Axiomatic Foundations of Mathematics

Ryan Melton
Dr. Clint Richardson, Faculty Advisor
Stephen F. Austin State University

As Bertrand Russell once said,

 Pure mathematics is the subject in which we do not know what we are talking about, or whether what we are saying is true.

Russell's statement begs from us one major question:

What is Mathematics founded on?

Axioms and Axiom Systems

An **axiom** is a belief taken without proof, and thus an **axiom system** is a set of beliefs taken without proof.

Consistent? Complete?

An axiom system is:

 ---**consistent** if no valid statement in the theory is both provably true and provably false.

 ---**complete** if each valid statement in the theory is provably true or provably false.

The Issue

Can an axiomatic system be both consistent and complete?

The answer (first attempt):
 Principia Mathematica!

The NEW Issue

However, according to *Principia Mathematica*, the consistency of a given system relies on the consistency of formal logic itself.

Thus, we (still) know **NOTHING** about consistency!

What is Metamathematics?

According to Hilbert, metamathematics is the language ABOUT mathematics.

Metamathematical statements are statements about the signs occuring within a formalized mathematical system.

Consider the expression

$$2 + 3 = 5$$

This expression is mathematical; it belongs to the field we call arithmetic and is composed of basic arithmetic symbols.

On the other hand, the sentence

'2 + 3 = 5' is an arithmetical formula.

is metamathematical; it is constructed outside of mathematics and labels the expression above as a formula in arithmetic.

Since *Principia Mathematica* was such a bold leap in the right direction--although proving nothing about consistency--several attempts at proving consistency were made by other mathematicians of the time, the most notable of these was…

Kurt Gödel!

Kurt Gödel (1906 - 1978) was a German logician, and a personal friend of Einstein. As a refugee from the Nazi Party, Gödel fled to the United States, and came to the Institute of Advanced Study in New Jersey. There, he proceeded to develop a fascinating proof that would amaze all who were struggling with a solution to consistency.

At its heart, Gödel's argument hinges on the following question:

Can Metamathematics be discussed in the context of mathematics?

That is,

Can you speak ABOUT mathematics WITH mathematics?

Gödel's Method

First, Gödel assigned a unique natural number to each of the logical symbols and numbers.

For example: if the symbol '0' corresponds to the natural number 1, '+' to 2, and '=' to 3, then

'0 = 0'   '0 + 0 = 0'
 ↓ ↓ ↓   and   ↓ ↓ ↓ ↓ ↓
 1 3 1   1 2 1 3 1

so each expression corresponds to a sequence.

Then, for this new sequence $x_1 x_2 x_3 \ldots x_n$ of positive integers, we associate a Gödel number as follows:

$$enc(\, x_1 x_2 x_3 \ldots x_n\,) = 2^{x_1}\, 3^{x_2}\, 5^{x_3} \ldots p_n^{x_n}$$

where the encoding is the product of n factors, each of which is found by raising the j-th prime to the x_j power.

Thus, '0 = 0' corresponds to 1, 3, 1
 which is encoded as
 $2^1\, 3^3\, 5^1 = 270$
 and
'0 + 0 = 0' corresponds to 1, 2, 1, 3, 1
 which is encoded as
 $2^1\, 3^2\, 5^1\, 7^3\, 11^1 = 339{,}570.$

By the Fundamental Theorem of Arithmetic, this representation as a product of powers of primes is unique.

Thus, any formula can be uniquely represented as a Gödel number, and any given Gödel number can be used to produce the original formula.

This allowed Gödel to show a correspondence between statements about natural numbers and statements about the provability of theorems about natural numbers; this the key observation of the proof.

Ultimately, metamathematics could be addressed in the context of arithmetic!

CONCLUSION

Gödel's final result was the fact that no axiom system that could support arithmetic could be both consistent and complete.

Thus, consistency and completeness cannot coexist within the foundations of axiomatic mathematics. However, the goal is to understand that foundation.

References

Goldstein, Rebecca. Incompleteness: The Proof and Paradox of Kurt Gödel. New York: W.W. Norton & Company, 2005. Print.
Nagel, Ernest and James Newman. *Gödel's Proof*. New York: New York University Press, 2001. Print.
Weisstein, Eric W. "Complete Axiomatic Theory." From *MathWorld*--A Wolfram Web Resource 3 Mar 2010. Web. 20 Mar 2010.
Weisstein, Eric W. "Consistency." From *MathWorld*--A Wolfram Web Resource. 3 Mar 2010. Web. 20 Mar 2010.