2014

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Communications on Stochastic Analysis
Vol. 8, No. 3 (2014) 381-411

Serials Publications
www.serialspublications.com

GLOBAL EXISTENCE AND FINITE TIME BLOW-UP IN A CLASS OF STOCHASTIC NONLINEAR WAVE EQUATIONS

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Abstract. We consider a stochastic extension of a class of wave equations with nonlinear viscoelastic damping and nonlinear forcing. We show the global existence of the solution of the stochastic equation and, additionally, when the source term dominates the damping term and when the initial data are large enough, we show that the expected value of the L^p norm of the solution, blows up in finite time. In the presence of noise, we extend the previously known range of initial data corresponding to blow-up. Furthermore we use a spectral stochastic Galerkin method to perform numerical simulations that verify certain special cases of our theoretical results.

1. Introduction

Wave motion is a very common phenomenon in the physical world: vibrations of elastic strings and plates, propagation of water waves, electromagnetic waves and sound, shock waves and tsunamis are well-known examples [4, 12, 38], which have been extensively studied. In certain cases the external forcing terms modeling such phenomena, may contain a random component. For example, waves arising in composite material structures pass through a host of different layers [28], which have randomly varying physical properties. Understanding the role played by stochastic effects in wave dynamics, is thus an important theoretical as well as practical problem. Given a mathematical model for non-linear wave propogation, one can investigate the question of whether there is a globally existing solution, or if perhaps finite time blow-up occurs. That is:

$$\lim_{t\to T^*<\infty} ||u||_X = +\infty \tag{1.1}$$

where X is a certain function space. Classical examples where finite time blow-up is seen include Burgers' equation [38], and wave equations with a nonlinear source term [16, 35]. Understanding the conditions that lead to blow-up of solutions in

Received 2014-2-20; Communicated by the editors.
2010 *Mathematics Subject Classification.* Primary 60H15 ; 35L05 ; 35L70 ; 35B44; Secondary 78M22.
Key words and phrases. Stochastic nonlinear wave equation, global existence, finite time blow-up, spectral stochastic Galerkin method.
* Research partially supported by KAUST-Stanford Academic Excellence Alliance Grant by KAUST.
† Research supported by King Abdullah University of Science and Technology (KAUST).

stochastic nonlinear wave equations, continues to be intensely investigated [2, 5, 7, 10, 20, 27, 31].

Our goal in this work is to demonstrate global existence as well as blow-up (depending on the class of the initial data and the exponents on the source and damping terms), for the following nonlinear damped and forced wave equation with a stochastic noise term

$$\begin{cases} \partial_{tt}u - \Delta\partial_t u - \mathrm{div}(|\nabla u|^{\alpha-2}\nabla u) - \mathrm{div}(|\nabla\partial_t u|^{\beta-2}\nabla\partial_t u) + a|\partial_t u|^{q-2}\partial_t u \\ = b|u|^{p-2}u + \sigma(x,t)\partial_t W(x,t), \\ u(x,t) = 0, \quad \text{on } \partial D, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), \end{cases} \tag{1.2}$$

for $x \in D$ and $t > 0$, where $D \subset \mathbb{R}^n$ is a bounded domain with the boundary ∂D, which is assumed to be smooth. The constants in (1.2) are such that $a, b > 0$ and $\alpha,\ \beta,\ q,\ p \geq 2$. We also require that $p \leq r_\alpha$, where r_α is defined as follows:

$$\begin{array}{ll} r_\alpha = n\alpha/(n-\alpha), & \text{if } n > \alpha, \\ r_\alpha > \alpha, & \text{if } n = \alpha, \\ r_\alpha = \infty, & \text{if } n < \alpha. \end{array} \tag{1.3}$$

The restriction of $p \leq r_\alpha$ is related to the criticality of the Sobolev embedding, $W_0^{1,\alpha}(D) \hookrightarrow L^p(D)$, which is necessary for the analysis below. Furthermore, for global existence, we require that $p \leq q$. The term $\sigma(x,t)\partial_t W(x,t)$ represents a perturbation via a Wiener random field [9].

The energy functional associated with system (1.2) is defined as

$$\mathcal{E}(t) = \frac{1}{2}||\partial_t u(t)||_2^2 + \frac{1}{\alpha}||\nabla u(t)||_\alpha^\alpha - \frac{b}{p}||u(t)||_p^p. \tag{1.4}$$

We note that, in this functional, the first two terms represent the kinetic and potential energies of deformation and they are always positive. The last term arises from the external forcing in (1.2) and it can be negative. Even though this definition allows for negative values of $\mathcal{E}(t)$, it is the natural energy functional for (1.2).

In the absence of the stochastic term, the PDE in (1.2) takes the form [39, 30]:

$$\partial_{tt}u - \mathrm{div}\left[(|\nabla u|^{\alpha-2})\nabla u + \left(1 + |\nabla\partial_t u|^{\beta-2}\right)\nabla\partial_t u\right] + a|\partial_t u|^{q-2}\partial_t u = b|u|^{p-2}u. \tag{1.5}$$

The energy functional associated with (1.5) is defined as

$$\tilde{\mathcal{E}}(t) = \frac{1}{2}||\partial_t u(t)||_2^2 + \frac{1}{\alpha}||\nabla u(t)||_\alpha^\alpha - \frac{b}{p}||u(t)||_p^p. \tag{1.6}$$

However, the key difference between (1.4) and (1.6) is that in (1.4) the terms are functions of the random component ω as well as t. Thus, Ito's lemma [19] has to be applied in order to compute the time derivative of $\mathcal{E}(t)$.

The nonlinearities in (1.5) and (1.2), are motivated by the Kelvin-Voigt models of viscoelastic deformations, which assume that the stress, S, depends on the strain, ε, and the rate of strain, $\partial_t\varepsilon$, as $S = E\varepsilon + \eta\partial_t\varepsilon = E\nabla u + \eta\partial_t\nabla u = E\nabla u + \eta\nabla\partial_t u$, where E is the Young's modulus and η is the coefficient of viscosity, both of which are usually constants. However, in (1.5) and (1.2) to model the the stress (the terms inside the square brackets in (1.5)) we have $E = E(\nabla u) = |\nabla u|^{\alpha-2}$ and $\eta = \eta(\nabla \partial_t u) = 1 + |\nabla \partial_t u|^{\beta-2}$. That is we have a stress dependent Young's modulus and a strain dependent viscosity, as is often seen in certain viscoelastic materials [8, 17, 22].

We begin with a recap of the literature in the deterministic setting. Equation (1.5) was considered in [39] where the author showed that its solution blows up in finite time, T^*, under the condition $\max\{\alpha, q\} < p < r_\alpha$ (r_α is defined in (1.3)), $\alpha > \beta$, and the initial energy is sufficiently negative (see condition *(ii)* in [39, Theorem 2.1]). In fact, this condition makes it clear that there exists a certain relation between the blow-up time and $|D|$, the measure of D (see [39, Remark 2]). Messaoudi and Said-Houari [30] improved the result in [39] and showed that the blow-up of solutions of problem (1.5) takes place for negative initial data regardless of $|D|$.

Equation (1.5) is a generalized version of the wave equation with damping and source terms

$$\partial_{tt} u - \Delta u + |\partial_t u|^{q-2}\, \partial_t u = |u|^{p-2}\, u, \qquad \text{in } (0,T) \times D, \tag{1.7}$$

with Dirichlet boundary conditions on ∂D. The properties of the solution of (1.7) are well understood. Indeed, in the absence of the damping term, Ball [1] and Kalantarov and Ladyzhenskaya [18] showed that the source term $|u|^{p-2}u$ causes finite-time blow-up of solutions with large initial data (negative initial energy); that is, the L^p norm of the solution tends to infinity as time, t, approaches a finite value, T^*. On the other hand, and as it was shown by several authors (see for example Haraux and Zuazua [13]), the damping term $|\partial_t u|^{q-2}\, \partial_t u$ in the absence of the source term extends the lifespan of the solutions to the whole interval, $[0, +\infty)$.

Analysis of the competition between the source and damping terms was the subject of a series of papers beginning in the early 1970s. Levine [23, 24] investigated (1.7), when the damping is linear ($q = 2$) and showed that solutions with negative initial energy blow up in finite time, via the "concavity method". Note, the concavity method fails if the nonlinear damping term (i.e., $q > 2$) is present. Georgiev and Todorova [11] solved this problem in 1994, and extended Levine's result to the nonlinear damping case. In [11], the authors considered (1.7) and introduced a new method to show that solutions with small initial data continue to exist globally in time if the damping term dominates the source term (i.e., if $p \le q$) and blows up in finite time in the other case (i.e., if $p > q$), provided that the initial data are large, that is, for sufficiently negative initial energy. Their method is based on the construction of an auxiliary function, L, which is a perturbation of the total energy of the system. However, the blow-up result in [11] was not optimal in terms of the initial data. Thus, several improvements followed (see, for example, [25, 26, 29, 37]). In particular, Vitillaro in [37] combined the arguments in [11] and [26] to extend the result in [11] to situations where the damping is nonlinear and the solution has positive initial energy.

There have been a number of papers dealing with local and global existence or finite time blow-up, when the stochastic source term is present, [2, 5, 7, 10, 20, 27, 31]. The primary tool in most of these works is Ito's lemma. Essentially, the equations considered are viewed as a deterministic equation plus a stochastic perturbation, which has to be handled via Ito's calculus and the use of various martingale inequalities, [9, 19]. In [7], Chow investigated a wave equation with a polynomial nonlinearity perturbed by white noise. He showed explosion of the solutions when the nonlinearity was cubic. He also showed existence and uniqueness of global solutions when the nonlinearity satisfied certain restrictions.

The study of the stochastic nonlinear wave equations with nonlinear damping terms was initiated by Pardoux in [32]. There are also recent results to this end by Kim [20] and Barbu et al. [2]. The following system,

$$\begin{cases} \partial_{tt}u - \Delta u + |\partial_t u|^{q-2}\partial_t u = |u|^{p-2}u + \sigma(x,t)\partial_t W(x,t), \\ u(x,t) = 0 \quad \text{on } \partial D, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), \end{cases} \tag{1.8}$$

was recently considered in [10]. They established global existence for $p \leq q$. For $p > q$, they showed that the L^2 norm of the solution blows up in finite time with positive probability. Their results are valid when the initial energy (the same as (1.4) for $\alpha = 2$) is sufficiently negative; namely, they require $\mathcal{E}(0) \leq -\beta$ for some $\beta > 0$. Note that in [11] for the deterministic equation, finite time blow-up takes place if $\mathcal{E}(0) < 0$. The recent works on stochastic nonlinear wave equations [5, 27, 10] deal only with the standard wave operator with damping and source terms. However the nonlinear stochastic viscoelastic wave equation (that is with gradient type nonlinearity or α-laplacian type operators) has not been investigated, despite the many applications of viscoelastic materials [8, 17, 22]. Furthermore, the blow-up results in [27, 10] are proved only under the condition that the initial energy is sufficiently negative. We know in the deterministic setting [37] that blow-up can also occur for positive initial energy, but no effort has been made to extend this to the stochastic setting. Lastly, the recent works on the stochastic wave equation are purely theoretical, and no numerical simulations have been performed to visualise the blow-up and global existence results. Our primary contributions in the present work are the following,

(1) We show show local existence for (1.2), which is a generalised form of (1.8). Note here we are dealing with α-laplacian type operators, and not the standard wave operator.
(2) We show that if $q \geq p$, the local solution can be extended for all time, and is thus global.
(3) We show that finite time blow up is possible for (1.2), if $p > q$, and the initial data is large enough (that is if the initial energy is sufficiently negative).
(4) We improve upon the results in [10], by showing that blow-up is possible *even if the initial energy is positive.*
(5) We verify global existence and blow-up results numerically in $1d$ and $2d$, in certain parameter range, via a stochastic spectral Galerkin method, for the equation considered in [10]. This also verifies a special case of our global existence result Theorem 4.1.

(6) In both [10] and [27], it is remarked that the stochastic forcing acts as a damping term. Essentially in order to have finite time blow-up in the presence of the stochastic term, the initial energy is required to be *more* negative than the deterministic case, as the stochastic calculus leads to certain positive quantities appearing in the energy inequalities. We verify numerically that the stochastic forcing, does indeed act as damping.

Our manuscript is organized as follows. In section 2, we recall certain stochastic preliminaries that are used throughout the paper. Section 3 is devoted to the study of the local existence of weak solutions. In section 4, global existence is demonstrated. In section 5, we prove that under certain assumptions the local solution ceases to exist, and blows up in the L^p norm in finite time. In section 6 we verify our results numerically in a certain parameter range, via a stochastic spectral Galerkin method. Lastly, in all estimates made henceforth, $C, C_i,\ \{i = 1, 2, ...\}$ are generic constants that can change in value from line to line, and sometimes within the same line, if so required.

2. Preliminaries

For the sake of completeness, we introduce here certain stochastic preliminaries. For complete details, the reader is refered to [9] and [19].

Let $(\Omega, \mathbb{P}, \mathcal{F})$ be a complete probability space for which a filtration $\{\mathcal{F}_t, t \geq 0\}$ is defined. An atom in Ω will be denoted as ω and $\mathbb{E}$ stands for the expectation with respect to the probability measure, $\mathbb{P}$. We consider $\{W(x,t), t \geq 0\}$, a $H = L^2(D)$-valued Wiener random field on the defined probability space, with covariance operator $\mathcal{R}$ satisfying Tr $\mathcal{R} < \infty$, such that

$$\mathcal{R}e_i = \lambda_i e_i, \quad i = 1, 2, ... \tag{2.1}$$

where λ_i are eigenvalues of $\mathcal{R}$ satisfying $\sum_{i=1}^{\infty} \lambda_i < \infty$ and $\{e_i\}$ are the corresponding eigenfunctions that form an orthonormal basis in H. Therefore,

$$W(x,t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} B_i(t) e_i, \tag{2.2}$$

where $\{B_i(t)\}$ are independent copies of standard Brownian motions in one dimension. We also define $\mathcal{H}$ to be the set of $L_2^0 = L^2(\mathcal{R}^{\frac{1}{2}}H, H)$-valued processes that possess the norm

$$||\psi(t)||_H = \left(\mathbb{E}\left[\int_0^t ||\psi(s)||^2_{L_2^0} ds\right]\right)^{\frac{1}{2}} = \left(\mathbb{E}\left[\int_0^t \mathrm{Tr}(\psi(s)\mathcal{R}\psi^*(s))ds\right]\right)^{\frac{1}{2}} < \infty, \tag{2.3}$$

where $\psi^*(s)$ denotes the adjoint operator of $\psi(s)$.

Since (1.2) contains an α-Laplacian operator, $F(u) = \mathrm{div}(|\nabla u|^{\alpha-2}\nabla u)$, $(\alpha > 2)$, the natural space that one searches for a solution is $W_0^{1,\alpha}(D)$, and not $H_0^1(D)$. This is easily seen via a cursory integration by parts, upon multiplying (1.2) by a suitable test function. In order to proceed we recap the notion of a Gelfand triple.

Definition 2.1. A *Gelfand triple* is datum of the form

$$B \overset{J}{\hookrightarrow} H \overset{K}{\hookrightarrow} B^*, \tag{2.4}$$

where H is a separable Hilbert space, B is a Banach space (or more general topological vector space (TVS)), B^* is a dual TVS of B, $J : B \hookrightarrow H$ is an injective bounded operator with dense image, and K is the composition of the canonical isomorphism $H \cong H^*$ determined by the inner product and of the Banach transpose (dual) $J^* : H^* \to B^*$ of the operator J.

In the context of our problem then, we define the following Gelfand triple,

$$W_0^{1,\alpha}(D) \hookrightarrow L^2(D) \hookrightarrow W^{-1,\alpha/(\alpha-1)}(D), \tag{2.5}$$

such that

$$F : [0,T] \times W_0^{1,\alpha}(D) \times \Omega \to W^{-1,\alpha/(\alpha-1)}(D). \tag{2.6}$$

This allows us to reconcile the inner product of a $W_0^{1,\alpha}(D)$-valued function with the $L^2(D)$-valued noise via methods in [34], in the energy estimates that follow. (See [34] for details on stochastic differential equations with α-Laplacian type operators).

Definition 2.2. Let $(\mathbb{X}, \mathcal{A})$ be a measurable space. A stochastic process $X(t,\omega) : [0,\infty) \times \Omega \to \mathbb{X}$ is said to be *progressively measurable*, if, for every time t, the map $X : [0,t] \times \Omega \to \mathbb{X}$ defined by $(s,\omega) \mapsto X_s(\omega)$ is Borel $(0,t) \otimes \mathcal{F}_t$ measurable.

Furthermore, we will assume that for a solution to equation (1.2), we have

$$(u_0, u_1) \in L^\alpha(\Omega; W_0^{1,\alpha}(D)) \times L^2(\Omega; L^2(D)) \tag{2.7}$$

and (u_0, u_1) are $\mathcal{F}_0$ measurable (here (u_0, u_1) denotes a pair of functions and it should not be confused with the L^2 scalar product, which is denoted by $\langle .,. \rangle$). Also the $\sigma(x,t)$ in (1.2) is assumed to be $H_0^1(D) \cap L^\infty(D)$-valued and progressively measurable such that

$$\mathbb{E}\left[\int_0^T \left(||\nabla\sigma(t)||_2^2 + ||\sigma(t)||_\infty^2\right) dt\right] < \infty. \tag{2.8}$$

Definition 2.3. Under the assumptions made in (2.7) and (2.8), we say that u is a *weak solution* of (1.2) on the time interval $[0,T]$ if

$$\begin{aligned}
&(u, \partial_t u) \text{ is } W_0^{1,\alpha}(D) \times L^2(D)\text{-valued progressively measurable},\\
&(u, \partial_t u) \in L^\alpha(\Omega; C([0,T]; W_0^{1,\alpha}(D))) \times L^2(\Omega; C([0,T]; L^2(D))),\\
&\partial_t u \in L^q((0,T) \times D), \text{ for almost all } \omega,\\
&u(0) = u_0, \qquad \partial_t u(0) = u_1, \text{ for almost all } \omega,
\end{aligned}$$

and the following holds

$$\begin{aligned}
\langle u_t(t), \phi \rangle &= \langle u_1, \phi \rangle - \int_0^t \left\langle |\nabla u|^{\alpha-2}\nabla u, \nabla\phi \right\rangle ds - \int_0^t \left\langle |\nabla u_t|^{\beta-2}\nabla u_t, \nabla\phi \right\rangle ds\\
&- \int_0^t \left\langle |u|^{p-2}u, \phi \right\rangle ds + \int_0^t \left\langle |u_s|^{q-2}u_s, \phi \right\rangle ds + \int_0^t \langle \phi, \sigma(x,s) dW_s \rangle .
\end{aligned}$$

For all test functions ϕ bounded, positive, compactly supported in D, and in $C^\infty(D \times [0,T])$, with $\phi(x,T) = \phi_t(x,T) = 0$.

Remark 2.4. Henceforth we use the following notation: we denote u by u_t and $\partial_t u$ by v_t. This is done to be consistent with the standard notation in the literature. Also, we use the following notation for the $L^2(D)$ inner product:

$$\int_D UV\,dx = \langle U, V\rangle\,. \tag{2.9}$$

3. Local Existence

In this section, we show the existence and uniqueness of a weak solution of problem (1.2). We follow the methods of [10, 20], and refer to them when necessary. We first consider the following regularized problem (without loss of generality, we set $a = 1$, $b = 1$ from now on):

$$\begin{cases} \partial_{tt}u - \Delta\partial_t u - \operatorname{div}(|\nabla u|^{\alpha-2}\nabla u) - \operatorname{div}(|\nabla\partial_t u|^{\beta-2}\nabla\partial_t u) + g_\lambda(\partial_t u) \\ \quad = f_N(u) + \sigma(x,t)\partial_t W(x,t), & (x,t)\in D\times(0,T), \\ u(x,t) = 0, & (x,t)\in\partial D\times(0,T), \\ u(x,0) = u_0(x), \qquad \partial_t u(x,0) = u_1(x), & x\in D, \end{cases} \tag{3.1}$$

where the functions f_N and g_λ are regularized approximations of f and g as defined below in (3.9) and (3.10), respectively. We assume that, in (3.1),

$$u_0 \in L^\alpha(\Omega; W_0^{1,\alpha}(D)),\ \ u_1 \in L^2(\Omega; L^2(D)), \tag{3.2}$$

and (u_0, u_1) are $\mathcal{F}_0$ measurable. Also, $\sigma(x,t)$ is $H_0^1(D)\cap L^\infty(D)$-valued progressively measurable such that

$$\mathbb{E}\left[\int_0^T \left(||\nabla\sigma(t)||_2^2 + ||\sigma(t)||_\infty^2|\right) dt\right] < \infty. \tag{3.3}$$

The idea is now to consider a Galerkin approximation of the solution to (3.1) of the form

$$u_{tm}(x,t) = \sum_{j=1}^m a_{m,j}(t)e_j(x) \tag{3.4}$$

and to obtain a priori estimates for u_{tm} and take the limit as $m\to\infty$ afterward. Here, $e_j(x)$ is taken to be a basis in $W_0^{1,\alpha}(D)$ that is orthogonal in $L^2(D)$ as shown in [40]. We state the following lemma.

Lemma 3.1. *Assume that* (3.2) *and* (3.3) *hold. Then there exists a time T and a pathwise unique solution of* (3.1) *such that*

$$u_t \in L^\alpha(\Omega; L^\infty(0,T; W_0^{1,\alpha}(D))) \cap L^2(\Omega; C([0,T]; H_0^1(D))), \tag{3.5}$$

$$v_t \in L^2(\Omega; L^\infty(0,T; L^2(D)))\cap L^\beta(\Omega; L^\beta(0,T; W_0^{1,\beta}(D)))\cap L^2(\Omega; C([0,T]; L^2(D))), \tag{3.6}$$

and

$$\mathbb{E}\left[||v_t||_{L^\infty(0,T;L^2(D))} + ||v_t||_{L^\beta(0,T;W_0^{1,\beta}(D))} + ||u_t||_{L^\infty(0,T;W_0^{1,\alpha}(D))}\right] \le C_N. \tag{3.7}$$

Essentially, in the regularized problem, we have replaced $f(u_t)$ by $f_N(u_t)$ and $g(v_t)$ by $g_\lambda(v_t)$. We begin the proof by recalling the basic theory for these approximations (see [20, 10] for details). Let $f(u_t) = |u_t|^{p-2}u_t$. For each $N \geq 1$, we define a function

$$\chi_N = \begin{cases} 1, & \text{if } x \leq N, \\ \in (0,1), & \text{if } N < x < N+1, \\ 0, & \text{if } x \geq N+1, \end{cases} \tag{3.8}$$

such that $||\chi_N'||_\infty \leq 2$, and let

$$f_N(u_t) = \chi_N(||\nabla u_t||_2) f(u_t). \tag{3.9}$$

Then we have

$$||f_N(u) - f_N(v)||_2 \leq C_N ||\nabla u - \nabla v||_2.$$

(See equation (11) in [5] for further details). Let $g(x) = |x|^{q-2}x$ and for any given $\lambda > 0$, let

$$g_\lambda(x) = \frac{1}{\lambda}\left(x - (I + \lambda g)^{-1}(x)\right) = g(I + \lambda g)^{-1}(x), \quad x \in \mathbb{R}, \tag{3.10}$$

where g_λ is the Yosida approximant of the mapping g.

We next recap the maximal monotonicity property

Definition 3.2. A set valued map T from a Banach space E into the subset of its dual E^* is said to be a *monotone operator* provided

$$\langle x^* - y^*, x - y \rangle \geq 0 \tag{3.11}$$

$\forall x, y \in E$ and $x^* \in T(x), y^* \in T(y)$.

Definition 3.3. A subset G of $E \times E^*$ is said to be *monotone* provided $\langle x^* - y^*, x - y \rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in G$. A set valued mapping $T : E \to 2^{E^*}$ is a monotone operator if and only if its graph

$$G(T) = \{(x, x^*) \in E \times E^* : x^* \in T(x)\} \tag{3.12}$$

is a monotone set. A monotone set is said to be *maximal monotone* if it is maximal in the family of monotone subsets of $E \times E^*$, ordered by inclusion. An element $(x, x^*) \in E \times E^*$ is said to be *monotonically related* to the subset G provided

$$\langle x^* - y^*, x - y \rangle \geq 0 \tag{3.13}$$

for all $(y, y^*) \in G$.

We say that a monotone operator T is maximal monotone provided its graph is a maximal monotone set.

Via the above, we see that $g(x)$ satisfies a maximal monotonicity property. Furthermore $g'(x) = (q-1)|x|^{q-2} \geq 0$ for any $x \in \mathbb{R}$, then $g_\lambda \in C^1(\mathbb{R})$ and satisfies the following:

$$0 \leq g_\lambda' \leq \frac{1}{\lambda}, \qquad |g_\lambda(x)| \leq |g(x)|, \qquad |g_\lambda(x)| \leq \frac{1}{\lambda}|x|, \quad \forall x \in \mathbb{R}.$$

We now recall the following lemma [33].

Lemma 3.4. *Let $\{\lambda_k\}$ be a sequence of positive numbers and $\{x_k\}$ be a sequence of real numbers such that $\lambda_k \to 0$ as $x_k \to x$. Then*

$$\lim_{k\to\infty} g_{\lambda_k}(x_k) = g(x). \tag{3.14}$$

(For details on derivations of manipulations on g_λ, see [20]). Using the above Galerkin approximation and applying Ito's lemma to $||v_{tm}||_2^2$, we obtain

$$\begin{aligned}
&||v_{tm}||_2^2 + 2\int_0^t ||\nabla v_{tm}||_2^2 ds + 2\int_0^t ||\nabla v_{sm}||_\beta^\beta ds + \frac{2}{\alpha}||\nabla u_{tm}||_\alpha^\alpha \le ||v_{tm}(0)||_2^2 \\
&+2\int_0^t ||\nabla v_{tm}(0)||_2^2 ds + \frac{2}{\alpha}||\nabla u_{tm}(0)||_\alpha^\alpha - 2\int_0^t\int_D g_\lambda(v_{sm})v_{sm}dxds \\
&+2\int_0^t\int_D f_N(v_{sm})v_{sm}dxds + 2\int_0^t \langle v_{tm}, \sigma dW_s\rangle \\
&+C^2\mathrm{Tr}\mathcal{R}\sum_{j=1}^m \int_0^t |\langle e_j, \sigma\rangle|^2 ds.
\end{aligned} \tag{3.15}$$

Via the form of the constructions for $f_N(u_t)$ and $g_\lambda(v_t)$ and Hölder's inequality, we find

$$\begin{aligned}
\int_D f_N(u_{tm})v_{tm}dx \quad &\le \int_D \chi_N(||\nabla u_{tm}||_2)||u_{tm}||^{p-1}||v_{tm}(s)||dx \\
&\le C_N||\nabla u_{tm}||_2||v_{tm}||_2.
\end{aligned} \tag{3.16}$$

For details on this estimate, see [5], but it essentially follows from the form of the construction of $f_N(u_t)$. We also have

$$-2\int_0^t\int_D g_\lambda(v_m(s))v_m(s)dxds \le 0. \tag{3.17}$$

To tackle the stochastic integral term, we define

$$M_t = \int_0^t \langle v_{tm}, \sigma(x,s)dW_s\rangle. \tag{3.18}$$

We can show that M_t is a local martingale; hence, to estimate its supremum, we can apply the Burkholder-Davis-Gundy inequality [19] as follows:

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \langle v_{tm}, \sigma(x,s)dW_s\rangle\right|\right] \\
&\le \quad C\mathbb{E}\left[\sup_{t\in[0,T]}||v_{tm}||_2\left(\sum_{j=1}^\infty\int_0^T \langle\sigma(x,t)\mathcal{R}e_i, \sigma(x,t)e_i\rangle\, dt\right)^{\frac{1}{2}}\right] \\
&\le \quad \frac{1}{2}\mathbb{E}\left[\sup_{t\in[0,T]}||v_{tm}||_2^2\right] + C\mathrm{Tr}\mathcal{R}\mathbb{E}\left[\int_0^t ||\sigma(t)||_2^2 dt\right] \\
&\le \quad \mathbb{E}\left[\sup_{t\in[0,T]}\frac{1}{2}||v_{tm}||_2^2\right] + C_1.
\end{aligned} \tag{3.19}$$

We take the expectation and supremum of (3.15), via the Sobolev embedding of $W_0^{1,\alpha}(\Omega) \hookrightarrow H_0^1(\Omega)$ and the estimate in (3.19), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} \frac{1}{2}||v_{tm}||_2^2\right] + \mathbb{E}\left[\sup_{t\in[0,T]} \frac{2}{\alpha}||\nabla u_{tm}||_\alpha^\alpha\right] \\
\leq \quad & C_N \int_0^t \left(\mathbb{E}\left[\sup_{t\in[0,T]} ||v_{tm}||_2^2\right] + \mathbb{E}\left[\sup_{t\in[0,T]} \frac{2}{\alpha}||\nabla u_{tm}||_\alpha^\alpha\right]\right) ds + C_1.
\end{aligned}$$

Consequently, Gronwall's lemma yields

$$\mathbb{E}\left[||v_{tm}||_{L^\infty(0,T;L^2(D))} + ||u_{tm}||_{L^\infty(0,T;W_0^{1,\alpha}(D))}\right] \leq C_N. \tag{3.20}$$

Now, we insert this bound back into (3.15) to obtain

$$\begin{aligned}
\int_0^t ||\nabla v_{sm}||_\beta^\beta ds \quad \leq \quad & ||v_{tm}(0)||_2^2 + ||\nabla v_{tm}(0)||_2^2 + \frac{1}{\alpha}||\nabla u_{tm}(0)||_\alpha^\alpha \\
& + C_N \int_0^t \left(\mathbb{E}\left[\sup_{t\in[0,T]} ||v_{tm}||_2^2\right] + \mathbb{E}\left[\sup_{t\in[0,T]} \frac{2}{\alpha}||\nabla u_{tm}||_\alpha^\alpha\right]\right) ds \\
& + C_2, \qquad (3.21)
\end{aligned}$$

Thus

$$\mathbb{E}\left[||v_{tm}||_{L^\beta(0,T;W_0^{1,\beta}(D))}\right] \leq C_N + C_2. \tag{3.22}$$

Next, we define the following quantity:

$$\begin{aligned}
A_\lambda \quad = \quad & ||v_{tm}||_{L^\infty(0,T;L^2(D))} + ||v_{tm}||_{L^\beta(0,T;W_0^{1,\beta}(D))} + ||u_t||_{L^\infty(0,T;W_0^{1,\alpha}(D))} \\
& + 2\int_0^T \int_D g_\lambda(v_{tm}(s)) v_m(s) dx ds. \qquad (3.23)
\end{aligned}$$

The idea is now to give a probabilistic estimate on the event that $A_\lambda \leq L$. This is done by considering a union over all truncations, u_{tm}, and then subsequently a union over all possible finite bounds, L. This is incorporated directly from the form of A_λ as

$$\mathbb{P}\left(\bigcup_{L=1}^\infty \bigcap_{j=1}^\infty \bigcup_{m=j}^\infty \{A_\lambda(u_{tm}) \leq L\}\right) = 1.$$

Next, we define P_m as the projection onto the subspace spanned by $\{e_1, ..., e_m\}$. From the earlier truncation and Ito's lemma, we have

$$\begin{aligned}
&\partial_t(v_{tm} - P_m M(t)) = -P_m(g_\lambda(v_{tm})) + P_m f_N(u_{tm}) \\
&+ P_m\left(\operatorname{div}(|\nabla u_{tm}|^{\alpha-2}\nabla u_{tm})\right) + P_m\left(\operatorname{div}(|\nabla v_{tm}|^{\beta-2}\nabla v_{tm})\right).
\end{aligned}$$

This holds in a distributional sense in $(0, T) \times D$ for almost all ω. Here, $M(t) = \int_0^t \langle \sigma(x, s), dW_s\rangle$. As $\sigma(x, t)$ is $H_0^1(D)$-valued and progressively measurable and $\{W(t, x) : t > 0\}$ is a V-valued process, there exits $\Omega_1 \subset \Omega$ of full measure, such that, for each $\omega \in \Omega_1$, we have

$$M \in C([0, T]; H_0^1(D))$$

Now, from the estimate derived in (3.20) and (3.22), we have for each $\omega \in \Omega_1$ that there is a subsequence, $\{u_{tm_k}\}_{k=1}^{\infty}$, such that

$$A_\lambda(u_{tm_k}) \leq L_\omega \tag{3.24}$$

for all k. Thus, we conclude that

$$u_{tm_k} \overset{*}{\rightharpoonup} u_t \text{ in } L^\infty(0,T;W_0^{1,\alpha}(D)), \tag{3.25}$$

$$v_{tm_k} \rightharpoonup v_t \text{ in } L^\beta(0,T;W_0^{1,\beta}(D)), \tag{3.26}$$

and

$$v_{tm_k} \overset{*}{\rightharpoonup} v_t \text{ in } L^\infty(0,T;L^2(D)). \tag{3.27}$$

In the above $\rightharpoonup$ denotes weak convergence, and $\overset{*}{\rightharpoonup}$ denotes weak-star convergence. Also these convergences are with $\mathbb{P}$ $a.s.$ By defining an appropriate Gelfand triple $W_0^{1,\alpha}(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow W^{-1,\frac{\alpha}{\alpha-1}}(\Omega)$, and using the estimate in (3.20) we obtain

$$u_{tm_k} \to u_t \text{ in } C([0,T];H_0^1(D)). \tag{3.28}$$

Next, we note via the earlier assumption on g_λ in (3.10), the embedding of $L^{q/(q-1)}(D) \hookrightarrow W^{-1,\alpha/(\alpha-1)}(D)$, and the estimates in (3.23) and (3.24) that

$$||g_\lambda(v_{tm_k})||_{L^{q/(q-1)}(0,T;W^{-1,\alpha/(\alpha-1)}(D))} \leq CL_\omega.$$

To show the convergence of the terms with divergence structures, we are required to show that, for almost all ω,

$$\lim_{m\to\infty}\int_0^t \left\langle \operatorname{div}(|\nabla u_{tm}|^{\alpha-2}\nabla u_{tm}, w_j\right\rangle ds = \int_0^t \left\langle \operatorname{div}(|\nabla u_t|^{\alpha-2}\nabla u_t, w_j\right\rangle ds$$

and

$$\lim_{m\to\infty}\int_0^t \left\langle \operatorname{div}(|\nabla v_{tm}|^{\beta-2}\nabla v_{tm}, w_j\right\rangle ds = \int_0^t \left\langle \operatorname{div}(|\nabla v_t|^{\beta-2}\nabla v_t, w_j\right\rangle ds.$$

Indeed, consider

$$U_\epsilon^m = \epsilon u_{tm} + (1-\epsilon)u_t,\ V_\epsilon^m = \epsilon v_{tm} + (1-\epsilon)v_t.$$

Then we can make the following estimates:

$$\lim_{m\to\infty}\left|\int_0^t \left\langle \operatorname{div}(|\nabla u_{tm}|^{\alpha-2}\nabla u_{tm} - |\nabla u_t|^{\alpha-2}\nabla u_t, w_j\right\rangle ds\right| \tag{3.29}$$

$$= \lim_{m\to\infty}\left|\int_0^t \left\langle \int_0^1 \frac{d}{d\epsilon}\nabla U_\epsilon^{m^{\alpha-1}}\nabla U_\epsilon^m d\epsilon, \nabla w_j\right\rangle ds\right| \tag{3.30}$$

$$\leq \lim_{m\to\infty}\alpha\int_0^t \left(|\nabla(u_{tm}-u_t)|\int_0^1 |\nabla U_\epsilon^m|^{\alpha-1}d\epsilon, |\nabla w_j|\right) ds \tag{3.31}$$

$$\leq \lim_{m\to\infty} C\int_0^t ||\nabla(u_{tm}-u_t)||\,||\nabla w_j||_{2\alpha}\int_0^1 ||\nabla U_\epsilon^m||_{2\alpha}^{\alpha-1} d\epsilon ds \tag{3.32}$$

$$\leq \lim_{m\to\infty} C\int_0^t ||\nabla(u_{tm}-u_t)||_2 ds \tag{3.33}$$

$$\to 0. \tag{3.34}$$

These inequalities follow from the compact Sobolev embedding of

$$W_0^{1,\alpha}(D) \hookrightarrow H_0^1(D),\ W_0^{1,\beta}(D) \hookrightarrow H_0^1(D)$$

The convergence for v_{tm} is established similarly.

These estimates enable

$$||v_{tm} - P_m M(t)||_{W^{1,q/(q-1)}(0,T;W^{-1,\alpha/(\alpha-1)}(D))} \leq CL_\omega. \tag{3.35}$$

Hence, via (3.25), (3.27) and (3.35) and an appropriately defined Gelfand triple as $W_0^{1,\alpha}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W^{-1,\frac{\alpha}{\alpha-1}}(\Omega)$, we obtain

$$(v_{tm} - P_m M(t)) \to (v_t - M(t)) \text{ in } C([0,T]; L^2(\Omega)).$$

Thus, we have via the Riesz-Fischer theorem [36] a subsequence, still denoted as v_{tm_k}, that

$$v_{tm_k}(t,x) \to v_t(t,x)$$

for almost all $(t,x) \in (0,T) \times D$. Thus, we have

$$g_\lambda(v_{tm_k}) \rightharpoonup g_\lambda(v_t) \quad \text{in } L^{q/(q-1)}((0,T) \times D).$$

This demonstrates that (1.2) is satisfied in the sense of distributions. We show uniqueness via contradiction. Assume there exists another subsequence that converges to $\grave{u} = \grave{u}(\omega)$. We consider $w = u - \grave{u}$, and derive an equation for w. Multiplying this by w', and via earlier estimates via (3.16), (3.29), we derive an inequality essentially of the form

$$|w'(t)|_2^2 + |\nabla w(t)|_\alpha^\alpha \leq C \int_0^t (|w'(s)|_2^2 + |\nabla w(s)|_\alpha^\alpha) ds$$

Gronwall's lemma now yields $w = 0$ or $u = \grave{u}$. One can also see [10, 20, 6]. We now state without proof a progressive measurability result on (u_t, v_t). The proof follows via mimicking the methods in [10].

Lemma 3.5. *Consider (u_t, v_t) that solve* (3.1). *Then (u_t, v_t) is $W_0^{1,\alpha}(D) \times L^2(D)$-valued progressively measurable for any $t \in [0,T]$. Furthermore,*

$$\mathbb{E}[A_\lambda(u_t)] \leq C_N.$$

We next consider the following problem

$$\begin{cases} \partial_{tt} u - \Delta \partial_t u - \operatorname{div}(|\nabla u|^{\alpha-2} \nabla u) - \operatorname{div}(|\nabla \partial_t u|^{\beta-2} \nabla \partial_t u) + g(\partial_t u) \\ = f_N(u) + \sigma(x,t) \partial_t W(x,t), \\ u(x,t) = 0, \\ u(x,0) = u_0(x), \qquad \partial_t u(x,0) = u_1(x). \end{cases} \tag{3.36}$$

Applying the methods as we did to derive our earlier estimates (3.25), (3.26), (3.27) , (3.28) and (3.16) we can derive similar estimates for (3.36) and show that the following lemma holds.

Lemma 3.6. *Assume that* (3.2) *and* (3.3) *hold. Then there exists a time T and a pathwise unique local solution of* (3.36) *such that*

$$u_t \in L^\alpha(\Omega; L^\infty(0,T; W_0^{1,\alpha}(D))) \cap L^2(\Omega; C([0,T]; H_0^1(D))),$$

$$v_t \in L^2(\Omega; L^\infty(0,T; L^2(D))) \cap L^\beta(\Omega; L^\beta(0,T; W_0^{1,\beta}(D))) \cap L^2(\Omega; C([0,T]; L^2(D))),$$

and

$$\mathbb{E}\left[||v_t||_{L^\infty(0,T;L^2(D))} + ||v_t||_{L^\beta(0,T;W_0^{1,\beta}(D))} + ||u_t||_{L^\infty(0,T;W_0^{1,\alpha}(D))}\right] \le C_N.$$

We can now state the following result concerning (1.2).

Theorem 3.7. *Assume that* (3.2) *and* (3.3) *hold. Then there exists a time* T *and a pathwise unique local solution of* (1.2) *such that, for any* $t \in [0,T]$*, and almost all* ω*, the following energy equation holds,*

$$\begin{aligned}
&||v_t||_2^2 + 2\int_0^t ||\nabla v_t(s)||_2^2 ds + \frac{2}{\alpha}||\nabla u_t||_\alpha^\alpha + 2\int_0^t ||\nabla v_t(s)||_\beta^\beta ds \\
&+2\int_0^t ||v_t(s)||_q^q ds - \frac{2}{p}||u_t||_p^p = ||u_1||_2^2 + \frac{2}{\alpha}||\nabla u_0||_\alpha^\alpha \\
&+2\int_0^t \langle v_t(s), \sigma(x,s)dW_s\rangle + \sum_{i=1}^\infty \int_0^t \int_D \lambda_i e_i^2(x)\sigma^2(x,s)dxds.
\end{aligned}$$

Proof. The proof proceeds similarly as in the earlier truncated problems. We consider Galerkin truncation sequences $\{u_{tm}\}$, $\{v_{tm}\}$, and $\{\sigma_m(x,t,\omega)\}$ such that

$$u_{0,m} \in L^\alpha(\Omega; W_0^{1,\alpha}(D)),\ \ u_{1,m} \in L^2(\Omega; L^2(D)),$$

$$\sigma_m(x,t,\omega) \in L^2(\Omega; L^2(0,T; H_0^1(D) \cap L^\infty(D))).$$

Using the a priori estimates made in Lemma 3.6, we see that the following energy equation holds:

$$\begin{aligned}
&||v_{tm}||_2^2 + 2\int_0^t ||\nabla v_{tm}(s)||_2^2 ds + \frac{2}{\alpha}||\nabla u_{tm}||_\alpha^\alpha + 2\int_0^t ||\nabla v_{tm}(s)||_\beta^\beta ds \\
&+2\int_0^t ||v_{tm}(s)||_q^q ds - 2\int_0^t \int_D \chi(||\nabla u_{tm}||_2)|u_{tm}|^{p-2}u_m v_{tm} dxds \\
&= ||u_1||_2^2 + \frac{2}{\alpha}||\nabla u_0||_\alpha^\alpha + 2\int_0^t \langle v_{tm}(s), \sigma_m(x,s)dW_s\rangle \\
&+\sum_{i=1}^\infty \int_0^t \int_D \lambda_i e_i^2(x)\sigma_m^2(x,s)dxds. \qquad (3.37)
\end{aligned}$$

We set

$$M_m(t) = \int_0^t \langle \sigma_m(x,s), dW_s\rangle. \qquad (3.38)$$

In order to show that the energy equation in Theorem 3.7 holds, we take the limit of each of the terms in (3.37). The convergence of most terms is standard, as shown earlier. For the stochastic terms, notice that

$$\begin{aligned}
& \mathbb{E}\left[\left|\int_0^T \langle v_{tm}, \sigma_m dW_s\rangle - \int_0^T \langle v_{tN}, \sigma_N dW_s\rangle\right|\right] \\
\leq\ & \mathbb{E}\left[\left|\int_0^T \langle v_{tm} - v_{tN}, \sigma_m dW_s\rangle\right|\right] + \mathbb{E}\left[\left|\int_0^T \langle v_{tN}, (\sigma_m - \sigma_N) dW_s\rangle\right|\right] \\
\leq\ & C\mathbb{E}\left[\sup_{0\leq t\leq T} ||v_{tm} - v_{tN}||_2 \left(\sum_{i=1}^{\infty}\int_0^T (\sigma_m \mathcal{R} e_i, \sigma_m e_i) dt\right)^{\frac{1}{2}}\right] \\
& + C\mathrm{Tr}\mathcal{R}\mathbb{E}\left[\sup_{0\leq t\leq T} ||v_{tN}||_2\right]\mathbb{E}\left[\left(\int_0^T ||\sigma_m - \sigma_N|_2^2 dt\right)^{\frac{1}{2}}\right] \\
& \to 0. \qquad (3.39)
\end{aligned}$$

These follow via the Burkholder-Davis-Gundy inequality [19]. We therefore have

$$\lim_{m\to\infty}\int_0^T \langle v_{tm}, \sigma_m dW_s\rangle \to \int_0^T \langle v_{tN}, \sigma_N dW_s\rangle . \qquad (3.40)$$

Here, u_{tN} is the solution to (3.36). We thus obtain

$$\begin{aligned}
& ||v_{tN}||_2^2 + 2\int_0^t ||\nabla v_{tN}(s)||_2^2 ds + \frac{2}{\alpha}||\nabla u_{tN}||_\alpha^\alpha + 2\int_0^t ||\nabla v_{tN}(s)||_\beta^\beta ds \\
& + 2\int_0^t ||v_{tN}(s)||_q^q ds - 2\int_0^t\int_D \chi(|\nabla u_{tN}|)|u_{tN}|^{p-2}u_{tN}v_{tN}\,dxds \\
& = ||u_1||_2^2 + \frac{1}{\alpha}||\nabla u_0||_\alpha^\alpha + 2\int_0^t \langle v_{tN}(s), \sigma(x,s) dW_s\rangle \\
& + \sum_{i=1}^{\infty}\int_0^t\int_D \lambda_i e_i^2(x)\sigma^2(x,s)\,dxds. \qquad (3.41)
\end{aligned}$$

Next, we define a stopping time as

$$\tau_N = \inf\{t > 0; ||\nabla u_{tN}||_2 \geq N\}. \qquad (3.42)$$

Clearly, for $t \in [0, \tau_N \wedge T)$, $u_t = u_{tN}$ is the local solution to (1.2). We let

$$\tau_\infty = \lim_{N\to\infty}\tau_N.$$

Thus, we have constructed a unique local solution,

$$u_t = \lim_{N\to\infty} u_{tN},$$

to (1.2) on $[0, T\wedge\tau_\infty]$, which solves the energy equation in Theorem 3.7. □

4. Global Existence

In this section, we show that the local solution derived in the previous section can be continued beyond the local time of existence if $p \leq q$. We establish a uniform bound on the following functional:

$$\Psi(t) := ||v_t||_2^2 + \frac{2}{\alpha}||\nabla u_t||_\alpha^\alpha + \frac{2}{p}||u_t||_p^p, \tag{4.1}$$

which will prevent unlimited growth [11, 10, 20].

Theorem 4.1. *Assume that* (3.2) *and* (3.3) *hold. If $p \leq q$, then for any time $T > 0$, there is a pathwise unique solution of* (3.1) *on the time interval $[0, T]$ such that for any $t \in [0, T]$,*

$$\mathbb{E}\left[\Psi(t)\right] < \infty. \tag{4.2}$$

Proof. We proceed by showing that

$$\lim_{N\to\infty} u_{tN} = u_t(t \wedge \tau_N) \to u_t. \tag{4.3}$$

To do this, it suffices to prove that

$$\lim_{N\to\infty} \mathbb{P}(\tau_N \to \infty) = 1. \tag{4.4}$$

Here τ_N is as defined in (3.42). To this end, the Borel-Cantelli lemma is employed, [19]. Equation (4.4) demonstrates that the local solution can be continued to be global. Via Theorem 3.7, the following energy equation holds

$$\begin{aligned}
&||v_t||_2^2 + 2\int_0^t ||\nabla v_t(s)||_2^2 ds + \frac{2}{\alpha}||\nabla u_t||_\alpha^\alpha + 2\int_0^t ||\nabla v_t(s)||_\beta^\beta ds \\
&+2\int_0^t ||v_t(s)||_q^q ds - \frac{2}{p}||u_t||_p^p = ||u_1||_2^2 + \frac{2}{\alpha}||\nabla u_0||_\alpha^\alpha \\
&+2\int_0^t \langle v_t(s), \sigma(x,s)dW_s\rangle + \sum_{i=1}^\infty \int_0^t \int_D \lambda_i e_i^2(x)\sigma^2(x,s)dxds.
\end{aligned} \tag{4.5}$$

This is, of course, true only on the local time interval, $[0, t \wedge \tau_N]$. We add

$$\frac{4}{p}||u_t(t\wedge\tau_N)||_p^p = 4\int_0^{t\wedge\tau_N}\int_D |u_t|^{p-2}u_t v_t dxds \tag{4.6}$$

to both sides of (4.5) to obtain

$$\begin{aligned}
\Psi(u_t(t\wedge\tau_N)) \quad \leq \quad & \Psi(u_t(0)) + 4\int_0^{t\wedge\tau_N}\int_D |u_t|^{p-2}u_t v_t dxds \\
& -2\int_0^{t\wedge\tau_N} ||\nabla v_t(s)||_2^2 ds - 2\int_0^{t\wedge\tau_N} |\nabla v_t(s)|_\beta^\beta ds \\
& -2\int_0^{t\wedge\tau_N} ||v_t(s)||_q^q ds - 2\int_0^{t\wedge\tau_N} \langle v_t(s), \sigma(x,s)dW_s\rangle \\
& -\sum_{i=1}^\infty \int_0^{t\wedge\tau_N}\int_D \lambda_i e_i^2(x)\sigma^2(x,s)dxds.
\end{aligned} \tag{4.7}$$

We now estimate, by using Young's inequality, that

$$\left|\int_D |u_t|^{p-2}u_t v_t dx\right| \leq \gamma||v_t||_p^p + C(\gamma)||u_t||_p^p. \tag{4.8}$$

Since we are considering $q \geq p$, we have $||v_t||_p^p \leq C||v_t||_q^p$. Using this inequality in (4.7) we obtain,

$$\begin{aligned}\Psi(u_t(t\wedge\tau_N)) \quad &\leq \quad \Psi(u_t(0)) + 4C\gamma\int_0^{t\wedge\tau_N} ||v_t||_q^p ds - 2\int_0^{t\wedge\tau_N} ||v_t(s)||_q^q ds \\ &\quad +4C(\gamma)\int_0^{t\wedge\tau_N} ||u_t||_p^p ds + 2\int_0^{t\wedge\tau_N} \langle v_t(s), \sigma(x,s)dW_s\rangle \\ &\quad +C_0 \mathrm{Tr}\mathcal{R}\int_0^{t\wedge\tau_N} ||\sigma||_2^2 ds. \qquad (4.9)\end{aligned}$$

Now, there are two possible cases for $q \geq p$. Either we have $||v_t||_q^q > 1$, in which case we choose a small enough γ, such that

$$-2||v_t(s)||_q^q + 4C\gamma||v_t||_q^p \leq 0, \qquad (4.10)$$

or we have $||v_t||_q^q < 1$, and we trivially have

$$-2||v_t(s)||_q^q + 4C\gamma||v_t||_q^p \leq 4C\gamma. \qquad (4.11)$$

In either case, we have

$$\begin{aligned}\Psi(u_t(t\wedge\tau_N)) \quad &\leq \quad \Psi(u_t(0)) + 4C\gamma(t\wedge\tau_N) + 4C(\gamma)\int_0^{t\wedge\tau_N} ||u_t||_p^p ds \\ &\quad +2\int_0^{t\wedge\tau_N} \langle v_t(s), \sigma(x,s)dW_s\rangle \\ &\quad +C_0 \mathrm{Tr}\mathcal{R}\int_0^{t\wedge\tau_N} ||\sigma||_2^2 ds. \qquad (4.12)\end{aligned}$$

We now take expectations in (4.12) to obtain

$$\begin{aligned}\mathbb{E}[\Psi(u_t(t\wedge\tau_N))] \quad &\leq \quad \Psi(u_t(0)) + 4C\gamma(t\wedge\tau_N) + 4C(\gamma)\int_0^{t\wedge\tau_N} \mathbb{E}\left[||u_t||_p^p\right] ds \\ &\quad +2\int_0^{t\wedge\tau_N} \mathbb{E}\left[\langle v_t(s), \sigma(x,s)dW_s\rangle\right] + \Psi(u_t(0)) + 4C\gamma(t\wedge\tau_N) \\ &\quad +C_0 \mathrm{Tr}\mathcal{R}\int_0^{t\wedge\tau_N} \mathbb{E}\left[||\sigma||_2^2\right] ds + K\int_0^{t\wedge\tau_N} \mathbb{E}[\Psi(u_t(t\wedge\tau_N))]ds \\ &\quad +C_0 \mathrm{Tr}\mathcal{R}\int_0^{t\wedge\tau_N} \mathbb{E}\left[||\sigma||_2^2\right] ds.\end{aligned}$$

This follows by the local martingale property of $\int_0^{t\wedge\tau_N} \langle v_t(s), \sigma(x,s)dW_s\rangle$ and by the definition of the functional, $\Psi(t)$. We now use the integral version of Gronwall's lemma and assume that (3.2) and (3.3) hold to obtain

$$\mathbb{E}[\Psi(u_t(T\wedge\tau_N))] \leq (\Psi(u(0) + CT)e^{KT} \leq C_T. \qquad (4.13)$$

We note that for a characteristic function, $\chi(\tau_N \leq T)$, we obtain

$$\begin{aligned}\mathbb{E}[\Psi(u_t(T\wedge\tau_N))] \quad &\geq \quad \mathbb{E}\left[\chi(\tau_N \leq T)\Psi(u_t(\tau_N))\right] \\ &\geq \quad C\mathbb{E}[\chi(\tau_N \leq T)||u_{\tau_N}||_2^2] \\ &\geq \quad CN^2\mathbb{P}(\tau_N \leq T). \qquad (4.14)\end{aligned}$$

Combining (4.13) and (4.14) gives us

$$\mathbb{P}(\tau_\infty \leq T) \leq \mathbb{P}(\tau_N \leq T) \leq \frac{C_T}{N^2}. \tag{4.15}$$

We apply the Borel-Cantelli lemma to obtain

$$\mathbb{P}(\tau_\infty \leq T) = 0, \tag{4.16}$$

which implies that

$$\lim_{N\to\infty} \tau_N = \infty. \tag{4.17}$$

Thus,

$$\lim_{N\to\infty} u_{tN} = u_t \tag{4.18}$$

is the global solution on

$$[0, \tau_\infty \wedge T] = [0, T]. \tag{4.19}$$

To check the energy bound in Theorem 4.1, we note that

$$\begin{aligned} \mathbb{E}[\sup_{0\leq t\leq T} \Psi(u_t(t))] \quad &\leq \quad \Psi(u_t(0)) + (4C\gamma + C_1)(T) \\ &\quad +4C(\gamma)\int_0^{t\wedge\tau_N} \mathbb{E}[\sup_{0\leq t\leq T} \Psi(u_t(T))]ds \\ &\quad +\mathbb{E}[\sup_{0\leq t\leq T}\int_0^t \langle v_t(s), \sigma(x,s)dW_s\rangle]. \end{aligned} \tag{4.20}$$

It follows via the Burkholder-Davis-Gundy inequality that

$$\begin{aligned} \mathbb{E}[\sup_{0\leq t\leq T}\left|\int_0^t \langle v_t(s), \sigma(x,s)dW_s\rangle\right|] \quad &\leq \quad \mathbb{E}[\sup_{0\leq t\leq T} ||v_t||_2^2] + C_1\int_0^T \mathbb{E}\left[||\sigma(t)||_2^2\right] dt \\ &\leq \quad C_2. \end{aligned}$$

Therefore, by inserting the above inequality into (4.20), we find

$$\mathbb{E}[\sup_{0\leq t\leq T} \Psi(u_t(t))] \leq C_3 + C_4\int_0^T \mathbb{E}[\sup_{0\leq s\leq T} \Psi(u_t(s))]ds. \tag{4.21}$$

Then employing the integral version of Gronwall's lemma, we obtain

$$\mathbb{E}[\sup_{0\leq t\leq T} \Psi(u_t(t))] \leq C_3 e^{C_4 T}, \tag{4.22}$$

which validates the energy bound in Theorem 4.1. □

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5. Finite Time Blow-up

In this section, we show that if the source term dominates the damping term and if the initial data are large enough, then the solution of problem (1.2) ceases to exist and blows up in finite time. To do this, we use certain estimates from [30], where the deterministic problem has been investigated. Using the same notation as in the previous section, (1.2) can be rewritten as the following Ito system:

$$\begin{cases} du_t = v_t dt, \\ dv_t = (\Delta v_t + \operatorname{div}(|\nabla u_t|^{\alpha-2}\nabla u_t) + \operatorname{div}(|\nabla v_t|^{\beta-2}\nabla v_t) \\ -|v_t|^{q-2}v_t + |u_t|^{p-2}u_t)dt + \sigma(x,t)dW(t,x), & \text{in} \in D\times(0,t), \\ u_t = 0, & \text{on } \partial D, \\ u_0(x,0) = u_0,\ v_0(x,0) = u_1(x), & \text{in } D. \end{cases} \quad (5.1)$$

Here, $(u_0, u_1) \in L^\alpha(\Omega; W_0^{1,\alpha}(D)) \times L^2(\Omega; L^2(D))$. The energy functional associated with (5.1) as defined in (1.4) takes the form

$$\mathcal{E}(t) = \frac{1}{2}||v_t||_2^2 + \frac{1}{\alpha}||\nabla u_t||_\alpha^\alpha - \frac{1}{p}||u_t||_p^p. \quad (5.2)$$

The following lemma holds true.

Lemma 5.1. *Let (u_t, v_t) be the solution to system (5.1) with initial data $(u_0, u_1) \in L^\alpha(\Omega; W_0^{1,\alpha}(D)) \times L^2(\Omega; L^2(D))$. Also, assume that (3.3) holds. Then we have*

$$\frac{d}{dt}\mathbb{E}\left[\mathcal{E}(t)\right] = -\mathbb{E}[||\nabla v_t||_2^2 + ||\nabla v_t||_\beta^\beta + ||v_t||_q^q] + \frac{1}{2}\sum_{i=1}^\infty \int_D \lambda_i e_i^2(x)\sigma^2(x,t)dx. \quad (5.3)$$

Furthermore,

$$\begin{aligned} &\mathbb{E}\left[\langle u_t(t), v_t(t)\rangle\right] = \langle u_t(0), v_t(0)\rangle - \int_0^t \mathbb{E}\left[||\nabla u_s||_2^2\right] ds \\ &- \int_0^t \mathbb{E}\left[\langle |v_s|^{q-2}v_s, u_s\rangle\right] ds - \int_0^t \mathbb{E}\left[||\nabla u_s||_\alpha^\alpha\right] ds - \int_0^t \mathbb{E}||\nabla v_s||_\beta^\beta ds + \\ &\int_0^t \mathbb{E}\left[||u_s||_p^p\right] ds + \int_0^t \mathbb{E}\left[||v_s||_2^2\right] ds. \end{aligned} \quad (5.4)$$

Proof. Using Ito's lemma, it follows that

$$\begin{aligned} ||v_t||_2^2 &= 2\mathcal{E}(0) - 2\int_0^t ||\nabla v_s||_2^2 ds - 2\int_0^t ||\nabla v_s||_\beta^\beta ds - \frac{2}{\alpha}||\nabla u_t||_\alpha^\alpha \\ &\quad -2\int_0^t ||v_s||_q^q ds + \frac{2}{p}||u_t||_p^p + 2\int_0^t \langle v_t(s), \sigma(x,s)dW_s\rangle \\ &\quad + \sum_{i=1}^\infty \int_0^t \int_D \lambda_i e_i^2(x)\sigma^2(x,s)dxds \end{aligned} \quad (5.5)$$

Now (5.3) follows by taking expectations in the above. Next we see that

$$\begin{aligned}\langle u_t, v_t\rangle &= \langle u_0, v_0\rangle + \int_0^t \langle u_s(s), dv_s(s)\rangle\, ds + \int_0^t \langle v_s(s), du_s(s)\rangle\, ds \\ &= \langle u_0, v_0\rangle - \int_0^t ||\nabla v_s||_2^2 ds - \int_0^t \left\langle |v_s|^{q-2} v_s, u_s\right\rangle ds + \int_0^t ||u_s||_p^p ds \\ &\quad + \int_0^t ||v_s||_2^2 ds - \int_0^t ||\nabla u_s||_\alpha^\alpha ds - \int_0^t ||\nabla v_s||_\beta^\beta ds \\ &\quad +2\int_0^t \langle u_t(s), \sigma dW_s\rangle, \end{aligned} \tag{5.6}$$

and (5.4) follows by taking expectations in the above. □

Let B be the optimal constant for the embedding $W_0^{1,\alpha}(\Omega) \hookrightarrow L^p(\Omega)$. Let us also define

$$E_1 = \left(\frac{1}{\alpha} - \frac{1}{p}\right)\zeta_1^\alpha, \qquad \zeta_1 = B^{-p/(p-\alpha)}. \tag{5.7}$$

We now state the following lemma, which is an adaptation of [37, Lemma 1].

Lemma 5.2. *Suppose that $\alpha \le p \le r_\alpha$ and let u be the solution of (1.2). Assume that $\mathcal{E}(0) < E_1$ and $\mathbb{E}\left[||\nabla u_0||_\alpha\right] > \zeta_1$. Then there exists a constant $\zeta_2 > \zeta_1$ such that*

$$\mathbb{E}\left[||\nabla u_t(.,t)||_\alpha\right] \ge \zeta_2, \qquad \forall t \in [0,T), \tag{5.8}$$

$$\mathbb{E}\left[||u_t||_p\right] \ge B\zeta_2, \qquad \forall t \in [0,T). \tag{5.9}$$

Proof. Using the definition of the energy functional, we obtain

$$\begin{aligned}\mathbb{E}\left[\mathcal{E}(t)\right] &\ge \frac{1}{\alpha}\mathbb{E}\left[||\nabla u_t||_\alpha^\alpha - \frac{1}{p}\mathbb{E}||u_t||_p^p\right] \\ &\ge \frac{1}{\alpha}\mathbb{E}\left[||\nabla u_t||_\alpha^\alpha\right] - \frac{B^p}{p}\mathbb{E}\left[||\nabla u_t||_\alpha^p\right] \\ &= \frac{1}{\alpha}\eta^\alpha - \frac{B^p}{p}\eta^p \\ &= h(\eta), \end{aligned} \tag{5.10}$$

where $\eta = \mathbb{E}\left[||\nabla u_t||_\alpha\right]$. It is easy to verify that h is increasing for $0 < \eta < \zeta_1$, decreasing for $\eta > \zeta_1$ and $h(\eta) \to -\infty$ as $\eta \to \infty$. Also

$$h(\zeta_1) = E_1. \tag{5.11}$$

Therefore, as $\mathbb{E}[\mathcal{E}(0)] < E_1$, there exists $\zeta_2 > \zeta_1$ such that $h(\zeta_2) = E(0)$. If we set

$$\zeta_0 = \mathbb{E}\left[||\nabla u_0||_\alpha\right],$$

then, by (5.10), we have

$$h(\zeta_0) \le E(0) = h(\zeta_2), \tag{5.12}$$

which implies that $\zeta_0 \ge \zeta_2$. Now, to establish (5.8), we assume that, on the contrary,

$$\mathbb{E}\left[||\nabla u_t(t_0)||_\alpha\right] < \zeta_2 \tag{5.13}$$

for some t_0. By the continuity of $\mathbb{E}\,[||\nabla u_t(.)||_\alpha]$ in time, we choose t_0 such that $\mathbb{E}\,[||\nabla u_t(t_0)||_\alpha] > \zeta_1$. We now use (5.10) to obtain

$$E(t_0) \geq h(\mathbb{E}\,[||\nabla u_t(t_0)||_\alpha]) > h(\zeta_2) = E(0).$$

However, since $E(t) \leq E(0)$ for all $t \in [0, T)$, we arrive at a contradiction. This proves (5.8). Now,

$$\mathbb{E}\,[||\nabla u_t(t)||_\alpha^\alpha] \leq \mathbb{E}[\mathcal{E}(0)] + \frac{1}{p}\mathbb{E}\left[||u_t||_p^p\right].$$

Therefore,

$$\begin{aligned}
\frac{1}{p}\mathbb{E}\left[||\nabla u_t||_p^p\right] \quad &\geq \quad \frac{1}{\alpha}\mathbb{E}\,[||\nabla u_t||_\alpha^\alpha] - \mathbb{E}\,[\ \mathcal{E}(0)] \\
&\geq \frac{1}{\alpha}\eta^\alpha - \mathbb{E}\,[\ \mathcal{E}(0)] \\
&\geq \frac{1}{\alpha}\eta^\alpha - g(\eta)\frac{B^p}{p}\eta^p.
\end{aligned}$$

□

Our main result in this section reads as follows:

Theorem 5.3. *Let (u_t, v_t) be the solution of system (1.2) with initial data $(u_0, u_1) \in L^\alpha(\Omega; W_0^{1,\alpha}(D)) \times L^2(\Omega; L^2(D))$. Assume further that*

$$\mathbb{E}\left[\int_0^T \left(||\nabla\sigma(t)||_2^2 + ||\sigma(t)||_\infty^2|\right) dt\right] < \infty. \tag{5.14}$$

Suppose that the initial energy satisfies

$$\mathcal{E}(0) \leq E_1 - E_2, \tag{5.15}$$

where E_1 is defined as

$$E_1 = \left(\frac{1}{\alpha} - \frac{1}{p}\right) B^{-\alpha p/(p-\alpha)}. \tag{5.16}$$

Here, B is the optimal constant for the embedding $W_0^{1,\alpha}(\Omega) \hookrightarrow L^p(\Omega)$, and $2 \leq p \leq r_\alpha$, where r_α is given in (1.3) and E_2 is defined as

$$\begin{aligned}
E_2 \quad &= \quad \mathbb{E}\left[\frac{1}{2}C^2 \mathrm{Tr}\mathcal{R}\int_0^\infty \int_D \sigma^2(x,s)dxds\right] \\
&\geq \quad \mathbb{E}\left[\frac{1}{2}\sum_{i=1}^\infty \int_0^t \int_D \lambda_i e_i^2(x)\sigma^2(x,s)dxds\right] = F(t).
\end{aligned} \tag{5.17}$$

Then if $\beta < \alpha$ and $\max(\alpha, q) < p < r_\alpha$, there exists a finite positive time $T^* \in (0, T]$ such that

$$\lim_{t \to T^* < \infty} \mathbb{E}[||u_t||_p^p] = +\infty. \tag{5.18}$$

Remark 5.4. Note that Theorem 5.3 allows for blow-up with positive initial energies since, depending on the strength of the noise, we may have $E_1 - E_2 > 0$.

Proof. Following [11], we define the function $H(t)$ as follows:

$$H(t) = E_1 + F(t) - \mathbb{E}\left[\mathcal{E}(t)\right]. \tag{5.19}$$

Thus, we obtain

$$H'(t) = F'(t) - \frac{d}{dt}\mathbb{E}\left[\mathcal{E}(t)\right] = \mathbb{E}[||\nabla v_t||_2^2 + ||\nabla v_t||_\beta^\beta + ||v_t||_q^q] \geq 0. \tag{5.20}$$

Next, we consider the perturbed functional

$$L(t) = H^{1-s}(t) + \epsilon\mathbb{E}\left[\langle u_t, v_t\rangle\right], \tag{5.21}$$

where ϵ is a small positive constant that will be chosen later and s satisfies

$$0 < s \leq \min\left(\frac{\alpha-2}{p},\ \frac{\alpha-\beta}{p(\beta-1)},\ \frac{p-q}{p(q-1)},\ \frac{\alpha-2}{2\alpha}\right). \tag{5.22}$$

Our goal is to show that $L(t)$ satisfies the differential inequality

$$\frac{dL(t)}{dt} \geq \xi L^{1+\nu}(t), \quad t \in [0,\infty), \nu > 0. \tag{5.23}$$

Inequality (5.23) leads to a blow-up of the solutions in finite time, $t \geq L(0)^{-\nu}\xi^{-1}\nu^{-1}$, provided that $L(0) > 0$. Indeed, taking the time derivative of (5.21), we obtain

$$\begin{aligned} L'(t) \quad = \quad & (1-s)H^{-s}(t)H'(t) - \epsilon\mathbb{E}\left[\langle \nabla u_t, \nabla v_t\rangle\right] - \epsilon\mathbb{E}\left[\left\langle |\nabla v_t|^{\beta-2}\nabla v_t, u_t\right\rangle\right] \\ & -\frac{1}{\alpha}\epsilon\mathbb{E}\left[||\nabla u_t||_\alpha^\alpha\right] - \mathbb{E}\left[\left\langle |v_t|^{q-2}v_t, u_t\right\rangle\right] + \frac{1}{p}\mathbb{E}\left[||u_t||_p^p\right] \\ & +\mathbb{E}\left[||v_t||_2^2\right] - \epsilon F(t). \end{aligned} \tag{5.24}$$

Young's inequality yields, for some $\mu > 0$ and $\delta > 0$, that

$$\mathbb{E}\left[\langle \nabla u_t, \nabla v_t\rangle\right] \leq \frac{1}{4\mu}\mathbb{E}\left[||\nabla u_t||_2^2\right] + \mu\mathbb{E}\left[||\nabla v_t||_2^2\right], \tag{5.25}$$

$$\mathbb{E}\left[\left\langle |v_t|^{q-2}v_t, u_t\right\rangle\right] \leq \frac{\delta^q}{q}\mathbb{E}\left[||u_t||_q^q\right] + \frac{q-1}{q}\delta^{-q/(q-1)}\mathbb{E}\left[||v_t||_q^q\right], \tag{5.26}$$

and

$$\mathbb{E}\left[\int_D |\nabla v_t|^{\beta-1}\nabla u_t dx\right] \leq \frac{\lambda^\beta}{\beta}\mathbb{E}\left[||\nabla u_t||_\beta^\beta\right] + \frac{\beta-1}{\beta}\lambda^{-\beta/(\beta-1)}\mathbb{E}\left[||\nabla u_t||_\beta^\beta\right]. \tag{5.27}$$

Substituting (5.25)-(5.27) into (5.24) results in

$$\begin{aligned} L'(t) \quad \geq \quad & (1-s)H^{-s}H'(t) + \epsilon\mathbb{E}\left[||v_t||_2^2\right] - \frac{\epsilon}{4\mu}\mathbb{E}\left[||\nabla u_t||_2^2\right] \\ & -\mu\epsilon\mathbb{E}\left[||\nabla v_t||_2^2\right] - \epsilon\mathbb{E}\left[||\nabla u_t||_\alpha^\alpha\right] - \epsilon\frac{\lambda^\beta}{\beta}\mathbb{E}\left[||\nabla u_t||_\beta^\beta\right] \\ & -\epsilon\frac{\beta-1}{\beta}\lambda^{-\beta/(\beta-1)}\mathbb{E}\left[||\nabla v_t||_\beta^\beta\right] + \epsilon\mathbb{E}\left[||u_t||_p^p\right] - \epsilon\frac{\delta^q}{q}\mathbb{E}\left[||u_t||_q^q\right] \\ & -\epsilon\frac{q-1}{q}\delta^{-q/(q-1)}\mathbb{E}\left[||v_t||_q^q\right] - \epsilon F(t). \end{aligned} \tag{5.28}$$

We now choose

$$\begin{cases} \delta^{-q/(q-1)} = M_1 H^{-s}(t), \\ \mu = M_2 H^{-s}(t), \\ \lambda^{-\beta/(\beta-1)} = M_3 H^{-s}(t), \end{cases}$$

where $M_i\,(i=1,2,3)$ are large positive constants that can be choosen later. Let $M = M_2 + (\beta-1)M_3/\beta + (q-1)M_1/q$. Then we obtain from (5.28) that

$$\begin{aligned} L'(t) \quad &\geq ((1-s) - \epsilon M) H^{-s} H^{'}(t) + \epsilon \mathbb{E}\left[||v_t||_2^2\right] - \frac{\epsilon}{4M_2} H^s \mathbb{E}\left[||\nabla u_t||_2^2\right] \\ &\quad - \epsilon \mathbb{E}\left[||\nabla u_t||_\alpha^\alpha\right] - \epsilon \frac{M_3^{-(\beta-1)}}{\beta} H^{s(\beta-1)} \mathbb{E}\left[||\nabla u_t||_\beta^\beta\right] \\ &\quad + \epsilon \mathbb{E}\left[||u_t||_p^p\right] - \epsilon \frac{M_1^{-(q-1)}}{q} H^{s(m-1)} \mathbb{E}\left[||u_t||_q^q\right] - \epsilon F(t). \end{aligned} \tag{5.29}$$

Since we are dealing with the case $p > q$, we use the embedding of $L^p(D) \hookrightarrow L^q(D)$ to obtain

$$H^{s(q-1)}(t) \mathbb{E}\left[||u_t||_q^q\right] \leq C\,(1/p)^{s(q-1)} \left(\mathbb{E}\left[||u_t||_p^p\right]\right)^{(q+sp(q-1))/p}. \tag{5.30}$$

We also exploit the inequality

$$\mathbb{E}\left[||\nabla u_t||_2^2\right] \leq C\left(\mathbb{E}\left[||\nabla u_t||_\alpha^\alpha\right]\right)^{2/\alpha}$$

and the embedding $W_0^{1,\alpha}(D) \hookrightarrow L^p(D)$ to obtain

$$H^s(t)\,\mathbb{E}\left[||\nabla u_t||_2^2\right] \leq C\,(1/p)^s \left(\mathbb{E}\left[||\nabla u_t||_\alpha^\alpha\right]\right)^{(ps+2)/\alpha}. \tag{5.31}$$

Because $\alpha > \beta$, we have

$$\mathbb{E}\left[||\nabla u_t||_\beta^\beta\right] \leq C\left(\mathbb{E}\left[||\nabla u_t||_\alpha^\alpha\right]\right)^{\beta/\alpha}.$$

Consequently,

$$H^{s(\beta-1)}(t)\,\mathbb{E}\left[||\nabla u_t||_\beta^\beta\right] \leq C\,(1/p)^{s(\beta-1)} \left(\mathbb{E}\left[||\nabla u_t||_\alpha^\alpha\right]\right)^{(ps(\beta-1)+\beta)/\alpha},$$

where C is a constant depending only on D. By using (5.22) and the algebraic inequality

$$z^\nu \leq z + 1 \leq \left(1 + \frac{1}{a}\right)(z+a), \quad \forall z \geq 0,\ 0 < \nu \leq 1,\ a \geq 0, \tag{5.32}$$

we have the following

$$\begin{aligned} \left(\mathbb{E}\left[||u_t||_p^p\right]\right)^{(q+sp(q-1))/p} \quad &\leq \quad \left(\mathbb{E}\left[||\nabla u_t||_\alpha^\alpha\right]\right)^{(q+sp(q-1))/\alpha} \\ &\leq \quad d\left(\mathbb{E}\left[||\nabla u_t||_\alpha^\alpha\right] + H(0)\right), \\ &\leq \quad d\left(\mathbb{E}\left[||\nabla u_t||_\alpha^\alpha\right] + H(t)\right), \qquad \forall t \geq 0. \end{aligned} \tag{5.33}$$

Similarly, we have

$$\left(\mathbb{E}\left[||\nabla u_t||_\alpha^\alpha\right]\right)^{(ps+2)/\alpha} \leq d\left(\mathbb{E}\left[||\nabla u_t||_\alpha^\alpha\right] + H(t)\right), \qquad \forall t \geq 0 \tag{5.34}$$

and

$$\left(\mathbb{E}\left[||\nabla u_t||_\alpha^\alpha\right]\right)^{(ps(\beta-1)+\beta)/\alpha} \leq d\left(\mathbb{E}\left[||\nabla u_t||_\alpha^\alpha\right] + H(t)\right), \qquad \forall t \geq 0, \tag{5.35}$$

where $d = 1 + 1/H(0)$. Now, we use

$$-\epsilon F(t) \geq -\epsilon(\mathbb{E}[||\nabla u_t||_\alpha^\alpha] + H(t)) - \frac{1}{2}\mathbb{E}\left[||v_t||_2^2\right] \tag{5.36}$$

and insert the estimates (5.30), (5.31), and (5.33) into (5.29) to obtain

$$\begin{aligned} L'(t) \quad \geq \quad & ((1-s) - \epsilon M)\, H^{-s}(t)\, H'(t) + kH(t) + \epsilon\left(\frac{1+k}{2}\right)\mathbb{E}\left[||v_t||_2^2\right] \\ & -\epsilon p\mathbb{E}(E_1) - \frac{\epsilon(C_2+1)}{M_2}\left(\mathbb{E}[||\nabla u_t||_\alpha^\alpha] + H(t)\right) + \epsilon\left(\frac{p}{\alpha} - 1\right)\mathbb{E}[||\nabla u_t||_\alpha^\alpha] \\ & -\frac{\epsilon C_3}{M_3^{\beta-1}}\left(\mathbb{E}[||\nabla u_t||_\alpha^\alpha] + H(t)\right) + \frac{k}{\alpha}\mathbb{E}[||\nabla u_t||_\alpha^\alpha] \\ & -\frac{\epsilon C_1}{M_1^{q-1}}\left(\mathbb{E}[||\nabla u_t||_\alpha^\alpha] + H(t)\right) + \left(\epsilon - \frac{k}{p}\right)\mathbb{E}\left[||u_t||_p^p\right], \end{aligned}$$

for some constant k and

$$C_1 = \frac{Cd}{q}\left(\frac{1}{p}\right)^{s(q-1)}, \quad C_2 = \frac{Cd}{4}\left(\frac{1}{p}\right)^{s}, \quad C_3 = \frac{Cd}{\beta}\left(\frac{1}{p}\right)^{s(\beta-1)}.$$

Then we use (5.9) to obtain

$$\begin{aligned} L'(t) \quad \geq \quad & ((1-s) - \epsilon M)\, H^{-s}(t)\, H^{'}(t) + kH(t) + \epsilon\left(1 + \frac{p}{2}\right)\mathbb{E}\left[||v_t||_2^2\right] \\ & -\frac{\epsilon C_2}{M_2}\left(\mathbb{E}[||\nabla u_t||_\alpha^\alpha] + H(t)\right) + \epsilon\left(\frac{p}{\alpha} - 1 - pE_1\zeta_2^{-\alpha}\right)\mathbb{E}[||\nabla u_t||_\alpha^\alpha] \\ & -\frac{\epsilon C_3}{M_3^{\beta-1}}\left(\mathbb{E}[||\nabla u_t||_\alpha^\alpha] + H(t)\right) + \frac{k}{\alpha}\mathbb{E}[||\nabla u_t||_\alpha^\alpha] \\ & -\frac{\epsilon C_1}{M_1^{q-1}}\left(\mathbb{E}[||\nabla u_t||_\alpha^\alpha] + H(t)\right) + \epsilon\left(1 - \frac{k}{p}\right)\mathbb{E}\left[||u_t||_p^p\right]. \end{aligned}$$

This implies that

$$\begin{aligned} L'(t) \quad \geq \quad & ((1-s) - \epsilon M)\, H^{-s}(t)\, H'(t) + kH(t) + \epsilon\left(1 + \frac{p}{2}\right)\mathbb{E}\left[||v_t||_2^2\right] \\ & -\frac{\epsilon C_2}{M_2}\left(\mathbb{E}[||\nabla u_t||_\alpha^\alpha] + H(t)\right) + \epsilon c_0\mathbb{E}[||\nabla u_t||_\alpha^\alpha] \\ & -\frac{\epsilon C_3}{M_3^{\beta-1}}\left(\mathbb{E}[||\nabla u_t||_\alpha^\alpha] + H(t)\right) + \frac{k}{\alpha}\mathbb{E}[||\nabla u_t||_\alpha^\alpha] \\ & -\frac{\epsilon C_1}{M_1^{q-1}}\left(\mathbb{E}[||\nabla u_t||_\alpha^\alpha] + H(t)\right) + \epsilon\left(1 - \frac{k}{p}\right)\mathbb{E}\left[||u_t||_p^p\right], \end{aligned}$$

where

$$c_0 = \frac{p}{\alpha} - 1 - pE_1\zeta_2^{-\alpha} > 0, \quad \text{as } \zeta_2 > \zeta_1.$$

Thus, we arrive at

$$\begin{aligned} L'(t) &\geq ((1-s) - \epsilon M)\, H^{-s}(t)\, H'(t) + \epsilon \left(\frac{p+2}{2}\right) \mathbb{E}\left[||v_t||_2^2\right] \\ &+\epsilon \left(p - \frac{C_2}{M_2} - \frac{C_3}{M_3^{\beta-1}} - \frac{C_1}{M_1^{q-1}}\right) H(t) \\ &+\epsilon \left(c_0 - \frac{C_2}{M_2} - \frac{C_3}{M_3^{\beta-1}} - \frac{C_1}{M_1^{m-1}} - 1\right) \mathbb{E}\left[||\nabla u_t||_\alpha^\alpha\right]. \end{aligned}$$

At this point, we choose large enough M_1, M_2, and M_3 such that

$$L'(t) \geq ((1-s) - \epsilon M)\, H^{-s}(t)\, H'(t) + \gamma\epsilon \left[H(t) + \mathbb{E}\left[||v_t||_2^2\right] + \mathbb{E}\left[||\nabla u_t||_\alpha^\alpha\right]\right],$$

where γ is a positive constant (this is possible because $p > \alpha$ and hence $c_0 > 0$). By choosing $\varepsilon < (1-s)/M$ such that

$$L(0) = H^{1-s}(0) + \epsilon \mathbb{E}\left[\langle u_0, u_1\rangle\right] > 0,$$

we obtain

$$L(t) \geq L(0) > 0, \quad \forall t \geq 0$$

and

$$L'(t) \geq \gamma\epsilon \left[H(t) + \mathbb{E}\left[||v_t||_2^2\right] + \mathbb{E}\left[||\nabla u_t||_\alpha^\alpha\right]\right]. \tag{5.37}$$

Next, it is clear that

$$L^{1/(1-s)}(t) \leq 2^{1/(1-s)} \left\{H(t) + \epsilon^{1/(1-s)} \left(\mathbb{E}\left[\langle v_t, u_t\rangle\right]\right)^{1/(1-s)}\right\}.$$

By the Cauchy-Schwarz inequality, the embedding of $L^\alpha(D) \hookrightarrow L^2(D)$, and the properties of expectations, we obtain

$$\begin{aligned} \mathbb{E}\left[\langle v_t, u_t\rangle\right] &\leq \left(\mathbb{E}\left[||u_t||_2^2\right]\right)^{1/2} \left(\mathbb{E}\left[||v_t||_2^2\right]\right)^{1/2} \\ &\leq C \left(\mathbb{E}\left[||u_t||_\alpha^\alpha\right]\right)^{1/\alpha} \left(\mathbb{E}\left[||v_t||_2^2\right]\right)^{1/2}, \end{aligned}$$

which implies

$$\mathbb{E}\left[\langle v_t, u_t\rangle\right]^{1/(1-s)} \leq C \left(\mathbb{E}\left[||u_t||_\alpha^\alpha\right]\right)^{1/(1-s)\alpha} \left(\mathbb{E}\left[||v_t||_2^2\right]\right)^{1/2(1-s)}.$$

Also, Young's inequality gives

$$\mathbb{E}\left[\langle v_t, u_t\rangle\right]^{1/(1-s)} \leq C \left[\left(\mathbb{E}\left[||u_t||_\alpha^\alpha\right]\right)^{\mu/(1-s)\alpha} + \left(\mathbb{E}\left[||v_t||_2^2\right]\right)^{\theta/2(1-s)}\right]$$

for $1/\mu + 1/\theta = 1$. We take $\theta = 2(1-s)$ and hence $\mu = \frac{2(1-s)}{(1-2s)}$ to get

$$\mathbb{E}\left[\langle v_t, u_t\rangle\right]^{1/(1-s)} \leq C \left[\left(\mathbb{E}\left[||u_t||_\alpha^\alpha\right]\right)^{2/(1-2s)\alpha} + \mathbb{E}\left[||v_t||_2^2\right]\right].$$

By using (5.22) and (5.32), we deduce

$$\left(\mathbb{E}\left[||u_t||_\alpha^\alpha\right]\right)^{2/(1-2s)\alpha} \leq \left(1 + \frac{1}{H(0)}\right) \left(\mathbb{E}\left[||u_t||_\alpha^\alpha\right] + H(t)\right).$$

Therefore,

$$\mathbb{E}\left[\langle v_t, u_t\rangle\right]^{1/(1-s)} \leq C \left[H(t) + \mathbb{E}\left[||u_t||_\alpha^\alpha\right] + \mathbb{E}\left[||v_t||_2^2\right]\right], \quad \forall t \geq 0. \tag{5.38}$$

Consequently,

$$L^{1/(1-s)}(t) \leq \Gamma\left[H(t) + \mathbb{E}\left[\|u_t\|_\alpha^\alpha\right] + \mathbb{E}\left[\|v_t\|_2^2\right]\right],$$

where Γ is a positive constant. A combination of (5.37) and (5.38) thus yields

$$L'(t) \geq \xi L^{1/(1-s)}(t), \ \ \forall t \geq 0. \tag{5.39}$$

Integration of (5.39) over $(0,t)$ gives

$$L^{s/(1-s)}(t) \geq \frac{1}{L^{-s/(1-s)}(0) - \frac{\xi s}{(1-s)}t}.$$

Hence, $L(t)$ blows up in finite time: that is, if

$$T^* = \frac{1-s}{\xi s L^{\frac{s}{1-s}}(0)}, \tag{5.40}$$

then $L(t) \to +\infty$ as $t \to T^*$, which implies the blow-up of $H(t)$ and hence of $\mathbb{E}[\|u_t\|_p^p]$. $\square$

6. Numerical Experiments

In this section our goal is to perform various numerical experiments on equation (1.7) with noise, when $q = 2$. This is precisely a special case of the problem considered in [10]. This is also a special case of the problem we consider (1.2) (that is, if $\alpha = q = 2$, and there were no strong damping terms. The strong damping terms can always be done away with if we assume some coefficients in front of them, and then set those to be zero.). We perform numerical simulations in one and two dimensions using a stochastic spectral Galerkin method on

$$\partial_{tt}u + a\partial_t u = \Delta u + b|u|^{p-2}u + \sigma(x,t)dW, \ x \in \Omega, \ t > 0, \tag{6.1}$$

where $u = 0$ on the boundary of the domain.

For brevity, the development of the numerical scheme for a one dimensional domain $(0, L)$ is shown. The development for higher dimensions follows similarly. Define the inner product as

$$< v(x), w(x) >= \int_0^L v(x)w(x)^* dx.$$

Let $u(x,t)$ be approximated through a finite dimensional fourier expansion in the spatial coordinates

$$u(x,t) \approx u_N(x,t) = \sum_{j=1}^{N} u_j(t)h_j(x),$$

where $\{h_j(x)\}$ are orthogonal global basis functions of the form,

$$h_j(x) \quad = \quad \frac{1}{2}\frac{\sin\left(\pi(N+1)(x-x_j)/L\right)}{\sin\left(\pi(x-x_j)/L\right)},$$

for $x_j = jh$, $h = L/(N+1)$, and $u_j(t)$ are the time-dependent fourier coefficients to be determined. The global basis functions satisfy the Dirichlet boundary conditions and are known to have advantageous properties in numerical computations [14].

The stochastic term is expanded in the fourier basis,

$$\sigma(x,t)dW = \sum_{j=1}^{N} \beta_j(t) h_j(x) dW_j,$$

where dW_j is a random time-dependent variable obeying a standard Wiener process. The finite term expansions are substituted into (6.1). The inner product is taken with another basis function, $h_k(x)$. Upon integrating by parts, a system of stochastic initial value differential equations are established that determines the fourier coefficients, u_j, namely,

$$\begin{aligned}
\frac{1}{2}(\dot{v}_k + a\dot{v}_k) &= \sum_{j=1}^{N} -u_j < h_j', h_k' > +b < \left|\sum_{j=1}^{N} u_j h_j\right|^{p-2} \sum_{j=1}^{N} u_j h_j, h_k > \\
&\quad + \frac{1}{2}\beta_k dW_k,\ k = 1, \ldots, N \qquad (6.2)\\
\dot{u}_j &= v_j,\ j = 1, \ldots, N, \qquad (6.3)
\end{aligned}$$

where $\dot{} = d/dt$. Define the $2N-$dimensional vectors

$$\begin{aligned}
y &= (v_1, \ldots, v_N, u_1, \ldots, u_N)^\top, \\
y_0 &= (v_1(0), \ldots, v_N(0), u_1(0), \ldots, u_N(0))^\top, \\
d\hat{W} &= (\beta_1 dW_1, \ldots, \beta_N dW_N, 0, \ldots, 0)^\top, \\
f &= \Bigg(< \left|\sum_{j=1}^{N} u_j h_j\right|^{p-2} \sum_{j=1}^{N} u_j h_j, h_1 >, \ldots, \\
&\qquad < \left|\sum_{j=1}^{N} u_j h_j\right|^{p-2} \sum_{j=1}^{N} u_j h_j, h_N >, 0, \ldots, 0 \Bigg)^\top.
\end{aligned}$$

Let $P = (< h_j', h_k' >)$, a $N \times N$ symmetric matrix. Let I_N and I represent the $N \times N$ and $2N \times 2N$ identity matrices, respectively. Then (6.2)-(6.3) can be written in vector form,

$$\dot{y} = Cy + Mf + I_0 d\hat{W},$$

where

$$C = \begin{pmatrix} -aI & -2P \\ I & 0 \end{pmatrix},\ M = \begin{pmatrix} 2I & 0 \\ 0 & I \end{pmatrix},\ I_0 = \begin{pmatrix} I_N & 0 \\ 0 & 0 \end{pmatrix}.$$

The implicit solution to the evolution equation is

$$y(t) = \exp(Ct) \left(y_0 + \int_0^t \exp(-C\tau) M f d\tau + \int_0^t \exp(-C\tau) I_0 d\hat{W} d\langle\tau\rangle \right).$$

We employ a Padé 1-1 approximation to the exponential operator, a trapezoidal rule for the nonlinear integral, and an Euler-Maruyama method for the stochastic integral to develop the fully discretized equations to advance the solution [3, 15, 21],

$$y_{n+1} = \left(I - \frac{\Delta t}{2}C\right)^{-1}\left(I + \frac{\Delta t}{2}C\right)\left(y_n + \frac{\Delta t}{2}Mf_n\right) + \frac{\Delta t}{2}Mf_{n+1} + d\hat{W}, \quad (6.4)$$

where $t_{n+1} = t_n + \Delta t$, and y_n and f_n are approximations to $y(t_n)$ and $f(t_n)$, respectively. The first N terms of implicit term f_{n+1} is determined through an Euler prediction to u_{j+1}, that is, $u_{j+1} \approx \tilde{u}_{j+1} = u_j + \Delta t\ v_j$.

In the following examples the numerical solution is said to *blow-up* if the $\max_{\mathbf{x}\in\Omega}|u_N(\mathbf{x}, t_i)| > 10^6$ for some finite t_i. Thus we are considering blow-up in the $L^\infty(\Omega)$ norm. The numerical solution is said to *approach a steady-state solution* if $\max_{\mathbf{x}\in\Omega}|\frac{\partial}{\partial t}u_N(\mathbf{x}, t_i)| < 10^{-6}$ at some finite t_i. The computations for our experiments are implemented through a Matlab© platform on a Dell® XPS 8700, (Intel Core i7-4770, 3.9 GHz, 8M L2 cache, 1600 MHz FSB, 2 TB Hard Drive) 64-bit workstation.

Example 6.1. Consider the one dimensional domain $(0, 5)$, $p = 2$, and initial conditions $u(x, 0) = .1x(5 - x)$ and $u_t(x, 0) = 0$. The initial energy is $-.3125$. Let $h = .05$, $\Delta t = 10^{-3}$, and $a = b = 1$. Hence, for this spatial domain and grid size there are 99 basis functions in our fourier expansions. Consider $\sigma(x, t) = \exp\{(-t)\}$. Hence, the criteria of [10] are satisfied. This experiment establishes the bound for the energy functional $\Psi(t)$ defined in [10]. The energy functional is defined as in (1.4), that is,

$$\Psi(t) = \frac{1}{2}\left(\|u_t\|_2^2 + \|u_x\|_2^2 - \|u\|_2^2\right)$$

The functional is calculated at each iteration. The semi-log plot of the functional, as shown in Figure 1, clearly shows it is bounded by an exponential function in time.

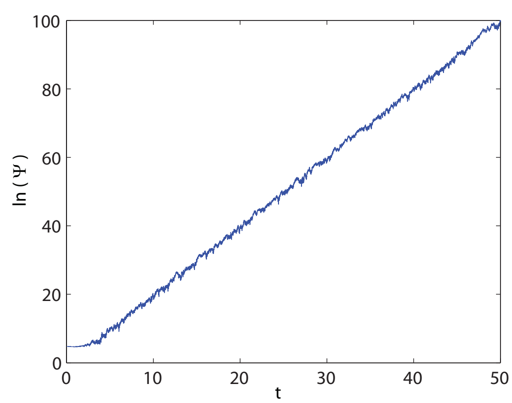

FIGURE 1. The natural logarithm of the functional $\Psi(t)$ is shown versus time. The noise is self-evident in the curve, however, the linear trend suggests that the functional is bounded by an exponential function.

Example 6.2. Consider a one dimensional domain $(0, 5)$, $p = 3$, and initial conditions $u(x, 0) = .1x(5 - x)$ and $u_t(x, 0) = 0$. Let $a = 1$. Again, we use a grid size of $h = .05$ and 99 basis functions.

In the deterministic case, $\sigma = 0$, blow-up will occur if $b > 0.035645 = b^*$. Consider the case $b = 0.035654 > b^*$. Hence, blow-up is expected and observed in numerical solution. Now, consider $\sigma(x, t) = \rho \exp\{(-\kappa t)\}$. It is observed for particular values of ρ and κ that on average (greater than 50% of the 1,000 simulations) the numerical solution *approaches a steady-state solution* and blow-up does not occur. The solution $u(x, t)$ is shown in Figure 2(a). The maximum location throughout the simulation is 2.5, hence a plot of $u(2.5, t)$ is shown in Figure 2(b). Note in this case we see blow-up even if the initial energy is positive, which is legitimate via the results of [37].

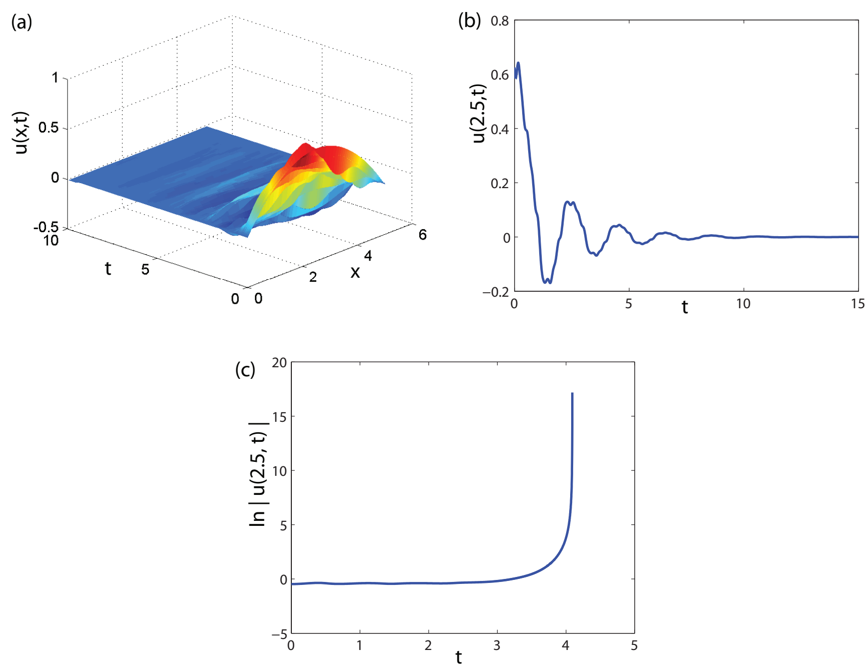

FIGURE 2. The solution in the (a) (x-t) plane and at $x = 2.5$ for the (b) stochastic ($\rho = 10, \kappa = 10$). A semi-log plot of the solution for the deterministic case is shown in (c). The deterministic solution will blow-up, however, on average, the stochastic term can prevent this from occurring and the solution will converge to a steady state.

This experimental observation is in support of the analysis of the previous sections, that is, a stochastic term may act as an additional damping and prevent blow-up. Note that this has been reported in [10, 27], but not numerically verified, to the best of our knowledge.

Remark 6.3. The experiments suggest that for a fixed ρ there exists a critical decay rate κ. For instance, if $\rho = 10$ and for values of $\kappa > \kappa^* = 7.726563$, then, on average, the numerical solution approaches a steady state. For values of $\kappa < \kappa^*$, then, on average, the numerical solution blows up.

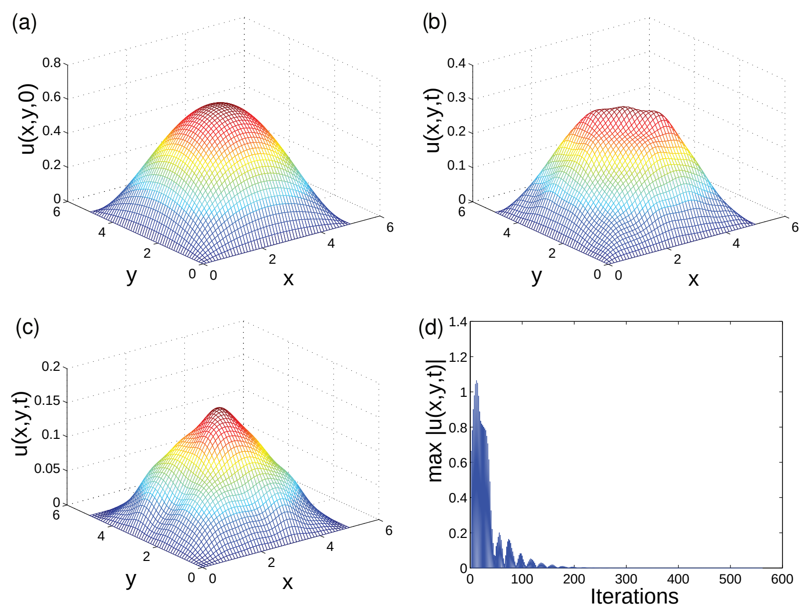

FIGURE 3. Plots of the numerical solution $u(x, y, t)$ (a) initially, (b) when the $\max_{x,y} |u(x, y, t)| = .5 \max_{x,y} |u(x, y, t)|$, (c) when the $\max_{x,y} |u(x, y, t)| = .25 \max_{x,y} |u(x, y, t)|$, and (d) $\max_{x,y\in\Omega} |u(x, y, t)|$ at t_i for $\mathrm{mod}(i, 100) = 0$. The solution converges to a steady state as a result of the introduction of the stochastic term of the form $\exp(-t)$. Fifty one equally spaced nodes are used in the x and y directions to create the two-dimensional basis functions. $\Delta t = 10^{-3}$, 49^2 basis functions, $a = 1$, and $b = 0.0684675$.

Example 6.4. Experiments were completed for a square domain $(0, 5) \times (0, 5)$. Let $u(x, y, 0) = 0.017xy(x - 5)(y - 5)$ and $u_t(x, y, 0) = 0$. The initial energy is on the same order as in the previous example in one-dimension, that is, $-.1325$. The grid size is .1 in both directions, hence we have 49^2 basis functions. Let $p = 3$ and $b = 1$. Blow-up is observed in the deterministic problem provided the nonlinear coefficient $b = b^* > 0.068467$. Let $b = 0.0684675 > b^*$, a value just above threshold, and let $\sigma(x,t) = \exp(-t)$. On average, over 50% of the $1,000$ simulations, the numerical solution was found to approach a steady-state rather than blow-up as in the deterministic case. This empirical evidence affirms the theoretical results of [10], and also a very special case of our global existence results (that is when $\alpha = p = q = 2$ and there is no strong damping). Select snapshots of the numerical solution in the stochastic case for particular iterates are shown in Figure 6.3(a)-(c). The maximum absolute value of the numerical solution at t_i for $\mathrm{mod}(i, 100) = 0$ is shown in Figure 6.3(d).

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