1971

A Variable Coeficient of Restitution Experiment on a Linear Air Track

R. Gruebel
Stephen F Austin State University

James C. Dennis
Stephen F Austin State University

L. Choate
Stephen F Austin State University

A Variable Coeficient of Restitution Experiment on a Linear Air Track

R. Gruebel, J. Dennis, and L. Choate

Citation: American Journal of Physics **39**, 447 (1971); doi: 10.1119/1.1986174
View online: https://doi.org/10.1119/1.1986174
View Table of Contents: https://aapt.scitation.org/toc/ajp/39/4
Published by the American Association of Physics Teachers

A Variable Coeficient of Restitution Experiment on a Linear Air Track

R. GRUEBEL
J. DENNIS
L. CHOATE
Stephen F. Austin State University
Nacogdoches, Texas 75961
(Received 13 January 1970; revision 6 July 1970)

A system consisting of two pendula attached to an air cart is mathematically analyzed, and the coefficient of restitution is shown to pass through a deep minimum. The solution to the small angle equation of motion is transcendental and provides an exercise in graphical methods for the beginning mechanics student.

A recent article in another journal[1] described a simple but interesting lossy collision experiment on a linear air track. The experimental–theoretical correlation needed for a good undergraduate lab experiment was missing. In this article a simple system is discussed which has been mathematically analyzed and has attracted a great deal of attention from advanced undergraduates. This exercise is illustrative of a number of real research problems met in the physics laboratory where a straightforward mathematical treatment of complex physical phenomena yields a simple result which can readily be compared to physical observations. The student is afforded the opportunity to use graphical methods for solving transcendental equations such as are encountered in the analysis of boundary value problems in quantum mechanics and classical wave phenomena.[2] The system referred to here consists of two pendula attached to an air cart. As in the previously mentioned arrangement, the coefficient of restitution passes through a deep minimum.

The coefficient of restitution, the ratio of the velocity after collision to the velocity before, is measured as a function of pendula mass (m_1) when the cart (mass m_2) plus pendula undergo a collision with an air track bumper spring. The spring constant must be small enough to allow the completion of a quarter-cycle of pendulum swing before the cart and spring separate. Therefore the restriction $2T_2 > T_1 > T_2$ is placed on the system, i.e., the pendulum should be in the second quarter cycle of its motion during separation.

Consider the motion of the arrangement when the pendulum is set into motion relative to the cart following the collision: The collision is inelastic with respect to the cart velocity to a degree dependent on the momentum of the pendula at the instant of separation. When $m_1 \ll m_2$ and $m_1 \sim m_2$, the pendulum has very little effect on the cart rebound velocity. The latter region is characterized by small in-phase oscillations of the pendula. The out-of-phase oscillations in the intermediate region produce a minimum in the cart rebound velocity and hence in e, as seen in Fig. 2.

DERIVATION

The Lagrangian of the system is

$$L = \tfrac{1}{2}(m_1+m_2)\dot{x}^2 + m_1 l\dot{x}\dot{\theta}\cos\theta + \tfrac{1}{2}m_1 l^2\dot{\theta}^2 - \tfrac{1}{2}kx^2 - m_1 gl(1-\cos\theta), \quad (1)$$

and the small angle equations of motion are

$$\ddot{x} + l\ddot{\theta} + g\theta = 0, \quad (2)$$

$$(m_1+m_2)\ddot{x} + m_1 l\ddot{\theta} + kx = 0, \quad (3)$$

where m_2 is the cart mass, l is the pendulum length, and k the spring constant [see Fig. 1]. After rearranging terms, Eqs. (2) and (3) become

$$\ddot{x} + \omega_2^2 x = m_1 g\theta/m_2, \quad (4)$$

$$\ddot{\theta} + \omega_1^2\theta = \omega_2^2 x/l, \quad (5)$$

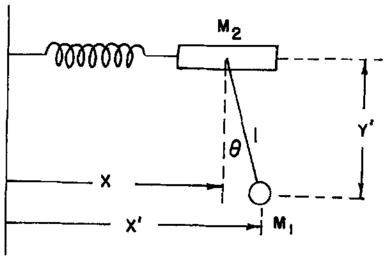

FIG. 1. Cart–pendulum–bumper spring system. The actual experimental arrangement had two pendula for reasons of convenience.

where ω_1 and ω_2 are defined by

$$\omega_1^2=(m_1+m_2)g/m_2l, \qquad \omega_2^2=k/m_2. \tag{6}$$

Assuming a solution of the form

$$x=A\sin(\alpha t), \qquad \theta=B\sin(\alpha t) \tag{7}$$

leads to

$$\begin{pmatrix} -\omega_2^2/l & \omega_1^2-\alpha^2 \\ \omega_2^2-\alpha^2 & -m_1g/m_2 \end{pmatrix}\begin{pmatrix} A_n \\ B_n \end{pmatrix}=0. \tag{8}$$

The solutions are

$$\alpha_{1,2}^2=\tfrac{1}{2}(\omega_1^2+\omega_2^2) \mp\tfrac{1}{2}[(\omega_1^2-\omega_2^2)^2+4m_1\omega_2^2g/m_2l]^{1/2}, \tag{9}$$

where α_1 and α_2 represent the modulated frequencies of the pendula and spring, respectively. In terms of the α_i, the general solutions become

$$x=A_1\sin(\alpha_1t)+A_2\sin(\alpha_2t),$$
$$\theta=B_1\sin(\alpha_1t)+B_2\sin(\alpha_2t). \tag{10}$$

At $t=0$, $\dot{x}=v_0$ and $\dot{\theta}=v_0/l$, where v_0 is the velocity of the cart–pendulum system just before collision with the spring. Therefore at $t=0$

$$v_0=A_1\alpha_1+A_2\alpha_2, \qquad v_0/l=B_1\alpha_1+B_2\alpha_2. \tag{11}$$

Solving from A_2 and B_2 in terms of A_1 and B_1 and substituting into Eq. (10), the solutions are

$$x=A_1[\sin(\alpha_1t)-(\alpha_1/\alpha_2)\sin(\alpha_2t)] +(v_0/\alpha_2)\sin(\alpha_2t), \tag{12}$$

$$\theta=B_1[\sin(\alpha_1t)-(\alpha_1/\alpha_2)\sin(\alpha_2t)] +(v_0/\alpha_2l)\sin(\alpha_2t). \tag{13}$$

From Eq. (8) we have

$$A_1=l(\omega_1^2-\alpha_1^2)B_1/\omega_2^2. \tag{14}$$

At $\theta=\theta_{\max}$, $\dot{\theta}=0$ and

$$B_1=\frac{v_0\cos(\alpha_2t')}{\{\alpha_1l[\cos(\alpha_2t')-\cos(\alpha_1t')]\}}, \tag{15}$$

where t', the time when $\dot{\theta}=0$, is found by substituting Eq. (15) into Eq. (13) and solving (13) for its maximum value. The latter is maximized at $t'=\pi/2\omega_1$. Thus

$$B_1=v_0/\alpha_1l, \tag{16}$$

and

$$\frac{x}{v_0}=\frac{(\omega_1^2-\alpha_1^2)}{\alpha_1\omega_2^2}\left[\sin(\alpha_1t)-\frac{\alpha_1}{\alpha_2}\sin(\alpha_2t)\right] +\frac{\sin(\alpha_2t)}{\alpha_2}. \tag{17}$$

To solve for the coefficient of restitution the time t'' when $x=0$ on the rebound must be determined, or from Eq. (17),

$$\sin(\alpha_1t'')=(\alpha_1/\alpha_2) \times[1-\omega_2^2/(\omega_1^2-\alpha_1^2)]\sin(\alpha_2t''). \tag{18}$$

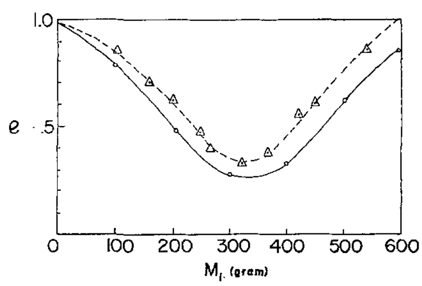

FIG. 2. Coefficient of restitution ($\dot{x}/v_0$ at $x=0$ for cart) as a function of pendula mass for $\omega_2^2=100$, $m_2=550$ g, and $l=20$ cm. The solid line represents theoretical values. Note that at $m_1=600$ g, $e>1$.

The above relationship is to be solved graphically for t''. Finally, the coefficient of restitution, $\dot{x}/v_0$, is

$$e = e_0[(1-R)\cos(\alpha_2 t'') + R\cos(\alpha_1 t'')], \quad (19)$$

where e_0 is the coefficient of restitution with $m_1 = 0$, and

$$R = (\omega_1^2 - \alpha_1^2)/\omega_2^2. \quad (20)$$

DISCUSSION

A comparison of Eq. (19) with experimental results is shown in Fig. 2. The natural spring frequency was measured by kinetic-potential conversion in a collision with $m_1 = 0$. Agreement lessens as m_1 increases due to increasing nonlinearity of the spring.

[1] J. L. Stull, Phys. Teacher **7**, 225 (1969).
[2] L. M. Clendenning, Amer. J. Phys. **36**, 879 (1968).