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EXTENDING THE SUPPORT THEOREM TO INFINITE DIMENSIONS

JEREMY J. BECNEL

ABSTRACT. The Radon transform is one of the most useful and applicable tools in functional analysis. First constructed by John Radon in 1917 [9] it has now been adapted to several settings. One of the principle theorems involving the Radon transform is the Support Theorem. In this paper, we discuss how the Radon transform can be constructed in the white noise setting. We also develop a Support Theorem in this setting.

Key Words. Radon Transform, Support Theorem, Gaussian Measure, Infinite Dimensional Distribution Theory, White Noise Analysis

1. INTRODUCTION

The Radon transform [9] associates to a function f on the finite-dimensional space $\mathbb{R}^n$ a function R_f on the set of all hyperplanes in $\mathbb{R}^n$ whose value on any hyperplane P is the integral of f over P:

$$(1.1) \qquad Rf(P) = \int_P f(x)\, dx,$$

the integration here being with respect to Lebesgue measure on P. This transform does not generalize directly to infinite dimensions because there is no useful notion of Lebesgue measure in infinite dimensions. However, there is a well-developed theory of Gaussian measures in infinite dimensions and so it is natural to extend the Radon transform to infinite dimensions using Gaussian measure:

$$(1.2) \qquad Gf(P) = \int f\, d\mu_P,$$

where μ_P is the Gaussian measure on any infinite dimensional hyperplane P in a Hilbert space H_0. A version of this transform was developed in [7] but we shall present a another account below. A central feature of the classical Radon transform R is the Support Theorem (see, for instance, Helgason [4]):

Theorem 1.1 (Support Theorem)**.** *If f is a rapidly decreasing continuous function for which $R_f(P)$ is 0 on every hyperplane P disjoint from some compact convex set K then $f(x) = 0$ for $x \notin K$.*

or more appropriately for the Gaussian measure we have

Theorem 1.2 (Support Theorem—Gaussian)**.** *If f is a exponentially bounded continuous function for which $G_f(P)$ is 0 on every hyperplane P disjoint from some compact convex set K then $f(x) = 0$ for $x \notin K$.*

In this paper we prove the infinite-dimensional version of this Support Theorem.

2. White Noise Setup

We begin by describing the setting under which White Noise Analysis takes place. Because of the absence of the Lebesgue measure in infinite dimensions we begin by constructing a Gaussian measure. From here we can develop appropriate sets of tests functions and distributions.

We work with a real separable Hilbert space H_0, and a positive Hilbert-Schmidt operator A on H_0 such that there is orthonormal basis $\{e_n\}_{n=1}^\infty$ of eigenvectors of A and eigenvalues $\{\lambda_n\}_{n=1}^\infty$ satisfying

(1) $Ae_n = \lambda_n e_n$
(2) $1 < \lambda_1 < \lambda_2 < \dots$
(3) $\sum_{n=1}^\infty \lambda_n^{-2} < \infty$

The example to keep in mind is

$$
\begin{aligned}
H_0 &= L^2(\mathbb{R}) \\
e_n &= \phi_n \\
A &= -\frac{d^2}{dx^2} + \frac{x^2}{4} + \frac{1}{2} \quad \text{with eigenvalues } \lambda_n = (n+1).
\end{aligned}
$$

We have the coordinate map

$$ J : H_0 \mapsto \mathbb{R}^W : f \mapsto \left(\langle f, e_n\rangle\right)_{n\in W} $$

where we use the notation $W = \{1, 2, ...\}$. Let

$$ (2.1) \qquad F_0 = J(H_0) = \{(x_n)_{n\in W} : \sum_{n\in W} x_n^2 < \infty\} $$

Now, for each $p \in W$, let

$$ (2.2) \qquad F_p = \{(x_n)_{n\in W} : \sum_{n\in W} \lambda_n^{2p} x_n^2 < \infty\} $$

On F_p we have the inner-product $\langle \cdot, \cdot \rangle_p$ given by

$$ \langle a, b\rangle_p = \sum_{n\in W} \lambda_n^{2p} a_n b_n $$

This makes F_p a real Hilbert space, unitarily isomorphic to $L^2(W, \mu_p)$ where μ_p is the measure on W specified by $\mu_p(\{n\}) = \lambda_n^{2p}$. Moreover, we have

$$ (2.3) \qquad F \overset{\text{def}}{=} \bigcap_{p\in W} F_p \subset \cdots F_2 \subset F_1 \subset F_0 = L^2(W, \mu_0) $$

and each inclusion $F_{p+1} \to F_p$ is Hilbert-Schmidt.

Now we pull all this back to H_0. First set

$$ (2.4) \qquad H_p = J^{-1}(F_p) = \{x \in H_0 : \sum_{n\in W} \lambda_n^{2p} |\langle x, e_n\rangle|^2 < \infty\} $$

It is readily checked that $H_p = \{x \in H_0 \,;\, |x|_p < \infty\}$ where $|x|_p = |Ax|_0$ and also $H_p = A^{-p}(H_0)$. On H_p we have the pull back inner-product $\langle \cdot, \cdot\rangle_p$, which works out to

$$ (2.5) \qquad \langle f, g\rangle_p = \langle A^p f, A^p g\rangle $$

Then we have the chain

$$(2.6) \qquad \mathcal{H} \stackrel{\text{def}}{=} \bigcap_{p \in W} H_p \subset \cdots H_2 \subset H_1 \subset H_0,$$

with each inclusion $H_{p+1} \to H_p$ being Hilbert-Schmidt.

Equip $\mathcal{H}$ with the topology generated by the norms $|\cdot|_p$ (i.e. the smallest topology making all inclusions $\mathcal{H} \to H_p$ continuous). Then $\mathcal{H}$ is, more or less by definition, a nuclear space. The vectors e_n all lie in $\mathcal{H}$ and the set of all rational-linear combinations of these vectors produces a countable dense subspace of $\mathcal{H}$.

Consider a linear functional on $\mathcal{H}$ which is continuous. Then it must be continuous with respect to some norm $|\cdot|_p$. Thus the topological dual $\mathcal{H}'$ is the union of the duals H_p'. In fact, we have:

$$(2.7) \qquad \mathcal{H}' = \bigcup_{p \in W} H_p' \supset \cdots H_2' \supset H_1' \supset H_0' \simeq H_0,$$

where in the last step we used the usual Hilbert space isomorphism between H_0 and its dual H_0'.

Going over to the sequence space, H_p' corresponds to

$$(2.8) \qquad F_{-p} \stackrel{\text{def}}{=} \{(x_n)_{n \in W} : \sum_{n \in W} \lambda_n^{-2p} x_n^2 < \infty\}$$

The element $y \in F_{-p}$ corresponds to the linear functional on F_p given by

$$x \mapsto \sum_{n \in W} x_n y_n$$

which, by Cauchy-Schwarz, is well-defined and does define an element of the dual F_p' with norm equals to $|y|_{-p}$.

2.1. **Gaussian measure in infinite dimensions.** Consider now the product space $\mathbb{R}^W$, along with the coordinate projection maps

$$\hat{X}_j : \mathbb{R}^W \to \mathbb{R} : x \mapsto x_j$$

for each $j \in W$. Equip $\mathbb{R}^W$ with the product σ–algebra, i.e. the smallest σ–algebra with respect to which each projection map $\hat{X}_j$ is measurable. Kolmogorov's theorem on infinite products of probability measures provides a probability measure ν on the product σ–algebra such that each function $\hat{X}_j$, viewed as a random variable, has standard Gaussian distribution. Thus,

$$\int_{\mathbb{R}^W} e^{it\hat{X}_j} \, d\nu = e^{-t^2/2}$$

for $t \in \mathbb{R}$, and every $j \in W$. The measure ν is the product of the standard Gaussian measure $e^{-x^2/2}(2\pi)^{-1/2}dx$ on each component $\mathbb{R}$ of the product space $\mathbb{R}^W$.

Since, for any $p \geq 1$, we have

$$\int_{\mathbb{R}^W} \sum_{j \in W} \lambda_j^{-2p} x_j^2 \, d\nu(x) = \sum_{j \in W} \lambda_j^{-2p} < \infty,$$

it follows that $\nu(F_{-p}) = 1$ for all $p \geq 1$. Thus $\nu(F') = 1$.

We can, therefore, transfer the measure ν back to $\mathcal{H}'$, obtaining a probability measure μ on the σ–algebra of subsets of $\mathcal{H}'$ generated by the maps

$$\hat{e}_j : \mathcal{H}' \to \mathbb{R} : f \mapsto f(e_j),$$

where $\{e_j\}_{j \in W}$ is the orthonormal basis of H_0 we started with (note that each e_j lies in $\mathcal{H} = \bigcap_{p \geq 0} H_p$). This is clearly the σ–algebra generated by the weak topology on $\mathcal{H}'$ (which happens to be equal also to the σ–algebras generated by the strong/inductive-limit topology [1]).

Specialized to the example $H_0 = L^2(\mathbb{R})$, and $A = -\frac{d^2}{dx^2} + \frac{x^2}{4} + \frac{1}{2}$, we have the standard Gaussian measure on the distribution space $\mathcal{S}'(\mathbb{R})$.

The above discussion gives a simple direct description of the measure μ. Its existence is also obtainable by applying the Minlos theorem:

Theorem 2.1 (Minlos theorem). *A complex value function ϕ on a nuclear space $\mathcal{H}$ is the characteristic function of a unique probability measure ν on $\mathcal{H}'$, i.e.,*

$$\phi(v) = \int_{E'} e^{i \langle x, y \rangle} \, d\nu(x) = \int_{E'} e^{i \hat{y}(x)} \, d\nu(x), \quad y \in \mathcal{H}$$

if and only if $\phi(0) = 1$, ϕ is continuous, and ϕ is positive definite.

For a proof of the Minlos theorem refer to [3]. Applying the Minlos theorem to the characteristic function $\phi(y) = e^{-\frac{1}{2}|y|_0^2}$ gives us the standard Gaussian measure μ we just constructed.

There is also the useful standard setting of Abstract Wiener Spaces for Gaussian measures introduced by L. Gross (see the account in Kuo [5]).

To summarize, we can state the starting point of much of infinite-dimensional distribution theory (white noise analysis): Given a real, separable Hilbert space H_0 and a positive Hilbert-Schmidt operator A on H_0, we have constructed a nuclear space $\mathcal{H}$ and a unique probability measure μ on the Borel σ–algebra of the dual $\mathcal{H}'$ such that there is a linear map

$$H_0 \to L^2(\mathcal{H}', \mu) : x \mapsto \hat{x},$$

satisfying

$$\int_{\mathcal{H}'} e^{it\hat{x}} \, d\mu = e^{-t^2 |x|_0^2 / 2},$$

for every real t and $x \in H_0$. This Gaussian measure μ is often called the *white noise measure* and forms the background measure for white-noise distribution theory.

3. White Noise Distribution Theory

We can now develop the ideas of the preceding section further to construct a space of test functions over the dual space $\mathcal{H}'$, where $\mathcal{H}$ is the nuclear space related to a real separable Hilbert space H_0 as in the discussion in Section 2. We will use the notation, and in particular the spaces H_p, from Section 2.

The symmetric Fock space $\mathcal{F}_s(V)$ over a Hilbert space V is the subspace of symmetric tensors in the completion of the tensor algebra $T(V)$ under the inner–product given by

$$(3.1) \qquad \langle a, b \rangle_{T(V)} = \sum_{n=0}^{\infty} n! \langle a_n, b_n \rangle_{V^{\otimes n}},$$

where $a = \{a_n\}_{n \geq 0}, b = \{b_n\}_{n \geq 0}$ are elements of $T(V)$ with a_n, b_n in the tensor power $V^{\otimes n}$. Then we have

$$(3.2) \qquad \mathcal{F}_s(\mathcal{H}) \overset{\text{def}}{=} \bigcap_{p \geq 0} \mathcal{F}_s(H_p) \subset \cdots \subset \mathcal{F}_s(H_2) \subset \mathcal{F}_s(H_1) \subset \mathcal{F}_s(H_0).$$

Thus, the pair $\mathcal{H} \subset H_0$ give rise to a corresponding pair by taking symmetric Fock spaces:

$$(3.3) \qquad \mathcal{F}_s(\mathcal{H}) \subset \mathcal{F}_s(H_0).$$

3.1. **Wiener–Itô Isomorphism.** In infinite dimensions the role of Lebesgue measure is played by Gaussian measure μ. There is a standard unitary isomorphism, the *Wiener-Itô isomorphism* or wave-particle duality map, which identifies the complexified Fock space $\mathcal{F}_s(H_0)_c$ with $L^2(\mathcal{H}', \mu)$. This is uniquely specified by

$$(3.4) \qquad I : \mathcal{F}_s(H_0)_c \to L^2(\mathcal{H}', \mu) : \mathrm{Exp}(x) \mapsto e^{\hat{x} - \frac{1}{2}|x|_0^2}$$

where $x \in \mathcal{H}$ and

$$\mathrm{Exp}(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^{\otimes n}.$$

Indeed, it is readily checked that I preserves inner–products (the inner–product is as described in (3.1)). Using I, for each $\mathcal{F}_s(H_p)$ with $p \geq 0$, we have the corresponding space $[H]_p \subset L^2(\mathcal{H}', \mu)$ with the norm $\| \cdot \|_p$ induced by the norm on the space $\mathcal{F}_s(H_p)_c$. The chain of spaces (3.2) can be transferred into a chain of function spaces:

$$(3.5) \qquad [\mathcal{H}] = \bigcap_{p \geq 0} [H]_p \subset \cdots \subset [H]_2 \subset [H]_1 \subset [H]_0 = L^2(\mathcal{H}', \mu).$$

Observe that $[\mathcal{H}]$ is a nuclear space with topology induced by the norms $\{\| \cdot \|_p \, ; \, p = 0, 1, 2, \dots\}$. Thus, starting with the pair $\mathcal{H} \subset H_0$ one obtains a corresponding pair $[\mathcal{H}] \subset L^2(\mathcal{H}', \mu)$.

As before, the identification of H_0' with H_0 leads to a complete chain

$$(3.6) \qquad \mathcal{H} = \bigcap_{p \geq 0} H_p \subset \cdots \subset H_1 \subset H_0 \simeq H_{-0} \subset H_{-1} \subset \cdots \subset \bigcup_{p \geq 0} H_{-p} = \mathcal{H}'.$$

In the same way we have a chain for the 'second quantized' spaces $\mathcal{F}_s(H_q)_c \simeq [H]_q$. The unitary isomorphism I extends to unitary isomorphisms

$$(3.7) \qquad I : \mathcal{F}_s(H_{-p})_c \to [H]_{-p} \overset{\mathrm{def}}{=} [H]_p' \subset [\mathcal{H}]',$$

for all $p \geq 0$. In more detail, for $a \in \mathcal{F}_s(H_{-p})_c$ the distribution $I(a)$ is specified by

$$(3.8) \qquad \langle I(a), \phi \rangle = \langle a, I^{-1}(\phi) \rangle,$$

for all $\phi \in [\mathcal{H}]$. On the right side here we have the pairing of $\mathcal{F}_s(H_{-p})_c$ and $\mathcal{F}_s(H_p)_c$ induced by the duality pairing of H_{-p} and H_p; in particular, the pairings above are complex bilinear (not sesquilinear).

3.2. **Properties of test functions.** The following theorem summarizes the properties of $[\mathcal{H}]$ which are commonly used. The results here are standard (see, for instance, the monograph [6] by Kuo), and we compile them here for ease of reference.

Theorem 3.1. *Every function in $[\mathcal{H}]$ is μ-almost-everywhere equal to a unique continuous function on $\mathcal{H}'$. Moreover, working with these continuous versions,*

(1) *$[\mathcal{H}]$ is an algebra under pointwise operations;*
(2) *pointwise addition and multiplication are continuous operations $[\mathcal{H}] \times [\mathcal{H}] \to [\mathcal{H}]$;*

(3) *for any $\phi \in \mathcal{H}'$, the evaluation map*

$$\delta_\phi : [\mathcal{H}] \to \mathbb{R} : F \mapsto F(\phi)$$

is continuous;

(4) *the exponentials $e^{i\hat{x} - \frac{1}{2}|x|_0^2}$, with x running over $\mathcal{H}$, span a dense subspace of $[\mathcal{H}]$.*

A complete characterization of the space $[\mathcal{H}]$ was obtained by Y. J. Lee (see the account in Kuo [6, page 89]). The test functions in $[\mathcal{H}]$ also have a useful growth condition which can be imposed on them.

Theorem 3.2. *Let $\phi \in [\mathcal{H}]$. The ϕ satisfies the following growth condition for any $p \geq 0$,*

$$|\phi(x)| \leq K_p \exp\left[\frac{1}{2}|x|_{-p}^2\right], \qquad x \in H_p'$$

where K_p is a constant depending on the choice of p.

A proof of this exponential bound can be found in [6] (Theorem 6.8 page 55).

3.3. **The Segal–Bargmann Transform.** An important tool for studying test functions and distributions in the white noise setting is the Segal–Bargmann transform. The Segal–Bargmann transform takes a function $F \in L^2(\mathcal{H}', \mu)$ to the function SF on the complexified space $\mathcal{H}_c$ given by

$$(3.9) \qquad SF(z) = \int_{H'} e^{\tilde{z} - z^2/2} F \, d\mu, \qquad z \in \mathcal{H}_c$$

with notation as follows: if $z = a + ib$, with $a, b \in \mathcal{H}$ then

$$(3.10) \qquad \tilde{z}(x) \overset{\text{def}}{=} zx \overset{\text{def}}{=} \langle x, a \rangle + i \langle x, b \rangle, \qquad \text{for } x \in \mathcal{H}'$$

and $z^2 = zz$, where the product zu is specified through

$$(3.11) \qquad zu \overset{\text{def}}{=} \langle a, s \rangle - \langle b, t \rangle + i \left(\langle a, t \rangle + \langle b, s \rangle \right)$$

if $z = a + ib$ and $u = s + it$, where $a, b, s, t \in \mathcal{H}$.

Let μ_c be the Gaussian measure $\mathcal{H}_c'$ specified by the requirement that

$$(3.12) \qquad \int_{\mathcal{H}_c'} e^{ax + by} \, d\mu_c(x + iy) = e^{(a^2 + b^2)/4}$$

for every $a, b \in \mathcal{H}$. For convenience, let us introduce the renormalized exponential function $c_w = e^{\tilde{w} - w^2/2} \in L^2(\mathcal{H}', \mu)$ for all $w \in \mathcal{H}_c$. It is readily checked that for any $w \in \mathcal{H}_c$

$$(3.13) \qquad [Sc_w](z) = e^{wz}, \qquad \text{for all } z \in \mathcal{H}_c.$$

Thus we may take Sc_w as a function on $\mathcal{H}_c'$ given by $Sc_w = e^{\tilde{w}}$ where now $\tilde{w}$ is a function on $\mathcal{H}_c'$ in the natural way. Then $Sc_w \in L^2(\mathcal{H}_c', \mu_c)$ and one has

$$\langle Sc_w, Sc_u \rangle_{L^2(\mu_c)} = \langle c_w, c_u \rangle_{L^2(\mu_c)} = e^{w\overline{u}}.$$

This shows that S provides an isometry from the linear span of the exponentials c_w in $L^2(\mathcal{H}', \mu)$ onto the linear span of the complex exponentials $e^{\tilde{w}}$ in $L^2(\mathcal{H}_c', \mu_c)$. Passing to the closure one obtains the **Segal–Bargmann** unitary isomorphism

$$S : L^2(\mathcal{H}, \mu) \to Hol^2(\mathcal{H}_c', \mu_c)$$

where $Hol^2(\mathcal{H}'_c, \mu_c)$ is the closed linear span of the complex exponential functions $e^{\tilde{w}}$ in $L^2(\mathcal{H}'_c, \mu_c)$.

An explicit expression for $SF(z)$ is suggested by (3.9). For any $\phi \in [\mathcal{H}]$ and $z \in \mathcal{H}'_c$, we have

(3.14)
$$(S\phi)(z) = \langle I(\mathrm{Exp}(z)), \phi \rangle$$

where the right side is the evaluation of the distribution $I(\mathrm{Exp}(z))$ on the test function ϕ. Indeed it may be readily checked that if $S\phi(z)$ is defined in this way then $[Sc_w](z) = e^{wz}$.

In view of (3.14), it natural to extend the Segal-Bargmann transform to distributions: for $\Phi \in [\mathcal{H}]'$, define $S\Phi$ to be the function on $\mathcal{H}_c$ given by

(3.15)
$$S\Phi(z) \stackrel{\text{def}}{=} \langle \Phi, I(\mathrm{Exp}(z)) \rangle, \qquad z \in \mathcal{H}_c$$

One of the many applications of the the S–transform includes its usefulness in characterizing generalized functions in $[\mathcal{H}]'$.

Theorem 3.3 (Potthoff–Streit). *Suppose a function F on $\mathcal{H}_c$ satisfies:*

(1) *For any $z, w \in \mathcal{H}_c$, the function $F(\alpha z + w)$ is an entire function of $\alpha \in \mathbb{C}$.*
(2) *There exists nonnegative constants A, p, and C such that*

$$|F(z)| \le Ce^{A|z|_p^2} \qquad \text{for all } z \in \mathcal{H}_c.$$

Then there is a unique generalized function $\Phi \in [\mathcal{H}]'$ such that $F = S\Phi$. Conversely, given such a $\Phi \in [\mathcal{H}]'$, then $S\Phi$ satisfies (1) and (2) above.

For a proof see Theorem 8.2 in Kuo's book [6] on page 79.

The S-transform can also aid us in determining convergence in $[\mathcal{H}]'$.

Theorem 3.4. *Let $\Phi_n \in [\mathcal{H}]'$ and $F_n = S\Phi_n$. Then Φ_n converges strongly in $[\mathcal{H}]'$ if and only if the following conditions are satisfied:*

(1) $\lim_{n\to\infty} F_n(z)$ *exists for all $z \in \mathcal{H}_c$.*
(2) *There exists nonnegative constants A, p, and C such that*

$$|F_n(z)| \le Ce^{A|z|_p^2}, \text{ for all } n \in N, z \in \mathcal{H}_c.$$

For a proof see Kuo's book [6] (Page 86, Theorem 8.6).

3.4. **Translation of the Gaussian Measure.** The Gaussian measure μ on $\mathcal{H}'$ and its translation $\mu(\cdot - \xi)$ are related via the S–transform when $\xi \in \mathcal{H}$ [8]. Observe the following:

Proposition 3.5. *The Gaussian Measure μ is quasi-invariant under the translation by any $\xi \in \mathcal{H}$ and the Radon-Nikodym derivative is given by*

$$\frac{d\mu(\cdot - \xi)}{d\mu} = e^{\langle \cdot, \xi \rangle - \frac{1}{2}\langle \xi, \xi \rangle}.$$

Proof. Suppose $x \in \mathcal{H}$ and consider the measure given by

$$\lambda(A) = \int_A e^{\langle x, \xi \rangle - \frac{1}{2}\langle \xi, \xi \rangle / 2} \, d\mu(x).$$

We compute the characteristic equation

$$\hat{\lambda}(y) = \int_{\mathcal{H}'} e^{i\langle x, y \rangle} e^{\langle x, \xi \rangle - \frac{1}{2}\langle \xi, \xi \rangle} \, d\mu(x).$$

Then using the characteristic equation for the Gaussian measure we have that the above gives us

$$\hat{\lambda}(y) = e^{\frac{1}{2}\langle \xi + iy, \xi + iy \rangle - \frac{1}{2}\langle \xi, \xi \rangle} = e^{i\langle \xi, y \rangle - \langle y, y \rangle / 2}.$$

On the other hand, since

$$\int_{\mathcal{H}'} e^{i\langle x, y \rangle}\, d\mu(x - \xi) = \int_{\mathcal{H}'} e^{i\langle x + \xi, y \rangle}\, d\mu(x) = e^{i\langle \xi, y \rangle - \langle y, y \rangle / 2}$$

we have that

$$\hat{\lambda}(y) = \int_{\mathcal{H}'} e^{i\langle x, y \rangle}\, d\mu(x - \xi)$$

and hence $\frac{d\mu(\cdot - \xi)}{d\mu} = e^{\langle \cdot, \xi \rangle - \frac{1}{2}\langle \xi, \xi \rangle}$. $\hfill\square$

3.5. **Translation Operator.** An important operator acting on the space of test functions is the translation operator T_y with $y \in \mathcal{H}'$.

Definition 3.6. For any $y \in \mathcal{H}'$ the translation operator T_y on $[\mathcal{H}]$ is defined by

$$T_y\phi(x) = \phi(x + y).$$

Since the Gaussian measure is not translation invariant this operator is more intricate than it first appears. The properties of this operator are summarized in the following theorem.

Theorem 3.7. *For any $y \in \mathcal{H}'$, the translation operator T_y is continuous from $[\mathcal{H}]$ into itself. Moreover, if $y \in H'_p$ and $q > p$ satisfies $\lambda_1^{2(q-p)} > 2$ then for all $\phi \in [\mathcal{H}]$,*

$$\|T_y\phi\|_p \le \|\phi\|_q (1 - \tfrac{2}{\lambda_1^{2(q-p)}}) \exp\left[\tfrac{1}{\lambda_1^{2(q-p)}} |y|^2_{-p}\right].$$

A proof of this can be found in the book by Kuo [6] (page 138, Theorem 10.21).

4. Gaussian Measure on an Affine Subspace

For a subspace W of $\mathbb{R}^n$ and $a \in W^\perp$ we have the Gaussian measure on $a + W$ given by:

$$\int_{a+W} e^{i\langle x, y \rangle}\, d\mu_{a+W}(x) = \int_{a+W} e^{i\langle x, y \rangle} e^{-\frac{1}{2}|x-a|^2}\, \frac{dx}{(2\pi)^{\dim W/2}} = e^{i\langle a, y \rangle - \frac{1}{2}\langle y_W, y_W \rangle}$$

where $y \in \mathbb{R}^n$ and y_W is the projection of y onto W. We now describe how such a measure can be constructed in white noise setting. Of course, the Gaussian measure cannot live on H_0 or $a + W$. However, just as we used the Minlos theorem to form the Gaussian measure μ on $\mathcal{H}'$ (which we think of as the Gaussian measure on H_0), we can again use the Minlos theorem to form the Gaussian measure for the affine subspace $a + W$.

4.1. **Gaussian Measure on $a + V$.** For a vector $a \in H_0$ and a subspace V of H_0 we can use the Minlos theorem to find that there is a measure μ_{a+V} on $\mathcal{H}'$ with

$$(4.1) \qquad \int_{\mathcal{H}'} e^{i\langle x, y \rangle}\, d\mu_{a+V}(x) = e^{i\langle a, y \rangle - \frac{1}{2}\langle y_V, y_V \rangle}$$

for any $y \in \mathcal{H}$. This measure μ_{a+V} is the *Gaussian measure for the affine subspace* $a + V$. This measure was originally constructed in [2].

4.2. **Hida Measure.** The Gaussian measure μ_{a+V} is a special type of measure known as a Hida measure. In this section we define the notion of Hida measure and give an overview of some its properties.

Definition 4.1. A measure ν on $\mathcal{H}'$ is called a *Hida measure* if $\phi \in L^1(\nu)$ for all $\phi \in [\mathcal{H}]$ and the linear functional

$$\phi \mapsto \int_{\mathcal{H}'} \phi(x) \, d\nu(x)$$

is continuous on $[\mathcal{H}]$.

We say that a generalized function $\Phi \in [\mathcal{H}]'$ is *induced* by a Hida measure ν if for any $\phi \in [\mathcal{H}]$ we have

$$\langle\langle \Phi, \phi \rangle\rangle = \int_{\mathcal{H}'} \phi(x) \, d\nu(x).$$

The following theorem characterizes those generalized functions which are induced by a Hida measure.

Theorem 4.2. *Let $\Phi \in [\mathcal{H}]'$. Then the following are equivalent:*

(1) *For any nonnegative $\phi \in [\mathcal{H}]$, $\langle\langle \Phi, \phi \rangle\rangle \geq 0$.*
(2) *The function $\mathcal{T}(\Phi)(x) = \langle\langle \Phi, e^{i\langle \cdot, x \rangle} \rangle\rangle$ is positive definite on $\mathcal{H}$.*
(3) *Φ is induced by a Hida measure.*

A proof of this theorem can be found in [6] (page 320, Theorem 15.3).

Corollary 4.3. *Let ν be a finite measure on $\mathcal{H}'$ such that for any $x \in \mathcal{H}$*

$$\langle\langle \Phi, e^{i\langle \cdot, x \rangle} \rangle\rangle = \int_{\mathcal{H}'} e^{i\langle y, x \rangle} \, d\nu(y)$$

for some $\Phi \in [\mathcal{H}]'$. Then Φ is induced by ν.

Proof. Since $\langle\langle \Phi, e^{i\langle \cdot, x \rangle} \rangle\rangle = \int_{\mathcal{H}'} e^{i\langle y, x \rangle} \, d\nu(y)$ it is clear that $\langle\langle \Phi, e^{i\langle \cdot, x \rangle} \rangle\rangle$ is positive definite. So we can apply Theorem 4.2 to get a finite measure m which is induced by Φ. Hence for all $\phi \in [\mathcal{H}]$,

$$\langle\langle \Phi, \phi \rangle\rangle = \int_{\mathcal{H}'} \phi \, dm.$$

Letting $\phi = e^{i\langle \cdot, x \rangle}$ in the above equation, we see that the characteristic functions for m and ν are identical. Therefore $m = \nu$ and we have that Φ is induced by ν. $\square$

Here is another useful theorem which characterizes Hida measures.

Theorem 4.4. *A measure ν on $\mathcal{H}'$ is a Hida measure if and only if ν is supported in H'_p for some $p \geq 1$ and*

$$\int_{H'_p} \exp\left[\tfrac{1}{2}|x|^2_{-p}\right] \, d\nu(x) < \infty.$$

For a proof of this refer to Kuo's book [6] (page 333, Theorem 15.17).

4.3. **Definition of the distribution $\tilde{\delta}_{a+V}$.** We now prove that μ_{a+V} is a Hida measure and develop the corresponding distribution $\tilde{\delta}_{a+V}$ which we think of as the delta function for the affine subspace $a + V$ [2]. Observe the effect of μ_{a+V} on the renormalized exponential $e^{\langle \cdot, z \rangle - \frac{1}{2}\langle z, z \rangle}$,

$$\int_{\mathcal{H}'} e^{\langle x, z \rangle - \frac{1}{2}\langle z, z \rangle} \, d\mu_{a+V}(x) = e^{-\langle z, z \rangle} \int_{\mathcal{H}'} e^{\langle x, z \rangle} \, d\mu_{a+V}(x)$$

$$= e^{-\langle z, z \rangle} e^{\langle a, z \rangle + \frac{1}{2}\langle z_V, z_V \rangle}$$

$$= e^{\langle a, z \rangle - \frac{1}{2}\langle z_{V^\perp}, z_{V^\perp} \rangle}.$$

Although $\tilde{\delta}_{a+V}$ was originally developed for $a \in H_0$ we could also take $a \in H'_p$. Let the function $F(z)$ denote the result from the calculations above. That is,

(4.2) $$F(z) = e^{\langle a, z \rangle - \frac{1}{2}\langle z_{V^\perp}, z_{V^\perp} \rangle}$$

We will show that $F(z)$ satisfies properties (1) and (2) of Theorem 3.3.

For property (1) consider $F(\alpha z + w)$ where $z, w \in \mathcal{H}_c$ and $\alpha \in \mathbb{C}$. Then notice that

$$F(\alpha z + w) = e^{\langle a, \alpha z + w \rangle - \frac{1}{2}\langle \alpha z_{V^\perp} + w_{V^\perp}, \alpha z_{V^\perp} + w_{V^\perp} \rangle}$$

$$= \exp[\alpha \langle a, z \rangle + \langle a, w \rangle - \frac{1}{2}(\alpha^2 \langle z_{V^\perp}, z_{V^\perp} \rangle + 2\alpha \langle z_{V^\perp}, w_{V^\perp} \rangle + \langle w_{V^\perp}, w_{V^\perp} \rangle)]$$

$$= e^{-\frac{\alpha^2}{2}\langle z_{V^\perp}, z_{V^\perp} \rangle} e^{\alpha(\langle a, z \rangle - \langle z_{V^\perp}, w_{V^\perp} \rangle)} e^{\langle a, w \rangle - \frac{1}{2}\langle w_{V^\perp}, w_{V^\perp} \rangle}$$

which is an entire function of $\alpha \in \mathbb{C}$.

Now for property (2) of Theorem 3.3 we write z as $z = x + iy$ with $x, y \in \mathcal{H}$ and observe that

$$|F(z)| = |e^{\langle a, z \rangle - \frac{1}{2}\langle z_{V^\perp}, z_{V^\perp} \rangle}|$$

$$= |e^{\langle a, x + iy \rangle - \frac{1}{2}\langle x_{V^\perp} + iy_{V^\perp}, x_{V^\perp} + iy_{V^\perp} \rangle}|$$

$$= e^{\langle a, x \rangle} e^{-\frac{1}{2}|x_{V^\perp}|_0^2 + \frac{1}{2}|y_{V^\perp}|_0^2}$$

$$\leq e^{\langle a, x \rangle} e^{\frac{1}{2}|z_{V^\perp}|_0^2}$$

$$\leq e^{|a|_{-p}|x|_p} e^{\frac{1}{2}|z|_0^2}$$

$$\leq e^{\frac{1}{2}|a|_{-p}^2 + \frac{1}{2}|z|_p^2} e^{\frac{1}{2}|z|_p^2} \qquad \text{by Young's Inequality}$$

$$\leq e^{\frac{1}{2}|a|_{-p}^2} e^{\frac{3}{2}|z|_p^2}.$$

So property (2) of Theorem 3.3 is satisfied.

Therefore by Theorem 3.3 there exist some $\Phi \in [\mathcal{H}]'$ such that $S(\Phi)(z) = F(z)$. Then by Corrollary 4.3 we have that for $a \in H_0$, Φ is induced by μ_{a+V}. We simply denote this Φ by $\tilde{\delta}_{a+V}$. This leads us to the following definition: [2]

Definition 4.5. The *delta function for the affine subspace $a + V$* is the distribution in $[\mathcal{H}]'$ induced by the Hida measure μ_{a+V}. We denote this generalized function by $\tilde{\delta}_{a+V}$.

Thus for any test function $\phi \in [\mathcal{H}]$ we have

$$\langle\langle \tilde{\delta}_{a+V}, \phi \rangle\rangle = \int_{\mathcal{H}'} \phi \, d\mu_{a+V}.$$

4.4. **S–transformof $\tilde{\delta}_{a+V}$.** Using the definition of the distribution $\tilde{\delta}_{a+V}$ we can directly compute its S–transform. By the calculations directly preceding (4.2) we have

$$(4.3) \qquad S(\tilde{\delta}_{a+V})(z) = e^{\langle a,z \rangle - \frac{1}{2}\langle z_{V^\perp}, z_{V^\perp} \rangle} \qquad \text{for} \quad z \in \mathcal{H}_c.$$

Using this framework of the Hida measure μ_{a+V} and the corresponding distribution $\tilde{\delta}_{a+V}$ we have the following intuitive theorem:

Theorem 4.6. *Let V be a subspace of H_0 and $a \in V^\perp$, then for any $\phi \in [\mathcal{H}]$ we have*

$$(4.4) \qquad \int_{\mathcal{H}'} \phi(x) \, d\mu_{a+V}(x) = \int_{\mathcal{H}'} \phi(x+a) \, d\mu_V(x)$$

Proof. First we take the special case where $\phi(x) = e^{i\langle x,\xi \rangle}$ for some $\xi \in \mathcal{H}$. Then we have for the left hand side

$$\int_{\mathcal{H}'} \phi(x) \, d\mu_{a+V}(x) = \int_{\mathcal{H}'} e^{i\langle x,\xi \rangle} \, d\mu_V(x) = e^{i\langle a,\xi \rangle - \frac{1}{2}\langle \xi_V, \xi_V \rangle}$$

and for the right hand side

$$\int_{\mathcal{H}'} \phi(x+a) \, d\mu_V(x) = \int_{\mathcal{H}'} e^{i\langle x+a,\xi \rangle} \, d\mu_V(x)$$
$$= e^{i\langle a,\xi \rangle} \int_{\mathcal{H}'} e^{i\langle x,\xi \rangle} \, d\mu_V(x) = e^{i\langle a,\xi \rangle - \frac{1}{2}\langle \xi_V, \xi_V \rangle}$$

Thus we have that (4.4) agrees on the linear span of $\{e^{i\langle x,\xi \rangle} \, ; \, \xi \in \mathcal{H}'\}$.

For any arbitrary $\phi \in [\mathcal{H}]$ take a sequence ϕ_n in the linear space of $\{e^{i\langle x,\xi \rangle} \, ; \, \xi \in \mathcal{H}'\}$ such that ϕ_n converges to ϕ in $[\mathcal{H}]$. Then we have

$$\int_{\mathcal{H}'} \phi(x) \, d\mu_{a+V}(x) = \langle\langle \phi, \tilde{\delta}_{a+V} \rangle\rangle$$
$$= \lim_{n \to \infty} \langle\langle \phi_n, \tilde{\delta}_{a+V} \rangle\rangle$$
$$= \lim_{n \to \infty} \int_{\mathcal{H}'} \phi_n(x) \, d\mu_{a+V}(x)$$
$$= \lim_{n \to \infty} \int_{\mathcal{H}'} \phi_n(x+a) \, d\mu_V(x)$$
$$= \lim_{n \to \infty} \langle\langle T_a(\phi_n), \tilde{\delta}_V \rangle\rangle$$
$$= \langle\langle T_a(\phi), \tilde{\delta}_V \rangle\rangle \text{ using the continuity of } T_a$$
$$= \int_{\mathcal{H}'} \phi(x+a) \, d\mu_V(x)$$

giving us the desired result. $\square$

Now we prove a convenient and somewhat expected property of convergence amongst these delta functions on an affine subspace.

Proposition 4.7. *Let $\{x_n\}$ be a sequence in H_p' converging to x and suppose $\{S_n\}$ is a sequence of subspaces of H_0 converging to a subspace S, in the sense that for any $v \in H_0$, we have v_{S_n} converges to v_S in H_0. Then the generalized functions $\tilde{\delta}_{x_n+S_n}$ converges strongly to $\tilde{\delta}_{x+S}$ in $[\mathcal{H}]'$.*

Proof. We will apply Theorem 3.4. To see that the conditions of Theorem 3.4 are satisfied notice that for $z \in \mathcal{H}_c'$ we have

$$
\begin{aligned}
\lim_{n \to \infty} S(\tilde{\delta}_{x_n + S_n})(z) &= \lim_{n \to \infty} \langle\langle \tilde{\delta}_{x_n + S_n}, e^{\langle \cdot, z \rangle - \langle z, z \rangle} \rangle\rangle \\
&= \lim_{n \to \infty} e^{\langle x_n, z \rangle - \frac{1}{2} \langle z_{S_n^\perp}, z_{S_n^\perp} \rangle} \text{ by } (4.3) \\
&= e^{\langle x, z \rangle - \frac{1}{2} \langle z_{S^\perp}, z_{S^\perp} \rangle} \\
&= \langle\langle \tilde{\delta}_{x+S}, e^{\langle \cdot, z \rangle - \frac{1}{2}\langle z, z \rangle} \rangle\rangle \\
&= S(\tilde{\delta}_{x+S})(z).
\end{aligned}
$$

For the second condition of Theorem 3.4 notice that

$$
\begin{aligned}
S(\tilde{\delta}_{x_n + S_n})(z) &= e^{\langle x_n, z \rangle - \frac{1}{2}\langle z_{S_n^\perp}, z_{S_n^\perp} \rangle} \\
&\leq e^{|x_n|_{-p}|z|_p} e^{\frac{1}{2}|z_{S_n^\perp}|_0^2} \\
&\leq e^{|x|_{-p}|z|_p} e^{\frac{1}{2}|z|_0^2} \\
&\leq e^{\frac{1}{2}|x|_{-p}^2 + \frac{1}{2}|z|_p^2} e^{\frac{1}{2}|z|_0^2} \text{ by Young's Inequality} \\
&\leq e^{\frac{1}{2}|x|_{-p}^2 + \frac{1}{2}|z|_p^2} e^{\frac{1}{2}|z|_p^2} \\
&= e^{\frac{1}{2}|x|_{-p}^2} e^{\frac{3}{2}|z|_p^2}.
\end{aligned}
$$

$\square$

5. Gauss Radon Transform in Infinite Dimensions

We begin by constructing the Radon–Gauss Transform in $\mathbb{R}^n$. Recall that a hyperplane in $\mathbb{R}^n$ can be represented using a unit normal vector $v \in \mathbb{R}^n$ and a number $\alpha \in \mathbb{R}$ by way of

$$\alpha v + v^\perp.$$

The probability density function for the standard Gaussian measure $\mu_{\alpha v + v^\perp}$ on $\alpha v + v^\perp$ is given by

$$d\mu_{\alpha v + v^\perp}(x) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} e^{-|x - \alpha v|^2/2} \, dx$$

where $x \in \mathbb{R}^n$, but dx denotes the Lebesgue measure on $\alpha v + v^\perp$. The characteristic function of this measure is given by

$$(5.1) \qquad\qquad \hat{\mu}_{\alpha v + v^\perp}(k) = e^{i\alpha\langle k, v \rangle - \frac{1}{2}\langle k_{v^\perp}, k_{v^\perp} \rangle},$$

where $k_{v^\perp}$ is the orthogonal projection of k onto $v^\perp$.

Using the measure μ_{a+V} we can construct the Gauss–Radon transform in the white noise framework. (Note that the Gauss–Radon transform was originally constructed for a similar setting in [7].)

5.1. **Hyperplanes in H_0.** In infinite dimensions we define a hyperplane as follows:

Definition 5.1. A *hyperplane* of a infinite dimensional Hilbert space H_0 is given by the set

$$\alpha v + v^\perp = \{\alpha v + x \, ; \, x \in H_0, \langle x, v \rangle_0 = 0\}$$

where α is a real number and v is a non-zero unit vector in H_0.

For such an affine subspace the measure $\mu_{\alpha v + v^\perp}$ has the following characteristic equation and S–transform:

$$(5.2) \qquad \int_{\mathcal{H}'} e^{i\langle x, y\rangle} \, d\mu_{\alpha v + v^\perp}(x) = e^{i\alpha\langle v, y\rangle - \frac{1}{2}\langle y_{v^\perp}, y_{v^\perp}\rangle}, \quad y \in \mathcal{H}$$

and

$$(5.3) \qquad \int_{\mathcal{H}'} e^{\langle x, z\rangle - \langle z, z\rangle} \, d\mu_{\alpha v + v^\perp}(x) = e^{\alpha\langle v, z\rangle - \frac{1}{2}\langle z, v\rangle^2}, \quad z \in \mathcal{H}_c.$$

Notice that the above is analogous to what we have observed in $\mathbb{R}^n$. Using this measure $\mu_{\alpha v + v^\perp}$ we can now define the Gauss–Radon transform in the white noise framework.

Definition 5.2. For a test function $\phi \in [\mathcal{H}]$ we define the Gauss–Radon transform to be the function on the hyperplanes of H_0 given by

$$G_\phi(\alpha v + v^\perp) = \int_{\mathcal{H}'} \phi(x) \, d\mu_{\alpha v + v^\perp}(x).$$

In [7] Mihai and Sengupta also demonstrated that this measure can be constructed using the Kolmogorov theorem and Gaussian measures μ_n on $\mathbb{R}^n$ specified by

$$\hat{\mu}_n(k) = e^{i\alpha\langle k, v_n\rangle - \frac{1}{2}(|k|^2 - |\langle k, v_n\rangle|^2)}$$

where $v_n = (\langle v, e_1\rangle, \ldots, \langle v, e_n\rangle)$. Note that if $|v_n| = 1$, then the above is the Gaussian measure on the hyperplane $\{x \in \mathbb{R}^n \, ; \, \langle v_n, x\rangle = \alpha\} = \alpha v + v^\perp$.

Putting these ideas together we have the following theorem

Proposition 5.3. Let $v \in span\{e_1, \ldots, e_n\} \subset H_0$ and $v_n = (\langle v, e_1\rangle, \ldots, \langle v, e_n\rangle) \in \mathbb{R}^n$. Then for any ϕ of the form $F(\langle \cdot, e_1\rangle, \ldots, \langle \cdot, e_n\rangle)$ where F is a integrable function with respect to the measure $\mu_{\alpha v_n + v_n^\perp}$ on $\mathbb{R}^n$ we have

$$G_\phi(\alpha v + v^\perp) = \int_{\mathcal{H}'} \phi \, d\mu_{\alpha v + v^\perp} = \int_{\alpha v_n + v_n^\perp} F\mu_{\alpha v_n + v_n^\perp}.$$

5.2. **Disintegration.** Here we demonstrate a Fubini like theorem for our Gaussian measure on the affine subspace $a + V$. The theorem allows us to break up the integral into integrals over subspaces making up V. This will be most useful when $a + V$ is a hyperplane as in the Gauss–Radon Transform.

Theorem 5.4. Let ϕ be a test function in $[\mathcal{H}]$ and consider the affine subspace $a + V$ in H_0. If

$$V = S \oplus S^\perp$$

where S is a subspace of H_0, then

$$(5.4) \qquad \int_{\mathcal{H}'} \phi \, d\mu_{a+V} = \int_{\mathcal{H}'} \int_{\mathcal{H}'} \phi(x + y) \, d\mu_{a+S}(x) \, d\mu_{S^\perp}(y).$$

Proof. We first show that the above holds for $\phi(x) = e^{i\langle x, \xi\rangle}$ where $\xi \in \mathcal{H}$. The lefthand side of (5.4) is simply the characteristic equation of μ_{a+V} given by (4.1)

$$(5.5) \qquad\qquad e^{i\langle a, \xi\rangle - \frac{1}{2}\langle \xi_V, \xi_V\rangle}$$

Now for the righthand side we have

$$\int_{\mathcal{H}'}\int_{\mathcal{H}'} e^{i\langle x+y,\xi\rangle}\,d\mu_{a+S}(x)d\mu_{S^\perp}(y) = \int_{\mathcal{H}'} e^{i\langle x,\xi\rangle}\,d\mu_{a+S}(x)\int_{\mathcal{H}'} e^{i\langle y,\xi\rangle}d\mu_{S^\perp}(y)$$

$$= e^{i\langle a,\xi\rangle-\frac{1}{2}\langle\xi_S,\xi_S\rangle}e^{-\frac{1}{2}\langle\xi_{S^\perp},\xi_{S^\perp}\rangle}$$

$$= e^{i\langle a,\xi\rangle-\frac{1}{2}\langle\xi_V,\xi_V\rangle}\text{ because }V = S\oplus S^\perp$$

So the above holds on the dense space given by the linear span of $\{e^{i\langle\cdot,\xi\rangle}\,;\,\xi\in\mathcal{H}\}$.

Now for an arbitrary ϕ, let ϕ_n be in the linear span of $\{e^{i\langle\cdot,\xi\rangle}\,;\,\xi\in\mathcal{H}\}$. For the lefthand side we have

$$\int_{\mathcal{H}'}\phi\,d\mu_{a+V} = \langle\langle\phi,\tilde{\delta}_{a+V}\rangle\rangle = \lim_{n\to\infty}\langle\langle\phi_n,\tilde{\delta}_{a+V}\rangle\rangle = \lim_{n\to\infty}\int_{\mathcal{H}'}\phi_n\,d\mu_{a+V}$$

using the relationship between the measure μ_{a+V} and the distribution $\tilde{\delta}_{a+V}$. The last term in the above equality is equal to

$$(5.6)\qquad\qquad \lim_{n\to\infty}\int_{\mathcal{H}'}\int_{\mathcal{H}'}\phi_n(x+y)\,d\mu_{a+S}(x)d\mu_{S^\perp}(y)$$

If we can pass the limit inside the integral then the proof will be complete. We will work inside out. First note that since $\mu_{S^\perp}$ is a Hida measure by Theorem 4.4 for some $p\geq 1$ we have that $\mu_{S^\perp}(H_p') = 1$ and

$$(5.7)\qquad\qquad \int_{H_p'}\exp\left[\tfrac{1}{2}|y|^2_{-p}\right]\,d\mu_{S^\perp}(y) < \infty$$

Thus we can rewrite the righthand side of (5.4) as

$$(5.8)\qquad\qquad \int_{H_p'}\int_{\mathcal{H}'}\phi(x+y)\,d\mu_{a+S}(x)d\mu_{S^\perp}(y).$$

Working inside out the inside part of the above integral can be written

$$\int_{\mathcal{H}'}\phi(x+y)\,d\mu_{a+S}(x) = \langle\langle T_y\phi,\tilde{\delta}_{a+S}\rangle\rangle.$$

with $y\in H_p'$. Since $\tilde{\delta}_{a+S}$ is in $[\mathcal{H}]'$ and T_y is continuous from $[\mathcal{H}]$ into itself we have that

$$\int_{\mathcal{H}'}\phi(x+y)\,d\mu_{a+S}(x) = \langle\langle T_y\phi,\tilde{\delta}_{a+S}\rangle\rangle$$

$$= \lim_{n\to\infty}\langle\langle T_y\phi_n,\tilde{\delta}_{a+S}\rangle\rangle = \lim_{n\to\infty}\int_{\mathcal{H}'}\phi_n(x+y)\,d\mu_{a+S}(x)$$

Thus (5.8) becomes

$$\int_{H_p'}\int_{\mathcal{H}'}\phi(x+y)\,d\mu_{a+S}(x)d\mu_{S^\perp}(y)$$

$$= \int_{H_p'}\lim_{n\to\infty}\int_{\mathcal{H}'}\phi_n(x+y)\,d\mu_{a+S}(x)\,d\mu_{S^\perp}(y).$$

We would like to use the dominated convergence theorem to pull the limit out once more. To do this notice that $\int_{\mathcal{H}'}\phi_n(x+y)\,d\mu_{a+S}(x)$ is measurable; here note that

$$\int_{\mathcal{H}'} e^{i\langle x+y,\xi\rangle}\,d\mu_{a+S}(x) = e^{i\langle y,\xi\rangle}e^{i\langle a,\xi\rangle-\frac{1}{2}\langle\xi_S,\xi_S\rangle}$$

which is measurable. Thus

$$\int_{\mathcal{H}'} \phi(x+y)\, d\mu_{a+S}(x) = \lim_{n\to\infty} \int_{\mathcal{H}'} \phi_n(x+y)\, d\mu_{a+S}(x)$$

is measurable. Also observe that choosing a k such that $k > p$ and $\tilde{\delta}_{a+S} \in [H_k]'$ we have

$$\left| \int_{\mathcal{H}'} \phi_n(x+y)\, d\mu_{a+S}(x) \right| = \left| \langle\langle T_y \phi_n, \tilde{\delta}_{a+S} \rangle\rangle \right|$$

$$\leq \|T_y \phi_n\|_k \|\tilde{\delta}_{a+S}\|_{-k}$$

$$\leq \|\phi_n\|_q \|\tilde{\delta}_{a+S}\|_{-k} \exp\left[\tfrac{1}{2}|x|^2_{-p}\right]$$

using Theorem 3.7 where q is chosen to ensure that $\lambda_1^{2(q-k)} > 2$. Now in the above we have $\|\phi_n\|_q$ is bounded because $\phi_n \to \phi$ in $[\mathcal{H}]$. Putting this altogether we have for some number M

$$\left| \int_{\mathcal{H}'} \phi_n(x+y)\, d\mu_{a+S}(x) \right| \leq M \exp\left[\tfrac{1}{2}|x|^2_{-p}\right]$$

and integral of the righthand side of the above using the measure $\mu_{S^\perp}$ is finite by (5.7). Therefore the dominated convergence theorem applies. □

For this work the above theorem proves most useful when the affine subspace is actually a hyperplane $\alpha v + v^\perp$. Then the above gives us a means by which to decompose the Gauss–Radon transform.

Corollary 5.5. *Let ϕ be a test function in $[\mathcal{H}]$ and consider the hyperplane $\alpha v + v^\perp$ in H_0. If*

$$v^\perp = S \oplus S^\perp$$

where S is a subspace of H_0, then

$$(5.9) \qquad G_\phi(\alpha v + v^\perp) = \int_{\mathcal{H}'} \phi\, d\mu_{\alpha v + v^\perp} = \int_{\mathcal{H}'} \int_{\mathcal{H}'} \phi(x+y)\, d\mu_{\alpha v + S}(x)\, d\mu_{S^\perp}(y).$$

Corollary 5.6. *Let V be a subspace of H_0. Then for any test function ϕ we have*

$$\int_{\mathcal{H}'} \phi(x)\, d\mu(x) = \int_{\mathcal{H}'} \int_{\mathcal{H}'} \phi(x+y)\, d\mu_V(x)\, d\mu_{V^\perp}(y).$$

5.3. **Coordinates.** Our goal here is to show that $\int_{\mathcal{H}'} \phi(x)\, d\mu_{a+V}(x)$ essentially only depends on the "projections" of the x-values to the subspace V. We first need the following lemma concerning our most popular dense set.

Lemma 5.7. *The linear span of $\{e^{i\langle \cdot, \xi \rangle} \,;\, \xi \in \mathcal{H}\}$ is dense in $L^1(\mu_{a+V})$.*

Proof. A result in [7] (Proposition 3.4) states that the linear span of $\{e^{i\langle \cdot, \xi \rangle} \,;\, \xi \in \mathcal{H}\}$ is dense in $L^2(\mu_{a+V})$. Now we simply show that $L^1(\mu_{a+V})$ is dense in $L^2(\mu_{a+V})$. Let $f \in L^1(\mu_{a+V})$ with f orthogonal to $L^2(\mu_{a+V})$. Our objective is to show that $f = 0$.

Note that for any measurable set A we have that 1_A in in $L^1(\mu_{a+V})$ and $L^2(\mu_{a+V})$. Since f is orthogonal to $L^2(\mu_{a+V})$ we must have

$$\int_{\mathcal{H}'} 1_A f\, d\mu_{a+V} = 0$$

In particular for the set $\{f \geq 0\}$ we have that

$$0 = \int_{\{f \geq 0\}} f \, d\mu_{a+S} = \int_{\{f \geq 0\}} f^+ \, d\mu_{a+S}.$$

Thus $f^+ = 0$ almost everywhere. Similarly, we can get $f^- = 0$ almost everywhere.
□

Theorem 5.8. *Let $a \in span\{e_1, \ldots, e_n\} \subset H_0$ and S be a subspace of H_0 with $S \subset span\{e_1, \ldots, e_n\}$. Then if $f \in L^1(\mu_{a+S})$, we have*

$$\int_{\mathcal{H}'} f(x) \, d\mu_{a+S}(x) = \int_{span\{e_1,\ldots,e_n\}} f(\langle x, e_1 \rangle e_1 + \cdots + \langle x, e_n \rangle e_n) \, d\mu_{a+S}(x)$$

Proof. Let P_S be the projection onto the subspace S. Observe that for any $k > n$ we have that

$$\int_{\mathcal{H}'} e^{it\hat{e}_k} \, d\mu_{a+S} = e^{i\langle a, te_k \rangle - \frac{1}{2}\langle tP_S e_k, tP_S e_k \rangle}$$

$$= e^0$$

$$= \int_{\mathbb{R}} e^{its} \, d\delta_0(s)$$

where δ_0 is the delta measure with $\delta_0(0) = 1$. Since the characteristic function of a random variable uniquely specifies the distribution, it follows that the random variable $\hat{e}_k$ has a distribution δ_0, i.e. $\hat{e}_k$ has the constant value 0 almost everywhere. Thus the measure of the set $\hat{e}_k^{-1}(0) = \{x \in \mathcal{H}' \, ; \, \langle x, e_k \rangle = 0\}$ has full measure with respect to μ_{a+S}. Therefore the set $\{\hat{e}_k \neq 0\} = \{x \in \mathcal{H}' \, ; \, \langle x, e_k \rangle \neq 0\}$ has $\mu_{a+S}-$ measure 0. Hence the set

$$\bigcup_{k=n+1}^{\infty} \{\hat{e}_k \neq 0\}$$

has μ_{a+S} measure 0. Likewise the complement

$$\left(\bigcup_{k=n+1}^{\infty} \{\hat{e}_k \neq 0\} \right)^c = \bigcap_{k=n+1}^{\infty} \{\hat{e}_k \neq 0\}^c = \bigcap_{k=n+1}^{\infty} \{\hat{e}_k = 0\} = span\{e_1, \ldots, e_n\}$$

has μ_{a+S}–measure 1. Therefore for any $f \in L^1(\mu_{a+S})$ we have

$$\int_{\mathcal{H}'} f(x) \, d\mu_{a+S}(x) = \int_{span\{e_1,\ldots,e_n\}} f(x) \, d\mu_{a+S}(x)$$

$$= \int_{span\{e_1,\ldots,e_n\}} f(\langle x, e_1 \rangle e_1 + \cdots + \langle x, e_n \rangle e_n) \, d\mu_{a+S}(x)$$

since $x = \langle x, e_1 \rangle e_1 + \cdots + \langle x, e_n \rangle e_n$ when $x \in span\{e_1, \ldots e_n\}$.
□

6. Support Theorem for Gauss–Radon Transform

Having the Gauss–Radon transform fully developed we take on the task of developing the Support Theorem in this setting. The Support Theorem in $\mathbb{R}^n$ requires that the Radon transform be zero outside of some convex compact set. The typical example in $\mathbb{R}^n$ are the closed balls. At some point we will to appeal the Support Theorem in $\mathbb{R}^n$. So the sets we consider in infinite dimensions must have "projections" which are convex and compact. The convexity issue is easily addressed. To have the property of compactness we desire leads us to the following definition:

Definition 6.1. A subset C of $\mathcal{H}'$ is *projectively compact* if the set
$$C_n = \{\vec{x}_n = (\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle)\, ; \, x \in C\}$$
is compact and $\vec{x}_n \in C_n$ for all n implies $x \in C$.

Most often we will be using the contrapositive of the above definition.

Remark 6.2. Let $C \subset \mathcal{H}'$ be a projectively compact set with corresponding sets $C_n = \{\vec{x}_n = (\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle)\, ; \, x \in C\}$. If $x \notin C$, then there exist an N such that for $n > N$ we have that $x_n \notin C_n$.

The following proposition discusses the properties of these projectively compact sets.

Proposition 6.3. *Let C be a projectively compact set with corresponding sets $C_n = \{\vec{x}_n = (\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle)\, ; \, x \in C\} \subset \mathbb{R}^n$. Then C is convex if and only if each C_n is convex.*

Proof. Suppose C is convex and consider the set C_n. Let $\vec{x}_n, \vec{y}_n$ be two points in C_n corresponding to $x, y \in C$. That is $\vec{x}_n = (\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle)$ and $\vec{y}_n = (\langle y, e_1 \rangle, \ldots, \langle y, e_n \rangle)$. We must show $\alpha \vec{x}_n + (1 - \alpha)\vec{y}_n$ is in C_n for any $\alpha \in [0, 1]$. Since C is convex we have that $\alpha x + (1 - \alpha)y \in C$. Thus
$$(\langle \alpha x + (1 - \alpha)y, e_1 \rangle, \ldots, \langle \alpha x + (1 - \alpha)y, e_n \rangle) \in C_n.$$
Notice
$$(\langle \alpha x + (1 - \alpha)y, e_1 \rangle, \ldots, \langle \alpha x + (1 - \alpha)y, e_n \rangle) = \alpha \vec{x}_n + (1 - \alpha)\vec{y}_n$$
and thus $\alpha \vec{x}_n + (1 - \alpha)\vec{y}_n \in C_n$ and we have C_n is convex.

On the other hand suppose that C_n is convex for each n. We must show that C is convex. Let $x, y \in C$. We will show that $\alpha x + (1 - \alpha)y \in C$. Since $x, y \in C$ we have that $\vec{x}_n, \vec{y}_n \in C_n$ for all n and by the convexity of each C_n we have that $\alpha \vec{x}_n + (1 - \alpha)\vec{y}_n \in C_n$ for all n. Thus we have that $\alpha x + (1 - \alpha)y \in C$. $\square$

We will now demonstrate that there are nontrivial sets which satisfy this criteria of being convex and projectively compact. In particular we demonstrate the closed ball in H'_p given by
$$B_r^{-p}(y) = \{x \in H'_p\, ; \, |x - y|_{-p} \le r\}$$
has positive measure for any $r > 0$ and $y \in H'_p$. Of course these sets are closed convex and projectively compact because their "projections" are essentially closed ellipses in $\mathbb{R}^n$. We must just demonstrate they have positive measure.

First we observe that every ball $B_r^{-p}(y)$ contains a ball centered around a "rational point" q, i.e. $q \in \mathrm{span}_{\mathbb{Q}}\{e_1, \ldots, e_n\} \subset \mathcal{H}$ for some n.

Lemma 6.4. *For any $y \in \mathcal{H}'$ and $r > 0$, the ball $B_r^{-p}(y)$ contains a ball $B_{r'}^{-p}(q)$ where $0 < r' < r$ and $q \in \mathrm{span}_{\mathbb{Q}}\{e_1, \ldots, e_n\}$ for some n.*

Proof. Consider the set $Q_n = \{r_1 e_1 + \cdots r_n e_n\, ; \, r_1, \ldots, r_n \in \mathbb{Q}\}$. Note that Q_n is a countable set. Now let $Q = \bigcup_{n=1}^{\infty} Q_n$. Again Q is countable. (This can be thought of as the set of rational points in $\mathcal{H}'$.) Observe if $y \in H'_p$, then there exists n such that
$$\left| y - \sum_{k=1}^{n} \langle y, e_n \rangle e_n \right|_{-p} < \frac{r}{2}$$

For each $k = 1, \ldots, n$, take $r_k \in \mathbb{Q}$ such that

$$|r_k - \langle y, e_k \rangle| < \frac{r}{2^k}.$$

Then let $q \in Q$ be given by $q = r_1 e_1 + \cdots + r_n e_n$ and we have

$$|y - q|_{-p} \leq |y - \sum_{k=1}^{n} \langle y, e_k \rangle e_k|_{-p} + |\sum_{k=1}^{n} \langle y, e_k \rangle e_k - \sum_{k=1}^{n} r_k e_k|_{-p}$$
$$< \frac{r}{2} + \frac{r}{2} = r$$

Thus for any $x \in B_{\frac{r}{2}}^{-p}(q)$ we have

$$|x - y|_{-p} \leq |x - q|_{-p} + |y - q|_{-p} \leq \frac{r}{2} + \frac{r}{2} \leq r.$$

Therefore $B_{\frac{r}{2}}^{-p}(q) \subset B_r^{-p}(y)$ $\hfill\square$

We now use the previous lemma along with the properties of the measure μ to deduce that any such ball in H_p' with positive radius must have positive measure.

Proposition 6.5. *For any $y \in H_p'$ and $r > 0$ we have that $\mu(B_r^{-p}(y)) > 0$.*

Proof. We let Q be as in Lemma 6.4. Since Q is dense in H_p' for any $r > 0$, H_p' can be written as a countable union of balls centered about rational points, i.e.

$$H_p' = \bigcup_{q \in Q} B_r^{-p}(q).$$

Since H_p' is of positive measure (actually full measure), we have that $B_r^{-p}(q)$ must have positive measure for some q. By Proposition 3.5 we have that $B_r^{-p}(q)$ is of positive measure for any q.

Every ball $B_r^{-p}(y)$ in H_p' contains a ball centered at a rational point by Lemma 6.4. Thus each ball $B_r^{-p}(y)$ must have positive measure. $\hfill\square$

The basis for the inductive topology for $\mathcal{H}'$ is given by the convex hull of the sets

$$\bigcup_{p=1}^{\infty} B_{r_p}^{-p}(x_p)$$

where $B_{r_p}^{-p}(x_p)$ denotes the unit ball in H_p' centered around $x_p \in H_p'$ with radius r_p [1]. Thus each nonempty open set contains an open ball $B_r^{-p}(x)$ for some $x \in H_{-p}'$ and $r > 0$. Since each $B_r^{-p}(x)$ has positive measure by Proposition 6.5, we must have that each nonempty open set in $\mathcal{H}'$ also has positive measure.

Now the inductive and strong topologies on $\mathcal{H}'$ are equivalent and of course the weak topology is coarser than either of these topologies [1]. This leads to the following corollary of Proposition 6.5.

Corollary 6.6. *The μ-measure of any nonempty open set U in any of the weak, strong, or inductive limit topologies is positive (i.e. $\mu(U) > 0$).*

The next theorem is the main result of this paper. It gives us a Support Theorem for the Gauss–Radon Transform.

Theorem 6.7 (Support Theorem for Gauss–Radon Transform). *Let ϕ be a test function and let C be a convex projectively compact set in $\mathcal{H}'$ with*

$$G_\phi(\alpha v + v^\perp) = 0$$

when $\alpha v \notin C$. Then $\phi(x) = 0$ for all $x \notin C$

Proof. Throughout the proof we make the following notational conventions: for a vector $w \in H_0$, we denote by w_n the projection of w onto $\mathrm{span}\{e_1, \ldots, e_n\}$. That is,

$$w_n = \langle w, e_1 \rangle e_1 + \langle w, e_2 \rangle e_2 + \cdots + \langle w, e_n \rangle e_n.$$

Also we denote by $\vec{w}_n$ the vector in $\mathbb{R}^n$ corresponding to w_n. That is,

$$\vec{w}_n = (\langle w, e_1 \rangle, \langle w, e_2 \rangle, \cdots, \langle w, e_n \rangle).$$

We also make the observation that if $w \notin C$, then there exist an N such that $w_n \notin C$ for all $n > N$. (For if $w_n \in C$ for all $n > N$, then $\vec{w}_n \in C_n$ for all n and hence $w \in C$.)

With this in mind we first take a $\alpha v \in V_n = \mathrm{span}\{e_1, \ldots, e_n\}$ with $\alpha v \notin C$. We know that

$$(6.1) \qquad 0 = G_\phi(\alpha v + v^\perp) = \int_{\mathcal{H}'} \phi(x)\, d\mu_{\alpha v + v^\perp}(x).$$

We can decompose the subspace $v^\perp$ as follows

$$v^\perp = (v^\perp \cap V_n) \oplus V_n^\perp.$$

Let $S_{n,v} = (v^\perp \cap V_n)$ and notice $V_n^\perp = \mathrm{span}\{e_{n+1}, e_{n+2}, \ldots\}$. The idea here is that V_n is in essence $\mathbb{R}^n$ and thus basically we have v "in" $\mathbb{R}^n$ and $S_{n,v}$ can be thought of as the orthogonal complement of v "in" $\mathbb{R}^n$.

We now use Theorem 5.4 to rewrite $G_\phi(\alpha v + v^\perp)$ as follows:

$$G_\phi(\alpha v + v^\perp) = \int_{\mathcal{H}'} \int_{\mathcal{H}'} \phi(x + y)\, d\mu_{V_n^\perp}(y)\, d\mu_{\alpha v + S_{n,v}}(x).$$

Let's write the inside of the above integral as

$$f_n(x) = \int_{\mathcal{H}'} \phi(x + y)\, d\mu_{V_n^\perp}(y)$$

and we have that

$$(6.2) \qquad G_\phi(\alpha v + v^\perp) = \int_{\mathcal{H}'} f_n(x)\, d\mu_{\alpha v + S_{n,v}}(x).$$

Since $v, S_{n,v}$ are in $\mathrm{span}\{e_1, \ldots, e_n\}$ the we can apply Theorem 5.8 to write the above as

$$(6.3) \quad \int_{\mathcal{H}'} f_n(x)\, d\mu_{\alpha v + S_{n,v}}(x) = \int_{\mathcal{H}'} f_n(\langle x, e_1 \rangle e_1 + \cdots + \langle x, e_n \rangle e_n)\, d\mu_{\alpha v + S_{n,v}}(x).$$

Moreover, $f_n(\langle x, e_1 \rangle e_1 + \cdots + \langle x, e_n \rangle e_n) = F_n(\hat{e}_1, \ldots, \hat{e}_n)$ where

$$F_n(x_1, \ldots, x_n) = \int_{\mathcal{H}'} \phi(x_1 e_1 + \cdots + x_n e_n + y)\, d\mu_{V_n^\perp}(y)$$

is a function on $\mathbb{R}^n$. Thus by Proposition 5.3

$$\int_{\mathcal{H}'} f_n(\langle x, e_1 \rangle e_1 + \cdots + \langle x, e_n \rangle e_n)\, d\mu_{\alpha v + S_{n,v}}(x) = \int_{\alpha \vec{v} + \vec{v}^\perp} F_n\, d\mu_{\alpha \vec{v} + \vec{v}^\perp}$$

where $\vec{v} = (\langle v, e_1 \rangle, \ldots, \langle v, e_n \rangle)$ is a vector in $\mathbb{R}^n$ corresponding to v. Thus combining the above with (6.2) and (6.3) we have that

$$G_\phi(\alpha v + v^\perp) = \int_{\alpha\vec{v}+\vec{v}^\perp} F_n \, d\mu_{\alpha\vec{v}+\vec{v}^\perp}.$$

For any $\alpha\vec{v} \notin C_n$ we must also that $\alpha v \notin C$. Thus by assumption we have $G_\phi(\alpha v + v^\perp) = 0$ and combining that with the above yields

$$\int_{\alpha\vec{v}+\vec{v}^\perp} F_n \, d\mu_{\alpha\vec{v}+\vec{v}^\perp} = 0.$$

Therefore the Gauss–Radon transform of the function F_n is 0 when for any hyperplane $\alpha\vec{v} + \vec{v}^\perp$ not intersecting the compact convex set C_n.

We would like to apply the original Support Theorem (for the Gaussian measure) to our function F_n to see that $F_n = 0$ outside of C_n. In order to do so we must check that F_n satisfies the other assumptions of Theorem 1.2.

Lemma 6.8. *The function*

$$F_n(x_1, \ldots, x_n) = \int_{\mathcal{H}'} \phi(x_1 e_1 + \cdots + x_n e_n + y) \, d\mu_{V_n^\perp}(y)$$

is continuous and bounded.

Proof. The continuity is easy to check, observe that if $\{\vec{x}^{(k)}\}$ converges to $\vec{x}$ in $\mathbb{R}^n$, then $x^{(k)} = x_1^{(k)} e_1 + \cdots + x_n^{(k)} e_n$ converges to $x = x_1 e_1 + \cdots + x_n e_n$ with respect to $|\cdot|_0$ (in fact, with respect to any $|\cdot|_p$ or $|\cdot|_{-p}$ norm). Therefore by Proposition 4.7 we have that $\tilde{\delta}_{x^{(k)}+V_n^\perp}$ converges to $\tilde{\delta}_{x+V_n^\perp}$ strongly as $k \to \infty$. Therefore

$$\lim_{k\to\infty} F_n(x_1^{(k)}, \ldots, x_n^{(k)}) = \lim_{k\to\infty} \int_{\mathcal{H}'} \phi(x_1^{(k)} e_1 + \cdots + x_n^{(k)} e_n + y) \, d\mu_{V_n^\perp}(y)$$

$$= \lim_{k\to\infty} \int_{\mathcal{H}'} \phi(y) \, d\mu_{x^{(k)}+V_n^\perp}(y) \qquad \text{by Theorem 4.6}$$

$$= \lim_{k\to\infty} \langle\langle \phi, \tilde{\delta}_{x^{(k)}+V_n^\perp} \rangle\rangle$$

$$= \langle\langle \phi, \tilde{\delta}_{x+V_n^\perp} \rangle\rangle \qquad \text{by Proposition 4.7}$$

$$= \int_{\mathcal{H}'} \phi(y) \, d\mu_{x+V_n^\perp}(y)$$

$$= \int_{\mathcal{H}'} \phi(x_1 e_1 + \cdots + x_n e_n + y) \, d\mu_{V_n^\perp}(y) \qquad \text{by Theorem 4.6}$$

$$= F_n(x_1, \ldots, x_n)$$

and thus F_n is continuous on $\mathbb{R}^n$.

We now need to verify that F_n is bounded. Again observe that

$$F_n(x_1, \ldots, x_n) = \int_{\mathcal{H}'} \phi(x_1 e_1 + \cdots + x_n e_n + y) \, d\mu_{V_n^\perp}(y) = \int_{\mathcal{H}'} \phi(y) \, d\mu_{x+V_n^\perp}(y).$$

Now ϕ is a test function and $\mu_{x+V_n^\perp}$ is a Hida measure. So we combining Theorem 3.2 and Theorem 4.4 to get that for some $p \geq 1$ we have

$$|F_n(x_1, \ldots, x_n)| = \left| \int_{\mathcal{H}'} \phi(y) \, d\mu_{x+V_n^\perp}(y) \right|$$

$$\leq \int_{\mathcal{H}'} |\phi(y)| \, d\mu_{x+V_n^\perp}(y)$$

$$\leq \int_{H_p'} K_p \exp\left[\frac{1}{2} |y|_{-p}^2 \right] d\mu_{x+V_n^\perp}(y)$$

and the last integral is finite by Theorem 4.4. $\qquad\square$

Thus by the original Support Theorem (Theorem 1.2) we have that $F_n(\vec{x}) = 0$ for all $\vec{x} \notin C_n = \{(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle) \mid x \in C\} \subset \mathbb{R}^n$. Thus

$$f_n(x_n) = \int_{\mathcal{H}'} \phi(x_n + y) \, d\mu_{V_n^\perp}(y)$$

is 0 when $\vec{x}_n \notin C_n$. To complete the proof we notice that $f_n(x_n) \to \phi(x)$. To this end notice that

$$f_n(x_n) = \int_{\mathcal{H}'} \phi(x_n + y) \, d\mu_{V_n^\perp}(y)$$

$$= \int_{\mathcal{H}'} \phi(y) \, d\mu_{x_n+V_n^\perp}(y) \text{ by Theorem 4.6}$$

(6.4) $$= \langle\langle \phi, \tilde{\delta}_{x_n+V_n^\perp} \rangle\rangle.$$

Combining Proposition 4.7 with (6.4) gives us that

$$\lim_{n\to\infty} f_n(x_n) = \lim_{n\to\infty} \langle\langle \phi, \tilde{\delta}_{x_n+V_n^\perp} \rangle\rangle = \langle\langle \phi, \tilde{\delta}_x \rangle\rangle = \phi(x).$$

We now have the tools to complete the proof. Take an $x \notin C$. Then by Remark 6.2 there exist an integer N such that for all $n > N$, we have $\vec{x}_n \notin C_n$. Thus $F_n(\vec{x}_n) = 0$ and likewise $f_n(x_n) = 0$ for all $n > N$. Taking the limit as n goes to infinity gives us $\phi(x) = 0$. $\qquad\square$

The following corollary restates the above when we simply take the closed ball $B_r^{-p}(x)$ as our closed convex projectively compact set.

Corollary 6.9 (Support Theorem for Gauss–Radon Transform)**.** *Let ϕ be a test function with*

$$G_\phi(\alpha v + v^\perp) = 0$$

when $|\alpha v|_{-p} > r$. Then $\phi(x) = 0$ for all $x \notin B_r^{-p}(0)$

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