\emptyset$ belong to $\mathcal{T}$,

(b) the intersection of any two elements of $\mathcal{T}$ is an element of $\mathcal{T}$, and

(c) the union of any sub-collection of $\mathcal{T}$ is an element of $\mathcal{T}$.

Definition 1.7. A **topological space** is a set X together with a topology $\mathcal{T}$ on X.

Definition 1.8. A set is **open** if it is a member of $\mathcal{T}$.

Definition 1.9. A set is **closed** if its complement is a member of $\mathcal{T}$.

Definition 1.10. Let $X \neq \emptyset$ be a set and $d : X \times X \to [0, \infty)$ be a function. Then d is a **metric** for X if for any points $x, y, z \in X$, the following are true:

(a) $d(x, y) \geq 0$

(b) $d(x, y) = 0$ if and only if $x = y$

(c) $d(x, y) = d(y, x)$

(d) $d(x, z) \leq d(x, y) + d(y, z)$

Note that for this thesis, $d(z, w) = |z - w|$.

Definition 1.11. Let $(X, \mathcal{T})$ be a space. A **base** or **basis** for $\mathcal{T}$ is a collection $\mathcal{B}$ of subsets of X such that:

(a) each member of $\mathcal{B}$ is also a member of $\mathcal{T}$, and

(b) if $U \in \mathcal{T}$ and $I \neq \emptyset$, then U is the union of sets belonging to $\mathcal{B}$.

Definition 1.12. Let X be a nonempty set of $\mathbb{C}$ and d a metric for X. The unique topology on X generated by the set of all open r-spheres in X, denoted $B(x, r)$ for some $r > 0$ and $x \in X$, and having these open r-spheres as a base is called the ***d-metric topology for X***. The d-metric topology is denoted $\mathcal{T}(d)$. The topological space $(X, \mathcal{T})$ is called a ***metric space*** if and only if there exists a metric d for X such that the d-metric topology $\mathcal{T}(d)$ on X is the same as $\mathcal{T}$. The notation for a metric space is (X, d).

Definition 1.13. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. A point z in set X is a **boundary point** of A if and only if every open set in X containing z contains at least one point of each of $X - A$ and A. The set of boundary points of A is called the **boundary** of A.

Definition 1.14. Let A be a subset of a topological space X. A point $x \in X$ is a **limit point** of A if every neighborhood of x contains at least one point of A different from x itself.

Theorem 1.15. *A set K is **open** if and only if it contains none of its boundary points.*

Theorem 1.16. *A set K is **closed** if and only if K contains all of its boundary points.*

Remark 1.17. Note that a set K is **non-closed** if and only if K lacks any of its boundary points.

Definition 1.18. A set K is **bounded** if and only if there exists an open disk that contains all of K.

Definition 1.19. Let $A, B \neq \emptyset$ be two subsets of the metric space (X, d). Then the **distance between** A **and** B, denoted $d(A, B)$, is the greatest lower bound of the set $\{d(x, y) : x \in A, y \in B\}$. If $A = \{a\}$, this is written $d(a, B)$ for $d(A, B)$.

Definition 1.20. Let A be a set in $\mathbb{C}$. The **closure** of A, denoted $\bar{A}$, is defined as follows:

$$\bar{A} = A \cup \{z : z \text{ is a boundary point of } A\}.$$

The following will be especially useful in subsequent work.

Theorem 1.21. *Let (X, d) be a metric space and $A \neq \emptyset$ a subset of X. Then $x \in \bar{A}$ if and only if $d(x, A) = 0$.*

Proof. Suppose $x \in \bar{A}$. Then, for each $r > 0$, $B(x, r) \cap A \neq \emptyset$ where $B(x, r)$ is any open disk of radius r centered at x. Therefore, for each $r > 0$ there exists a point $a_r \in A$ such that $d(x, a_r) < r$ and, as a consequence, the greatest lower bound of $\{d(x, a) : a \in A\}$ is zero. The conclusion is that $d(x, A) = 0$. For the converse, assume that $x \in X$ and $d(x, A) = 0$. Now if $x \in A$, certainly $x \in \bar{A}$ by the definition of $\bar{A}$. So suppose $x \in X - A$. It must be shown that x is a boundary point of A. Since $d(x, A) = 0$ for each $r > 0$ there exists a point $a_r \in A$ such that $d(x, a_r) < r$. It follows that for each $r > 0$, $B(x, r) \cap A \neq \emptyset$, showing $x \in \bar{A}$. †

Corollary 1.22. *If a subset of A of a metric space (X, d) is closed and $x \notin A$, then $d(x, A) > 0$. That is to say from any point not in the closed set, there is a positive distance that exists between the set and that point.*

Proof. This result is an immediate consequence of Theorem 1.21. †

Definition 1.23. A set K is compact if and only if every open cover of K has a finite sub-cover.

Theorem 1.24 (*Heine-Borel*)**.** *A set K is compact if and only if K is both closed and bounded.*

2 Linear Convex Hull of a Set

2.1 Geometric Linear Convex Hull of Compact a Set

A common interpretation of the linear convex hull is a geometric one. That is for any two points in the set Ω, the line segment containing those two points is also in the hull. For this thesis, the following definition is used.

Definition 2.1. The **geometric linear convex hull** of a compact set Ω is the set composed of all line segments connecting any two points $z_1, z_2 \in \Omega$. The geometric linear convex hull of Ω is denoted $\hat{\Omega}_G$.

The next several pages will include the development of the relationship between the geometric linear convex hull from Definition 2.1 and the functional linear convex hull from Definition 2.5.

Theorem 2.2. *A compact set* Ω *is a subset of its geometric linear convex hull,* $\hat{\Omega}_G$*. That is,* $\Omega \subseteq \hat{\Omega}_G$.

Proof. Let $z \in \Omega$. Then z is either an interior point of Ω or z is some boundary point of Ω.

Case 1: Suppose that z is an interior point of Ω. Then by definition, there is an open disk $B(z, r) \subseteq \Omega$. Then the closed disk $A = \overline{B(z, \frac{r}{2})} \subseteq \Omega$. Choose any diameter of A with points z_{1_d} and z_{2_d}. Note that $z_{1_d}, z_{2_d} \in \Omega$. Also, $z \in \overline{z_{1_d} z_{2_d}}$. Therefore, by

Definition 2.1, $z \in \hat{\Omega}_G$ and thus $\Omega \subseteq \hat{\Omega}_G$.

Case 2: Suppose that z is a boundary point of Ω. Let z_1 be any point in Ω such that $z_1 \neq z$. Then by Definition 2.1, $\overline{zz_1} \in \hat{\Omega}_G$. Therefore $z \in \hat{\Omega}_G$ and $\Omega \subseteq \hat{\Omega}_G$.

Therefore it has been shown in both cases that $\Omega \subseteq \hat{\Omega}_G$. †

Theorem 2.3. *The geometric linear convex hull of a compact set Ω is bounded.*

Proof. Let Ω be a compact set in $\mathbb{C}$ with geometric linear convex hull $\hat{\Omega}_G$. Let $r = \max_{z \in \Omega} \{|z|\}$. This maximum exists due to the maximum modulus theorem and the compactness of Ω. To prove the result it needs to be the case that $z_0 \in B(0, r)$ for all $z_0 \in \hat{\Omega}_G$. So let $z_0 \in \hat{\Omega}_G$. Then there exists $z_1, z_2 \in \Omega$ such that $|z_2| \leq |z_1| \leq r$ and $z_0 \in \overline{z_1 z_2}$. Since the line segment $\overline{z_1 z_2}$ can be written as $z = tz_1 + (1-t)z_2$ for $t \in [0, 1]$, we can write $z_0 = tz_2 + (1-t)z_2$ for some $t \in [0, 1]$. Now consider the following inequalities:

$$
\begin{aligned}
|z_0| &= |tz_1 + (1-t)z_2| \\
&\leq |tz_1| + |(1-t)z_2| && \text{by the triangle inequality,} \\
&\leq |tz_1| + |(1-t)z_1| && \text{since } |z_1| \geq |z_2|, \\
&= |t||z_1| + |(1-t)||z_1| \\
&= |z_1|(|t| + |1-t|) && \text{by the distributive property,} \\
&= |z_1|(t + 1 - t) && t, (1-t) > 0, t \in \mathbb{R}, \\
&= |z_1|(1) \\
&= |z_1| \\
&\leq r && \text{since } |z_1| \leq r. \\
&< r + 1
\end{aligned}
$$

Thus for any $z_0 \in \hat{\Omega}_G$, $|z_0| \leq r$. This means for all $z \in \hat{\Omega}_G$, $z \in B(0, r+1)$ and $\hat{\Omega}_G$ is bounded. †

Lemma 2.4. *The geometric linear convex hull, $\hat{\Omega}_G$, of a compact set Ω is closed.*

Proof. Let z_0 be a boundary point of $\hat{\Omega}_{\mathcal{G}}$. and suppose for contradiction that $z_0 \notin \hat{\Omega}_{\mathcal{G}}$. Then there is a sequence $\{z_n\} \subseteq \hat{\Omega}_{\mathcal{G}}$ such that $z_n \to z_0$. Associated with each z_n, there is a pair of points z_{1_n}, z_{2_n} such that $z_n \in \overline{z_{1_n} z_{2_n}}$ and $z_{1_n}, z_{2_n} \in \Omega$ (by the definition of geometric hull). Since Ω is compact, there is a convergent subsequence $\{q_{1_n}\}$ of $\{z_{1_n}\}$.

Similarly, let $\{q_{2_n}\}$ be the corresponding subsequence of $\{z_{2_n}\}$. Again there is a convergent subsequence $\{m_{2_n}\}$ of $\{q_{2_n}\}$ that converges to say m_2. This means the associated sequence $\{m_{1_n}\}$ from $\{q_{1_n}\}$ converges to say m_1. Define $\{w_n\}$ from $\{z_n\}$ to be the corresponding subsequence of points that converges to z_0. Let $r = \inf_{z \in \overline{m_1 m_2}} \{|z_0 - z|\}$ and choose $d = \frac{r}{3}$. Consider the disks $\overline{B(z_0, d)}, \overline{B(m_1, d)}, \overline{B(m_2, d)}$. Let l_1 and l_2 be the common external tangents to $\overline{B(m_1, d)}, \overline{B(m_2, d)}$. Define $t_{11} = l_1 \cap \overline{B(m_1, d)}, t_{12} = l_1 \cap \overline{B(m_2, d)}, t_{21} = l_2 \cap \overline{B(m_1, d)}$, and $t_{22} = l_2 \cap \overline{B(m_2, d)}$. Let $G = \overline{B(m_1, d)} \cup \overline{B(m_2, d)} \cup \overline{\square\ t_{11} t_{12} t_{22} t_{21}}$. Notice that the minimum distance from $\overline{B(z_0, d)}$ to G is $2d > d$. So there is an $N \in \mathbb{N} > 0$ such that for all $n \geq N$, $|w_n - z_0| < d$. This means that for all $n \geq N, r = \inf_{z \in \overline{q_{1_n} q_{2_n}}} \{|w_n - z|\} > d$, but $\overline{q_{1_n} q_{2_n}} \to \overline{m_1 m_2}$, which is a contradiction. Thus $z_0 \in \hat{\Omega}_{\mathcal{G}}$. Therefore all boundary points of $\overline{\hat{\Omega}}_{\mathcal{G}}$ are in $\hat{\Omega}_G$ and $\hat{\Omega}_G$ is closed. †

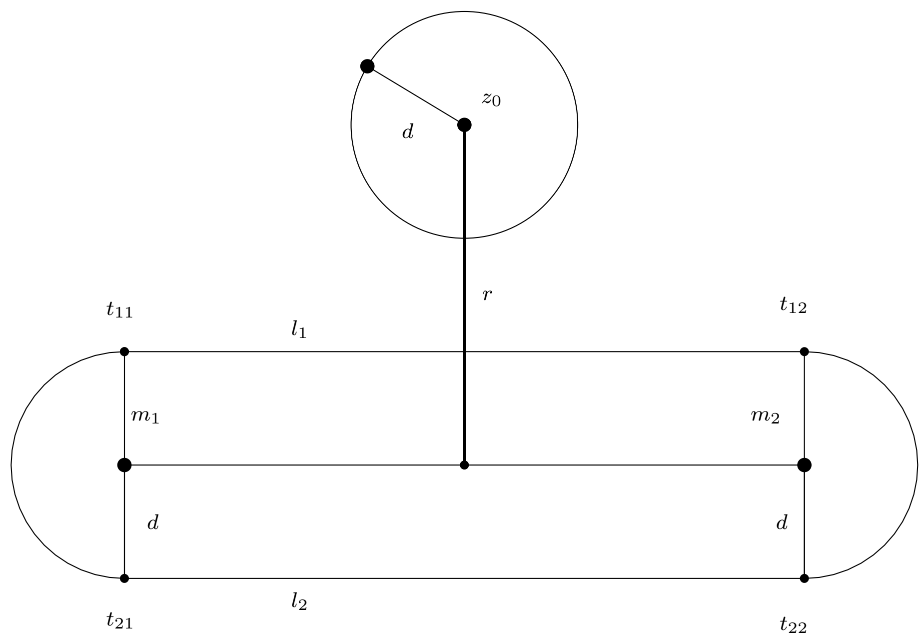

Figure 2.1: Proof of Lemma 2.4.

2.2 Functional Linear Convex Hull of a Compact Set

Another interpretation of the linear convex hull is a functional one, for which the definition is given below.

Definition 2.5. The **functional linear convex hull** of a compact set Ω, denoted $\hat{\Omega}_{\mathcal{F}}$ is defined as

$$\hat{\Omega}_{\mathcal{F}} = \bigcap_{a,b\in\mathbb{C}} \{w : |aw+b| \leq M_f\},$$

where $M_f = \max\limits_{z\in\Omega} \{|az+b|\}$.

Remark 2.6. In the complex plane, $|az + b| \leq r$ means $|a(z + \frac{b}{a})| \leq r$ or $|z + \frac{b}{a}| \leq \frac{r}{|a|}$, which is a closed disk centered at $-\frac{b}{a}$ of radius $\frac{r}{|a|}$. This means that a useful interpretation of Definition 2.5 is that the functional linear convex hull of a set Ω is the intersection of all closed disks containing Ω.

2.3 Determining the Relationship Between the Functional and Geometric Linear Convex Hull of a Compact Set

Theorem 2.7. *Let Ω be any compact set in $\mathbb{C}$. Then the geometric linear convex hull, $\hat{\Omega}_G$, is equivalent to the functional linear convex hull, denoted $\hat{\Omega}_{\mathcal{F}}$. That is to say $\hat{\Omega}_G = \hat{\Omega}_{\mathcal{F}}$.*

Proof. Let Ω be a compact set in $\mathbb{C}$. Let $\hat{\Omega}_G$ and $\hat{\Omega}_{\mathcal{F}}$ be the geometric and functional linear convex hulls of Ω, respectively.

To show that $\hat{\Omega}_{\mathcal{F}} \subseteq \hat{\Omega}_G$, choose a point $z_0 \notin \hat{\Omega}_G$. Then from Corollary 1.22,Theorem 2.3, and Lemma 2.4, there exists some minimum distance d_0 from z_0 to the boundary of $\hat{\Omega}_G$. Let z_1 be on the boundary of $\hat{\Omega}_G$ such that $|z_0 - z_1| = d_0$. Now, construct the line segment from z_0 to z_1. Let z_2 be the midpoint of $\overline{z_0 z_1}$. Let l be the line perpendicular to $\overline{z_0 z_1}$ and passing through z_1. It can be shown using Euclidean geometry that $\hat{\Omega}_{\mathcal{G}}$ is contained in the complement of the half-plane that contains z_0. Since Ω is compact, there exists a greatest distance, d, across Ω. Now consider the line l, containing z_1 perpendicular to $\overline{z_0 z_1}$ and choose points z_{1_d}, z_{2_d} on this line a distance of d away from

z_1 so that z_1 is between z_{1_d} and z_{2_d}. Since three distinct noncollinear points uniquely determine a circle, construct the disk determined by the circle formed by z_2, z_{1_d}, z_{2_d}. This disk captures all of $\hat{\Omega}_G$ and excludes z_0. Thus z_0 can be removed from contention for membership in $\hat{\Omega}_{\mathcal{F}}$. Since z_0 was arbitrary, any point not in $\hat{\Omega}_G$ will also not be in $\hat{\Omega}_{\mathcal{F}}$.

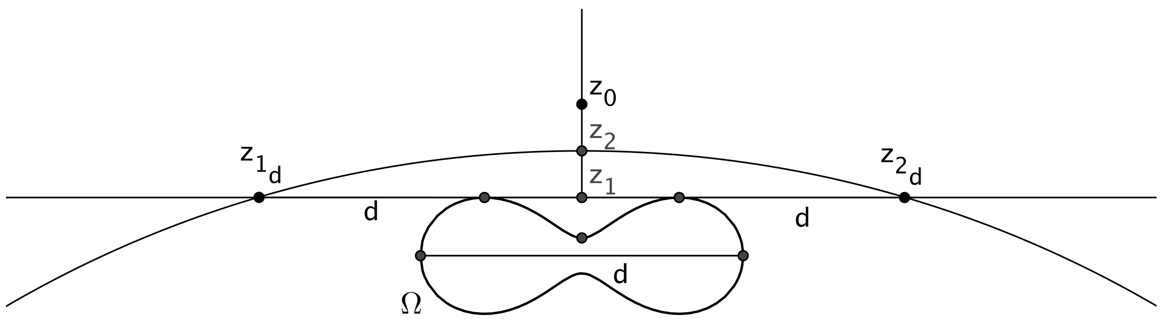

Figure 2.2: Proof of Theorem 2.7.

To show $\hat{\Omega}_G \subseteq \hat{\Omega}_{\mathcal{F}}$, let $z \in \hat{\Omega}_G$. Then there are two points $z_1, z_2 \in \Omega$ such that $z \in \overline{z_1 z_2}$. Then the segment $\overline{z_1 z_2}$ can be written as

$$z = tz_1 + (1-t)z_2,$$

for $t \in [0,1]$. Let $B(q,r)$ be any disk containing Ω. Without loss of generality, let

$|z_2 - q| \leq |z_1 - q| \leq r$. Now consider the following inequalities for any $z \in \overline{z_1 z_2}$.

$$
\begin{aligned}
|z - q| &= |t(z_1 - q) + (1-t)(z_2 - q)| \\
&\leq |t(z_1 - q)| + |(1-t)(z_2 - q)| && \text{by the triangle inequality,} \\
&\leq |t(z_1 - q)| + |(1-t)(z_1 - q)| && \text{since } |(z_1 - q)| \geq |(z_2 - q)|, \\
&= |t||(z_1 - q)| + |(1-t)||(z_1 - q)| \\
&= |(z_1 - q)|(|t| + |1-t|) && \text{by the distributive property,} \\
&= |(z_1 - q)|(t + 1 - t) && t, (1-t) > 0, t \in \mathbb{R}, \\
&= |(z_1 - q)|(1) \\
&= |(z_1 - q)| \\
&\leq r && \text{since } |(z_1 - q)| \leq r.
\end{aligned}
$$

Thus every point $z \in \overline{z_1 z_2}$ is contained in *every* closed disk containing Ω. Therefore $z_0 \in \hat{\Omega}_{\mathcal{F}}$. Therefore, both containments have been shown and for compact sets Ω in $\mathbb{C}$,

$$\hat{\Omega}_G = \hat{\Omega}_{\mathcal{F}}.$$

†

Why is there interest in looking at the relationship between geometric and functional linear convex hulls? As shown in the previous result, compactness makes the respective hulls equal. Relaxing the condition of compactness leads to some interesting results.

2.4 Linear Convex Hulls of Non-closed, Bounded Sets

Definition 2.8. The **geometric linear convex hull** of *any* set Ω is the set composed of all line segments connecting any two points $z_1, z_2 \in \Omega$. The geometric linear convex hull of Ω is denoted $\hat{\Omega}_G$.

To show that $\hat{\Omega}_{\mathcal{G}}$ is not necessarily equal to $\hat{\Omega}_{\mathcal{F}}$ for any set in $\mathbb{C}$, consider the following example.

Theorem 2.9. *Let Ω be a subset of the closed unit disk centered at the origin containing all of the interior of the disk. Then one of the following is true:*

1. *If Ω is open, then the geometric linear convex hull is itself. That is $\Omega = \hat{\Omega}_{\mathcal{G}}$*
2. *If Ω contains any number of its boundary points, where $\mathbf{B}$ is some subset of the boundary of Ω, then the geometric linear convex hull will be the interior of the disk with the boundary points added. That is $\hat{\Omega}_{\mathcal{G}} = \Omega \cup \mathbf{B}$.*

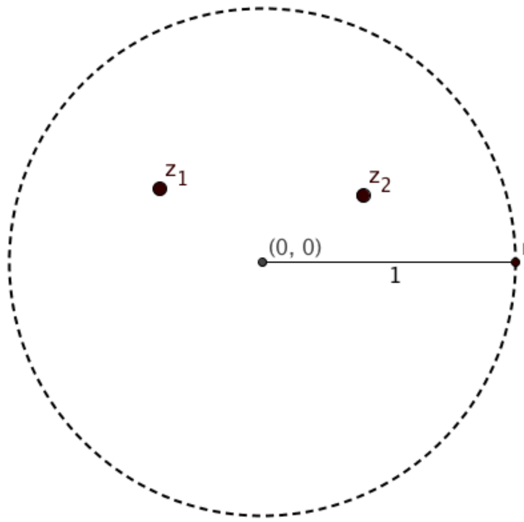

Figure 2.3: Case 1.

Case 1: Let Ω be open. Then the geometric linear convex hull of Ω is itself.

Proof. Let Ω be open. Further let $z_1, z_2 \in B(0,1)$. Then any point z on the line segment $\overline{z_1 z_2}$ can written in the following way:

$$z = z_1 \cdot (t) + (1-t) \cdot z_2 \text{ for some } t \in [0,1]$$

That is $\overline{z_1 z_2} = \{z : z = z_1 \cdot (t) + (1-t) \cdot z_2 \text{ for } t \in [0,1]\}$. Here, it is necessary to show that the entire segment is contained within Ω. That is to say, it is necessary to show the following inequality:

$$|z_1 \cdot (t) + (1-t) \cdot z_2| < 1$$

Now assume without loss of generality that $|z_1| \geq |z_2|$. Consider the following:

$$
\begin{aligned}
|z_1 \cdot (t) + (1-t) \cdot z_2| &\leq |t \cdot z_1| + |(1-t) \cdot z_2| && \text{by the triangle inequality,} \\
&= |t| \cdot |z_1| + |(1-t)| \cdot |z_2| && \text{since } |ab| = |a| \cdot |b|, \\
&\leq |t| \cdot |z_1| + |(1-t)| \cdot |z_1| && \text{by assumption that } |z_1| > |z_2|, \\
&= |z_1| \left(|t| + |1-t|\right) && \\
&= |z_1|(t + 1 - t) && \text{since } t, 1-t \text{ are positive,} \\
&= |z_1| \cdot (1) && \\
&= |z_1| && \\
&< 1 && \text{since } |z_1| < 1.
\end{aligned}
$$

Thus every point on every line segment has modulus strictly less than 1. That is to say that every point is on the interior of $B(0,1)$. Therefore the geometric linear convex hull of the open unit disk is itself.

†

Case 2: Let Ω contain any number of its boundary points. Let z_1 and z_2 be in Ω. If z_1 and z_2 are interior points, the previous proof shows that $\overline{z_1 z_2} \subseteq \Omega$. If not, either z_1, z_2 are both boundary points or one is a boundary point and the other is an interior point.

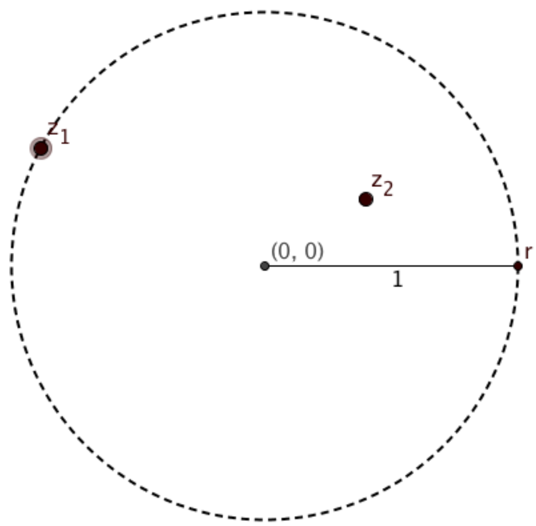

Figure 2.4: Non-closed disk Part I: Case 2

Proof. First, consider that z_1 is a boundary point and z_2 is an interior point of Ω. Then by assumption $|z_1| = 1$ and $|z_2| < 1$. Then any point z on the line segment $\overline{z_1 z_2}$ can be written in the following way:

$$z = z_1 \cdot (t) + (1-t) \cdot z_2 \text{ for some } t \in [0, 1].$$

Here, it is necessary to show that the only boundary points that are picked up in the hull for Ω are the boundary points in Ω. It is necessary to show the inequality below, with equality only at $t = 1$.

$$|z_1 \cdot (t) + (1-t) \cdot z_2| \leq 1$$

Consider the following:

$$\begin{aligned}
|z_1 \cdot (t) + (1-t) \cdot z_2| &\leq |t \cdot z_1| + |(1-t) \cdot z_2| && \text{by the triangle inequality,} \\
&= |t| \cdot |z_1| + |(1-t)| \cdot |z_2| && \text{since } |ab| = |a| \cdot |b|, \\
&< |t| \cdot |z_1| + |(1-t)| \cdot |z_1| && \text{by assumption that } |z_1| > |z_2|, \\
&= (|t| + |1-t|) && \text{since } |z_1| = 1 \\
&= (t + 1 - t) && \text{since } t, 1-t \text{ are positive,} \\
&= 1
\end{aligned}$$

Thus every point on every line segment has modulus less than or equal to 1. Equality occurs at $t = 1$, which is the boundary point z_1.

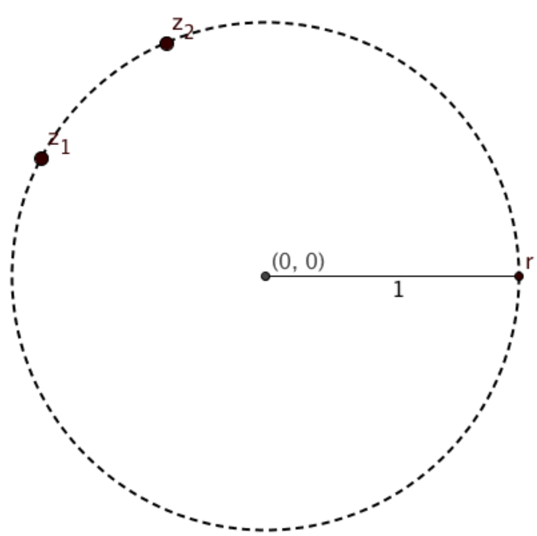

Figure 2.5: Non-closed disk Part II: Case 2

Now suppose z_1, z_2 are on the boundary of Ω. By assumption $|z_1| = 1$ and $|z_2| = 1$.

Then any point z on the line segment $\overline{z_1 z_2}$ can be written in the following way:

$$z = z_1 \cdot (t) + (1 - t) \cdot z_2 \text{ for some } t \in [0, 1].$$

Here, it is necessary to show that the segment $\overline{z_1 z_2}$ contains only interior points other than the endpoints. That is, it must be shown that

$$|z_1 \cdot (t) + (1 - t) \cdot z_2| \leq 1, \text{ with equality only occurring at the endpoints.}$$

Consider the following:

$$\begin{aligned}
|z_1 \cdot (t) + (1 - t) \cdot z_2| &\leq |t \cdot z_1| + |(1 - t) \cdot z_2| && \text{by the triangle inequality,} \\
&= |t| \cdot |z_1| + |(1 - t)| \cdot |z_2| && \text{since } |ab| = |a| \cdot |b|, \\
&= |t| + |(1 - t)| && \text{by assumption that } |z_1| = |z_2| = 1, \\
&= (|t| + |1 - t|) \\
&= (t + 1 - t) && \text{since } t, 1 - t \text{ are positive,} \\
&= 1
\end{aligned}$$

Thus every point on every line segment has modulus less than or equal to 1. Equality occurs at $t = 1$ (the endpoint z_1) and at $t = 0$ (the endpoint z_2). †

It has been shown that for any non-closed subset of the unit disk containing the interior of the disk and centered at the origin, that the geometric linear convex hull is the original set itself.

Theorem 2.10. *For the functional case, let* $\Omega = B(0,1) \cup \mathbf{B}$, *where* $\mathbf{B}$ *is some subset of the boundary of* $B(0,1)$. *Then the functional linear convex hull of* Ω *is the closed unit disk,* $\bar{\Omega}$. *That is* $\hat{\Omega}_{\mathcal{F}} = \bar{\Omega}$.

Proof. Let $\Omega = B(0,1) \cup \mathbf{B}$. Then the functional linear convex hull of Ω is the intersection of all disks containing all of Ω. To show $\hat{\Omega}_{\mathcal{F}} \subseteq \bar{\Omega}$, let $z \in \hat{\Omega}_{\mathcal{F}}$. Then z is in the intersection of all closed disks containing Ω. Since $\overline{\Omega}$ is a closed disk containing Ω, $z \in \overline{\Omega}$. Thus $\hat{\Omega}_{\mathcal{F}} \subseteq \bar{\Omega}$. To show $\bar{\Omega} \subseteq \hat{\Omega}_{\mathcal{F}}$, let $z \in \bar{\Omega}$. Then either $z \in \Omega$ or $z \notin \Omega$ but z is a boundary point of Ω.

Case 1: If $z \in \Omega$, then $z \in \Omega \subseteq \hat{\Omega}_{\mathcal{F}}$. Thus $z \in \hat{\Omega}_{\mathcal{F}}$.

Case 2: If z is a boundary point of Ω and $z \notin \Omega$, then $z \in \overline{B(0,1)}$. Note that the closed unit disk centered at the origin is in the intersection of all of the closed disks containing Ω. Thus, $z \in \hat{\Omega}_{\mathcal{F}}$ In either case, $\bar{\Omega} \subseteq \hat{\Omega}_{\mathcal{F}}$.

Therefore $\hat{\Omega}_{\mathcal{F}} = \bar{\Omega}$. †

To summarize, the functional linear convex hull is not always the same as the geometric linear convex hull for non-closed, bounded sets. This is the case because of the definition of the functional linear convex hull. The functional linear hull is defined as the intersection of all of the disks containing the original set. An arbitrary intersection of closed sets is closed, which implies that the functional hull is closed. This is in contrast to the geometric linear convex hull of the set in Theorem 2.9 and Theorem 2.10. The functional and geometric linear convex hulls always agree when the original set is compact.

3 Extensions to Other Functional Hulls

In the previous chapters, finding the $\mathcal{F}$-hull of a set required a set of functions $\mathcal{F}$ and a compact set $\Omega \subseteq \mathbb{C}$. Note that the functions that comprised $\mathcal{F}$ have been entire. For the remaining sections, the assumption is that all of the functions in $\mathcal{F}$ are analytic on Ω.

3.1 Inversion Convex Hull of a Set

The inversion convex hull of a set is found using the functions that consist of inversions of a complex numbers of the form $f(z) = \dfrac{a}{z}$, where $a \in \mathbb{C}$ is a constant and $z \in \mathbb{C}$ is a variable. It is useful to begin the exploration with the definition of the inversion convex hull of a set.

Definition 3.1. Let $\Omega \subseteq \mathbb{C} - \{0\}$ be compact and $\mathcal{I} = \left\{ f(z) : f(z) = \dfrac{a}{z};\ a \in \mathbb{C} \right\}$. The **inversion convex hull** of a set Ω, denoted $\hat{\Omega}_{\mathcal{I}}$, is defined as:

$$\hat{\Omega}_{\mathcal{I}} = \bigcap_{f \in \mathcal{I}} \left\{ w \in \mathbb{C} : |f(w)| \leq M_f \right\},$$

where $M_f = \max\limits_{z \in \Omega} \left\{ |f(z)| \right\}.$

Now that there is some basis to find an inversion convex hull of a set, begin with a finite set and find its hull.

Example 3.2. Consider the set $\Omega = \{1, 2\}$. Then for any $a \in \mathbb{C}$, $\max_{z\in\Omega}\left\{\left|\frac{a}{z}\right|\right\} = |a|$. That is for $|z| \geq 1$, $|\frac{a}{z}| \leq \max_{z\in\Omega}\left\{\left|\frac{a}{z}\right|\right\} = |a|$ and for $|z| < 1$, $\left|\frac{a}{z}\right| > |a|$ when $a \neq 0$. Therefore the inversion convex hull for Ω is $\hat{\Omega}_{\mathcal{I}} = \{|z| \geq 1; z \in \mathbb{C}\}$. A more visual description of this set is the complex plane with the open unit disk removed.

The previous example gives some intuition as to what the inversion convex hull of any two point set may be.

Theorem 3.3. *Let $\Omega = \{z_1, z_2\}$ be a two point set in $\mathbb{C} - \{0\}$. Then the inversion convex hull of the set Ω is the set $\hat{\Omega}_{\mathcal{I}} = \{|z| \geq r\}$ where $r = \min\{|z_1|, |z_2|\} \leq |z_i|$ for all $z_i \in \Omega$.*

Proof. Let $\Omega = \{z_1, z_2\}$, where $z_1, z_2 \in \mathbb{C}$ and $z_1, z_2 \neq 0$. Without loss of generality, let $r = |z_1| \leq |z_2|$ and let $K = \{z : |z| \geq |z_1|\}$. Let $z_0 \in K$. Note that $\max_{z\in\Omega}\left\{\left|\frac{a}{z}\right|\right\} = \left|\frac{a}{z_1}\right| = \frac{|a|}{r}$. Then $\left|\frac{a}{z_0}\right| \leq \left|\frac{a}{z_1}\right|$ and thus $z_0 \in \{w \in \mathbb{C} : |f(w)| \leq \left|\frac{a}{z_0}\right|\}$ is in the inversion convex hull of Ω. Now let w be in the inversion convex hull of Ω. That is to say $\left|\frac{a}{w}\right| \leq \max_{z\in\Omega}\left\{\left|\frac{a}{z}\right|\right\} = \left|\frac{a}{z_1}\right| = |a|$ for all $a \in \mathbb{C}$. Therefore $|w| \geq |z_1|$ and thus $w \in K$. Therefore, K is the inversion convex hull of Ω. †

Theorem 3.4. *Let $\Omega = \{z_1, z_2, ..., z_n\}$ be any finite set in $\mathbb{C} - \{0\}$. Then the inversion convex hull of the set Ω is the set $\hat{\Omega}_{\mathcal{I}} = \{|z| \geq r\}$ where $z_0 = \{\min\{|z_1|, |z_2|, ..., |z_n|\} \leq |z_i|$ for all $z_i \in \Omega$.*

Proof. Let $\Omega = \{z_1, z_2, ..., z_n\}$, where $z_1, z_2, ..., z_n \in \mathbb{C}$ and $z_1, z_2, ..., z_n \neq 0$. Without loss of generality, let $|z_1| \leq |z_i|$ for all $1 \leq i \leq n$. Then let $K = \{z : |z| \geq |z_1|\}$. Let

$z_0 \in K$. Note that $\max_{z\in\Omega}\left\{\left|\frac{a}{z}\right|\right\} = \left|\frac{a}{z_1}\right| = |a|$ for $z \in \Omega$. Then $\left|\frac{a}{z_0}\right| \leq \left|\frac{a}{z_1}\right|$ and thus z_0 is in the inversion convex hull of Ω, $\hat{\Omega}_{\mathcal{I}}$. Conversely, let w be in the inversion convex hull of Ω. That is to say $\left|\frac{a}{w}\right| \leq \max_{z\in\Omega}\left\{\left|\frac{a}{z}\right|\right\} = \left|\frac{a}{z_1}\right|$ for all $w \in \Omega$. Therefore $|w| \geq |z_1|$ and thus $w \in K$. Therefore, K is the inversion convex hull of Ω. †

Theorem 3.5. *Let Ω be a closed set in $\mathbb{C} - \{0\}$. Then the inversion convex hull of Ω is $\hat{\Omega}_{\mathcal{I}} = \mathbb{C} - B(0, r)$, where r is the distance from 0 to the nearest point in Ω.*

Proof. Using the same argument as the proof of Theorem 3.3, choose whichever point in Ω that is closest to the origin. This point is guaranteed to exist by Corollary 1.22. The result follows. †

3.2 Shift-Inversion Convex Hull of a Set

The shift-inversion convex hull of a set is found using the functions that consist of inversions of a complex numbers of the form $f(z) = \frac{a}{z-b}$, where $a, b \in \mathbb{C}$ are constants and $z \in \mathbb{C}$ is a variable. It is useful to begin with the definition of the shift-inversion convex hull of a set.

Definition 3.6. Let $\Omega \subseteq \mathbb{C}$ be compact and $\mathcal{S} = \left\{f(z) : f(z) = \frac{a}{z-b};\ a, b \in \mathbb{C}\right\}$. The **shift-inversion convex hull** of a set Ω, denoted $\hat{\Omega}_{\mathcal{S}}$, is defined as:

$$\hat{\Omega}_{\mathcal{S}} = \bigcap_{f\in\mathcal{S}} \{w \in \mathbb{C} : |f(w)| \leq M_f\},$$

where $M_f = \max_{z\in\Omega}\{|f(z)|\}$.

Now to get to the desired shift-inversion convex hull, the more useful way to proceed is to negate Definition 3.6 to find the complement of $\hat{\Omega}_{\mathcal{S}}$, $\hat{\Omega}_{\mathcal{S}}^c = \mathbb{C} - \hat{\Omega}_{\mathcal{S}}$.

Definition 3.7. The complement of the shift-inversion convex hull of a set Ω is the set $\mathbb{C} - \hat{\Omega}_{\mathcal{S}}$ defined as

$$\hat{\Omega}_{\mathcal{S}}^c = \bigcup_{f \in \mathcal{S}} \{w : |f(w)| > M_f\},$$

where $M_f = \max_{z \in \Omega} \{|f(z)|\}$.

The above definition allows the verification that a point z_0 is not in the hull by finding a single shift-inversion f so that $|f(z_0)| > \max_{z \in \Omega}\{|f(z)|\}$. With this knowledge, consider the following example.

Example 3.8. Let $\Omega = \{1, 2\}$. Then the shift-inversion convex hull of the set Ω is the original set $\Omega = \{1, 2\}$.

Proof. Let $\Omega = \{1, 2\}$ and let $z_0 \in \mathbb{C} - \Omega$. Define $d_1 = |z_0 - 1|$, $d_2 = |z_0 - 2|$, and let $\delta = \min\{d_1, d_2\}$. Suppose without loss of generality that $\delta = d_2$. Choose z_α on $\overline{2z_0}$ such that $|z_0 - 2| > |z_0 - z_\alpha|$. Choose b on the ray $\overrightarrow{z_\alpha z_0}$ such that $|b - z_0| = \frac{\delta}{4}$ and $|b - z_\alpha| > |z_\alpha - z_0|$. Consider $f(z) = \dfrac{1}{z - b}$. It is necessary to show that $|f(z_0)| > |f(z_\alpha)|$. Based on the choice of b, it is the case that $\dfrac{1}{|z_0 - b|} > \dfrac{1}{|z_\alpha - b|}$. Therefore, the only points that are in the shift-inversion convex hull are points that are in Ω to begin with. That is to say that Ω is its own shift-inversion convex hull. †

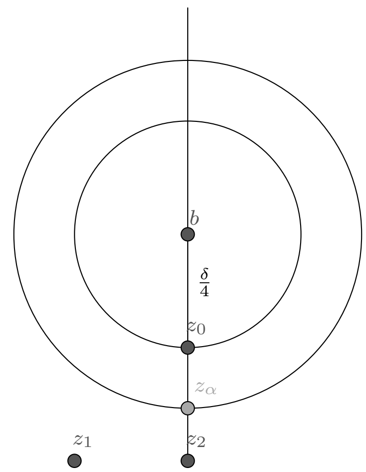

Figure 3.1: Construction for Theorem 3.9.

Theorem 3.9. *Let Ω be any finite two point set. Then the shift-inversion convex hull of the set Ω is the original set Ω.*

Proof. Let $\Omega = \{z_1, z_2\}$ and let $z_0 \in \mathbb{C} - \Omega$. Then $d_1 = |z_0 - z_1|$ and $d_2 = |z_0 - z_2|$. Let $\delta = \min\{d_1, d_2\}$. Suppose without loss of generality that $\delta = d_2$. Choose $z_\alpha \neq z_0$ on $\overline{z_2 z_0}$ such that $|z_0 - z_2| > |z_0 - z_\alpha|$. Choose b on the ray $\overrightarrow{z_\alpha z_0}$ such that $|b - z_0| = \frac{\delta}{4}$ and $|b - z_\alpha| > |z_\alpha - z_0|$. Consider $f(z) = \dfrac{1}{z - b}$. It is necessary to show that $|f(z_0)| > |f(z_\alpha)|$. Based on the choice of b, it is the case that $\dfrac{1}{|z_0 - b|} > \dfrac{1}{|z_\alpha - b|}$. Therefore, the only points that are in the shift-inversion convex hull are points that are in Ω to begin with. That is to say that Ω is its own shift-inversion convex hull. †

Theorem 3.10. *Let Ω be any finite set. Then the shift-inversion convex hull of the set Ω is the original set, Ω.*

Proof. Let $\Omega = \{z_1, z_2, ..., z_n\}$ where $z_i \in \mathbb{C}$ for all $1 \le i \le n$ and let $z_0 \in \mathbb{C} - \Omega$. Let $d_i = |z_0 - z_i|$ for all $1 \le i \le n$ and $\delta = \min\{d_i\}$. Suppose without loss of generality that $\delta = d_1$. Let $b \in \overline{z_0 z_1}$ such that $b \ne z_0$ and $|b - z_0| = \frac{\delta}{4}$. Then the following is true:

$$
\begin{aligned}
\delta &= |z_0 - b| + |b - z_1| && \text{Since } b \in \overline{z_0 z_1} \\
&\le |z_0 - z_i| && \text{Since } \delta \text{ is minimal over } \Omega, \\
&\le |z_0 - b| + |b - z_i| && \text{by the triangle inequality,}
\end{aligned}
$$

Also, $|b - z_1| \le |b - z_i|$ and since b is chosen closer to z_0 than z_1, $|z_0 - b| < |b - z_1|$, so $|z_0 - b| < |b - z_1| \le |b - z_i|$. The first and last parts of the previous inequality show that the point z_0 is closer to b than any point $z_i \in \Omega$. Consider the function $f(z) = \dfrac{1}{z - b}$. Then $|f(z_0)| = \frac{1}{|z_0 - b|} > \frac{1}{|z_i - b|} = |f(z_i)|$ for all $z_i \in \Omega$. Therefore, the only points that are in the shift-inversion convex hull are points that are in Ω to begin with. That is to say that Ω is its own shift-inversion convex hull. †

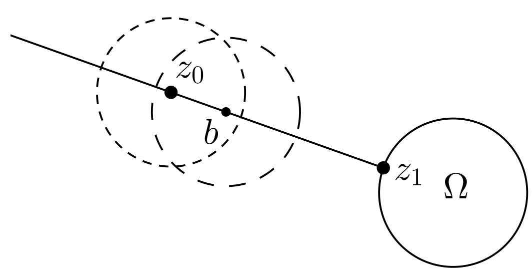

Figure 3.2: Constructions for Theorem 3.11

Theorem 3.11. *Let Ω be any closed bounded simply connected set. Then the shift-inversion convex hull of the set Ω is the original set, Ω.*

Proof. Let Ω be any closed bounded simply connected set and let $z_0 \in \mathbb{C} - \Omega$. The following constructions can be seen in Figure 3.2. From Theorem 1.21 and Corollary 1.22, there exists a smallest distance from z_0 to a point $z_1 \in \Omega$, which is on the boundary of Ω. Let this smallest distance be called δ. Consider $B(z_0, \frac{\delta}{4})$ and choose $b \in \overline{z_0 z_1} \cap B(z_0, \frac{\delta}{4})$, where $b \neq z_0$. Consider $f(z) = \dfrac{1}{z-b}$. It is necessary to show that $|f(z_0)| > |f(z_1)| \geq |f(z)|$ for all $z \in \Omega$. Based on the choice of b, it is the case that $\dfrac{1}{|z_0 - b|} > \dfrac{1}{|z_1 - b|} \geq \dfrac{1}{|z-b|}$ for all $z \in \Omega$. Therefore, the only points that are in the shift-inversion convex hull are points that are in Ω to begin with. That is to say that Ω is its own shift-inversion convex hull. †

Theorem 3.12. *Any closed set is its own shift inversion convex hull.*

Proof. Let Ω be a closed set and $\hat{\Omega}_{\mathcal{S}}$ be its shift-inversion convex hull. Then by definition of shift-inversion convex hull, $\Omega \subseteq \hat{\Omega}_{\mathcal{S}}$. Now for the other containment, assume for contradiction that $z_0 \in \hat{\Omega}_{\mathcal{S}}$ and $z_0 \notin \Omega$. As in a previous proof in this thesis, there is a neighborhood around z_0 such that a specific shift-inversion to exclude z_0 from the hull can be found. This contradicts $z_0 \in \hat{\Omega}_{\mathcal{S}}$. Therefore all $z \in \hat{\Omega}_{\mathcal{S}}$ are also in Ω, or $\hat{\Omega}_{\mathcal{S}} \subseteq \Omega$. Therefore $\Omega = \hat{\Omega}_{\mathcal{S}}$ and every closed set is its own shift-inversion convex hull. †

Theorem 3.13. *The shift-inversion convex hull of an open set is the closure of the set.*

Proof. Let Ω be an open set and $\hat{\Omega}_{\mathcal{S}}$ be its shift-inversion convex hull. Let z_1 be a boundary point for Ω. Then there is no neighborhood around z_1 where a shift-inversion to exclude z_1 from $\hat{\Omega}_{\mathcal{S}}$ could be found because any neighborhood would also include points inside Ω by Definition 1.13, which cannot be excluded. Thus every boundary point of an open set is included in $\hat{\Omega}_{\mathcal{S}}$. Therefore $\hat{\Omega}_{\mathcal{S}}$ is closed. Since $\hat{\Omega}_{\mathcal{S}}$ is closed, its shift-inversion convex hull is $\hat{\Omega}_{\mathcal{S}}$. Thus the shift-inversion convex hull of Ω is $\hat{\Omega}_{\mathcal{S}} = \bar{\Omega}$. †

Corollary 3.14. *The shift-inversion convex hull of a non-closed set is the closure of the set.*

Proof. For boundary points, use the same argument used in proof of Theorem 3.13. For points outside the boundary, use the same argument used in the proof of Theorem 3.12. †

Corollary 3.15. *A shift-inversion convex hull is always a closed set.*

Proof. The result follows from the fact that closed and non-closed sets have shift-inversion convex hulls that are closed. †

Remark 3.16. From the previous work in this section there is a complete classification of the shift-inversion convex hull for *any* set. In summary, for any set $\Omega \in \mathbb{C}$, $\hat{\Omega}_{\mathcal{S}} = \bar{\Omega}$.

3.3 Möbius Transformation Hull of a Set

The Möbius transformation convex hull of a set is found using the set of functions are called Möbius transformations, that is, functions of the form $f(z) = \dfrac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ are constants and $z \in \mathbb{C}$ is a variable with $ad - bc \neq 0$. Below is the definition of the shift-inversion convex hull of a set.

Definition 3.17. Let $\Omega \subseteq \mathbb{C}$ be compact and $\mathcal{M} = \left\{ f(z) : f(z) = \dfrac{az+b}{cz+d};\ a, b, c, d \in \mathbb{C} \right\}$. The **Möbius transformation hull** of a Set of a set Ω, denoted $\hat{\Omega}_{\mathcal{M}}$, is defined as:

$$\hat{\Omega}_{\mathcal{M}} = \bigcap_{f \in \mathcal{M}} \left\{ w \in \mathbb{C} : |f(w)| \leq M_f \right\},$$

where $M_f = \max_{z \in \Omega} \left\{ |f(z)| \right\}$.

Theorem 3.18. *A closed set Ω is its own Möbius transformation convex hull .*

Proof. Given that all shift-inversions are Möbius transformations and that Ω is a subset of its $\mathcal{M}$-convex hull, then the result is proven. †

The previous result leads to a larger, more general result for $\mathcal{F}$-convex hulls.

Theorem 3.19. *Suppose two sets of functions have the following relationship: $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Then the convex hull of a set Ω has the relationship $\hat{\Omega}_{\mathcal{F}_2} \subseteq \hat{\Omega}_{\mathcal{F}_1}$*

Proof. Let $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Let Ω be any set and $\hat{\Omega}_{\mathcal{F}_1}$ and $\hat{\Omega}_{\mathcal{F}_2}$ be the hulls under the respective sets of functions. Since there is potential for more functions in $\mathcal{F}_2$, there is a chance to exclude more points from the hull. Therefore $\hat{\Omega}_{\mathcal{F}_2} \subseteq \hat{\Omega}_{\mathcal{F}_1}$. †

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VITA

Chad Huckaby was born in Nacogdoches, Texas on 20 October 1992. He is a proud native East Texan. He is also a proud third generation SFA Lumberjack. He attended Martinsville High School in Martinsville, Texas, USA, graduating as Valedictorian in 2011. He attended Stephen F. Austin State University in Nacogdoches, Texas from 2011-2015 and received a Bachelor of Science degree in Mathematics and Political Science, with a minor in Chemistry in May 2015. In 2015, He was named Mr. SFA for his dedication and service to Stephen F. Austin State University. He began graduate studies in Mathematical Sciences at Stephen F. Austin State University in Fall 2015 and is expected to graduate in May 2017. In 2016, he was appointed by Texas Governor Greg Abbott to serve the State of Texas as the student representative on the Board of Regents of Stephen F. Austin State University for the academic year 2016-2017. When his studies are complete at SFA, he intends to pursue a doctoral degree in curriculum and instruction at Texas A & M University in College Station, Texas. He is married to Candra Huckaby, a special education teacher at Nacogdoches Independent School District. He and Candra live in Nacogdoches.

The style manual used in this thesis is A Manual For Authors of Mathematical Papers published by the American Mathematical Society.

This thesis was prepared by Chad Alan Huckaby using $\LaTeX$.