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Optimal width and depth for maximum breaking load of wood beams

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OPTIMAL WIDTH AND DEPTH FOR MAXIMUM BREAKING LOAD OF WOOD BEAMS

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CONTENTS

The model of rupture relationship for bending strength of solid wood beams implies that breaking load is a monotonic increasing function of beam width and depth. Where a single beam is cut from a log or bolt, a depth to width ratio equal to the $\sqrt{2}$ will yield the maximum breaking load. The ratio was derived using the Kuhn–Tucker conditions for characterizing optimal solutions to nonlinear programming problems. The ratio can be applied to determine optimal width and depth, and to calculate breaking load that can be obtained from a beam cut from a specific log.

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1. INTRODUCTION

The strength of solid wood beams is an important property to consider in construction. Such beams do not receive further primary processing after leaving the sawmill, and their quality and strength properties are therefore determined by their initial, primary breakdown at the sawmill. The most efficient means of controlling the quality of solid wood beams is thus at the sawmill level. This paper presents a method of determining the width and depth of solid beams that will maximize breaking load.

Strength and other mechanical properties of wood are extremely important in structural applications. Haygreen and Bowyer (1) describe standard procedures for measuring wood strength and other properties, as well as many uses of such measures in structural applications. Detailed descriptions of the engineering aspects of wood structures are provided by Hoyle (2) and Gurfinkel (3).

The bending strength of solid wood products is most often expressed in terms of the models of rupture (MOR), as calculated from the load to failure in a standard bending test. For wood beams with rectangular cross-sections, Haygreen and Bowyer (1) present the following MOR equation.

$$\mathrm{MOR} = (1.5)(\mathrm{P})(\mathrm{L})/\mathrm{wd}^2 \text{ (psi)} \tag{1}$$

where

P = the load to failure or breaking load (pounds),

L = the distance between supports or span (inches),

w = the width of the beam (inches),

d = the depth of the beam (inches).

In the present article, equation (1) is solved for P, and the Kuhn–Tucker conditions of nonlinear programming are used to determine optimal beam width and depth, given constraints on raw material size. Where a single beam is cut from a log or bolt, maximum breaking load is shown to be achieved where beam depth is a constant multiple of beam width ($d^* = \sqrt{2}\, w^*$.)

2. OPTIMAL WIDTH AND DEPTH

The MOR relationship reflects an important characteristic of the maximum load of wood beams. Equation (1) implies that maximum load is directly proportional to width and the square of the depth. This characteristic is shown by writing equation (1) in terms of P:

$$\mathrm{P} = \left[\frac{\mathrm{MOR}}{(1.5)(\mathrm{L})}\right] \mathrm{wd}^2 \tag{2}$$

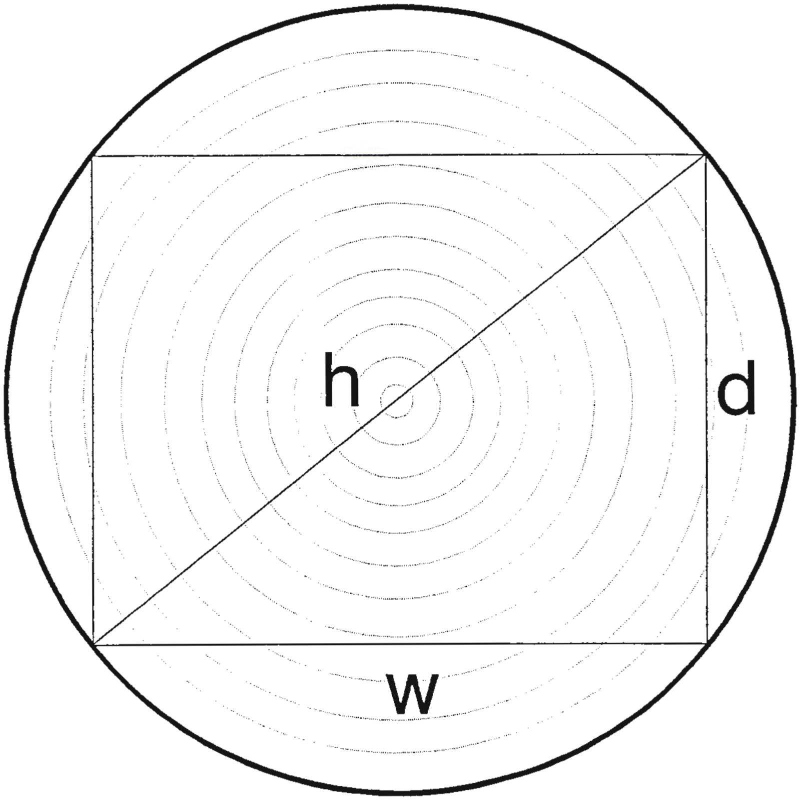

FIGURE 1. Log or bolt cross-section: w = beam width, d = beam depth, and h = hypotenuse of the right triangle formed by w and d; h also represents the diameter of the log or bolt.

The bracketed term in equation (2) is a parametric constant, and for beams of a given length, species, moisture content, and specific gravity, the general relationship may be represented by:

$$P = kwd^2 \tag{3}$$

For positive values of width and depth, breaking load is a monotonic increasing function. Where a single beam is to be cut from a log or bolt, however, width and depth are constrained by the diameter of the log to be sawn. Since width and depth form a right triangle, $w^2 + d^2 = h^2$ must hold, where h represents the hypotenuse of the triangle and the diameter of the log (Figure 1).

Optimal beam width and depth, the beam dimensions that yield the greatest breaking lead, may be obtained by solving:

$$\underset{\{w,\ d\}}{\text{Maximize}}\ P = kwd^2 \tag{4}$$

$$\text{subject to: } w^2 + d^2 \leq h^2 \tag{5}$$

$$w,\ d \geq 0 \tag{6}$$

Equations (4)–(6) represent a nonlinear programming problem with decision variables w and d. The program is a "convex" program since the objective function is monotonic increasing and for a given value of h, equation (5) represents the area inscribed by a circle (a convex set). For convex programs, the Kuhn–Tucker optimality conditions are necessary and sufficient to characterize a solution that is globally optimal (4).

The Kuhn–Tucker conditions state that the optimal solution is represented by a point where the gradient of P belongs to the cone spanned by the gradients of the problem's binding constraints. For the above problem, the Kuhn–Tucker conditions are:

$$\nabla P(x) + \sum_{i=1}^{m} U_i g_i(x) = 0 \tag{7}$$

$$U_1, \ldots, U_m \leq 0 \tag{8}$$

$$U_i g_i(x) = 0 \qquad 1, \ldots, m. \tag{9}$$

In equations (7)–(9), m represents the number of constraint equations [$g_i(x)$, $i = 1, \ldots, m$]. x represents the vector of decision variables, and U_i represents a dual variable for each constraint. Equation (7) is the required relationship between the gradient of P and the gradients of the binding constraints; equation (9) ensures that only binding constraints (where $g_i(x) = 0$) are considered in equation (7). Equation (9) ensures complementary slackness—if $g_i(x) = 0$, then U_i may be nonzero, and may be used to define the cone which spans the gradient of P in equation (7).

With respect to the maximum breaking load problem, the above conditions are:

$$\binom{kd^2}{2kwd} + U_1\binom{2w}{2d} + U_2\binom{-1}{0} + U_3\binom{0}{-1} = 0 \tag{10}$$

$$U_1(w^2 + d^2 - h^2) = 0 \tag{11}$$

$$U_2(-w) = 0 \tag{12}$$

$$U_3(-d) = 0 \tag{13}$$

$$U_1,\ U_2,\ U_3 \leq 0 \tag{14}$$

Since w and d must be positive, $U_2 = U_3 = 0$, and the remaining equations may be solved for w* and d*, optimal beam width and depth. The constraint-based equation, (11), must be binding in the optimal solution, and U_1 may therefore be nonzero. The gradient-based conditions are:

$$kd^2 + U_1 2w = 0 \tag{15}$$

$$2kwd + U_1 wd = 0 \tag{16}$$

From equation (16), $2d(kw + U_1) = 0$ must hold, and since d is positive, $U_1 = -kw$ must hold. Substituting $U_1 = -kw$ into equation (15) yields:

$$kd^2 - 2kw^2 = 0 \tag{17}$$

which implies that $d^* = \sqrt{2}\, w^*$. Regardless of the diameter of the log or bolt, the MOR relationship implies a constant relationship between the width and the depth that maximizes a single beam's braking load; the depth should be larger than the width by a factor of $\sqrt{2} \approx 1.4142$.

DISCUSSION

Many countries in the world process logs to produce solid beams. In Europe, for example, long and relatively small diameter logs are often sawn to produce high quality beams for construction. The solution above can be used to determine cutting dimensions for wood beams to achieve maximum breaking load. The results should not be interpreted without caution, however, since they depend entirely on the MOR relationships and the properties of specific logs. Logs are tapered rather than cylindrical, for example, and care must therefore be taken to measure diameter at the log's small end.

Also, in applying the results of optimization based on MOR relationships, one must recognize that the functional form of such relationships results in part from the length of the beams considered. If the ratio of beam length to depth is too high, for example, the beam would be slender and deflection could be a governing factor in service. In the opposite extreme, very deep beams may experience high levels of shear stress. According to Bodig and Jayne (5), length to depth ratios of around 21 are ideal. Our finding that maximum breaking load occurs where the depth to ratio is approximately 1.414 is well within the bounds for lateral stability—lateral support is recommended where such ratios are greater than 3.

Another area for potential concern in applying the optimization results is where internal defects may be present in the beam. Larger beams may have defects that cannot be seen, and that would not be reflected in the maximum breaking load. Logs with excessive heart checks are often more efficiently processed as beams; however, lumber for such logs may be of very low quality since the checks may open wider in lumber than when they remain enclosed inside the beam.

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