Solids of Revolution

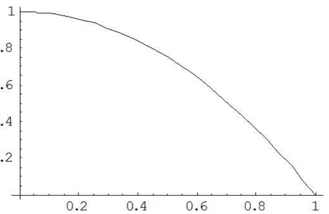
About the Line y=mx

Disk Method from Calculus

Consider the function $1-x^2$ on the interval [0,1]

Suppose that this function is continuous from [a,b]

Plot [1-x², {x,0,1}]



The volume of the solid is given by

$$\int_a^b A(x)\,dx$$ Where A(x) is the area at x

Therefore the volume of the function f on the interval [a,b] is

$$\int_a^b \pi R(x)^2\,dx$$

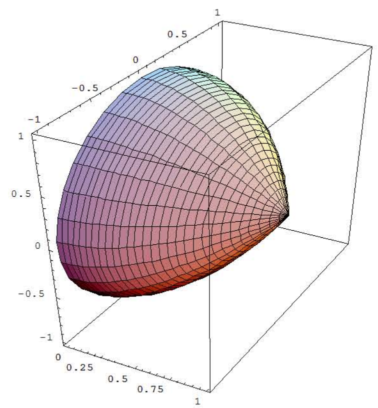
The formula with a radius of

$1-x^2$ on the interval [0, 1]

$A(x) = \pi\,(\text{radius})^2$

radius = F(x) - G(x); where F(x) ≥ G(x)

$(x) = 1 - x^2$ & $G(x) = 0$



This graphic shows the region that would be used to evaluate the volume of the function

$$\int_0^1 \pi\,(1-x^2)^2\,dx$$

The value $\frac{8\pi}{15}$ is the volume is in terms of π and the approximated value is 1.675516082

Notice we are "only" using 100 rectangles in the above problem and our answer agrees with the answer found using standard calculations to a large degree of accuracy.

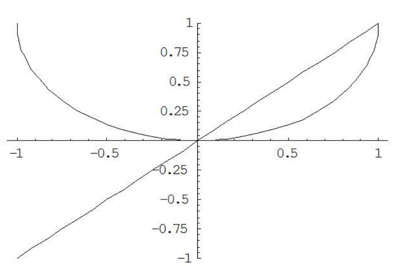
Limitations

The limitations for the methods that are used in Calculus II only allow the revolution of curves around the

- x-axis
- y-axis
- x=C (where C is some value in the integers)
- y=C (where C is some value in the integers)

The methods used in Calculus II are

- Shell Method
- Disk Method
- Washer Method



Each has their own purpose but non of them have the capability of revolving a curve around a line that has an equation of;

$y = m(x) + b$

Problem Description

This verifies our algorithm for finding the volume of a solid of revolution about the line $y=mx$. To verify our method works we make use of the symmetry of the circle. We rotate the chord from (0,0) to (1,1) on the unit $x^2 + (y-1)^2=1$ circle about the line $y = x$ (the line on which the chord sits) using the algorithm we developed.

We then use the conventional formula from Calculus II to rotate the chord from (0,0) to ($\sqrt{2}$, 0) on the unit circle

Notice both chords have the same length and are both on a circle of radius 1. Therefore the solids formed should be of the same volume.

$\left(x-\frac{\sqrt{2}}{2}\right)^2+\left(y-\frac{\sqrt{2}}{2}\right)^2=1$ about the line y=0 (the line on which the chord sits).

The variable m represents the slope of the line we are rotating about.

m=1;

Here we define the function which is being revolved.

(Here we have the bottom half of the circle $x^2 + (y-1)^2=1$.)

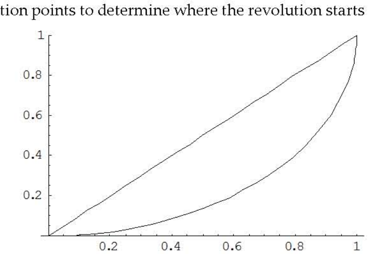
f[x_] := $-\sqrt{1-x^2}+1$;

Here we define the line of revolution (the line being revolved about).

l[x_]:=m*x;

Next we compute the two intersection points to determine where the revolution starts and end.

Solve[f[x]   l[x],x];
a=Min[x/.%]
0
Solve[f[x]   l[x],x];
b=Max[x/.%]
1

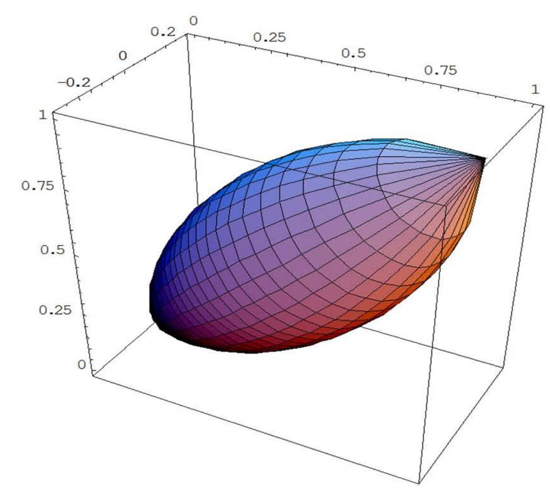


We begin by plotting the function $f(x)$ and the line $y = mx$.

Plot[{m*x,f[x]},{x,a,b},
PlotRange{l[a]-1,l[b]+0.5}];

The solid for which we are finding the volume is graphed below.

Graphics`SurfaceOfRevolution`SurfaceOfRevolution[f[x],{x,a,b},RevolutionAxis



The variable n represents the number of rectangles used in approximating the volume.

n=100;

The quantity Δx below computes the distance between elements of the partition of [a,b].

Δx:=(b-a)/n;

The function $p(k)$ is used to iterate through the partition of the segment [a,b]. The usually notation is for the partition is $x_0, x_1, \ldots, x_n$.

p[k_]:=a+(k-1)Δx;

The function $u(k)$ computes the x-coordinate of the intersection point between $f(x)$ and the line perpendicular to $y=mx$ at x_k. That is, the line perpendicular to $y=mx$ at $x = x_k$ is given by

$y-mx_k = (-1/m)(x-x_k)$

The function $u(k)$ is the x- coordinate of the intersection point between this line and the function f(x). Moreover it is the specific intersection point located in the interval [a, b]

u[k_]:=Min[{x}/.FindInstance[f[x]+(1/m)x = (m+(1/m))p[k]&&a≤x≤b, {x}]] ;

The coordinates points which compromise the "bottom" of the rectangle are given by (x_k, mx_k) and (x_{k+1}, mx_{k+1}). Using the distance formula we find the length of the bottom of the rectangle which we label w for width. (This can be considered to be the width of the rectangle or the width of the approximating disk.

$w = \sqrt{1+m^2}\,\Delta x;$

We now compute the radius used in the Disk method. Recall that the radius of the disk is the height of the rectangle being used in the approximation. So we find the distance between the point on the line of rotation (x_k, mx_k) (or p[k], l[p[k]]) and the point on the function in the direction perpendicular to the line (u_k, $f(u_k)$).

R2[k_] := (u[k] − p[k])^2 + (f[u[k]] − l[p[k]])^2

OurAnswer := N[$\sum_{k=1}^{n} \pi$ (R2[k]) w, 10]

OurAnswer: 0.2129703427

Notice we are 'only' using 100 rectangles in the above problem and our answer agrees with the answer found using standard calculations to a large degree of accuracy.

Verification Circle

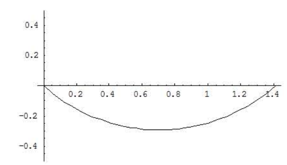
To verify the correctness of our method we now use the conventional formula from Calculus II to find the volume of the solid generated by the chord from (0,0) to ($\sqrt{2}$, 0) on the unit circle

$\left(x-\frac{\sqrt{2}}{2}\right)^2+\left(y-\frac{\sqrt{2}}{2}\right)^2=1$ about the line y=0 (the line on which the chord sits).

The function which yields the bottom half of the above circle is given by $g(x)$.

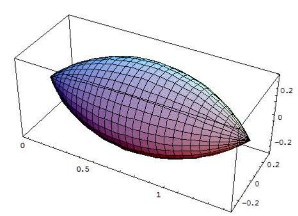
g[x_] = $-\sqrt{1-\left(x-\frac{\sqrt{2}}{2}\right)^2}+\frac{\sqrt{2}}{2}$;

Plot[g[x], {x, 0, $\sqrt{2}$}, PlotRange → {$-\frac{1}{2}$, $\frac{1}{2}$}];



Here we graph the solid formed by our rotation.

SurfaceOfRevolution[g[x],{x,0,(2)^(1/2)},RevolutionAxis{1,0,0},ViewVertical®{0,0,1}];



We now compute the volume of the above solid using the formula from Calculus II in order to verify our algorithm is correct.

CalI := N[Pi $\int_0^{\sqrt{2}}$ (g[x])^2 dx, 10]

CalI

0.2129703486

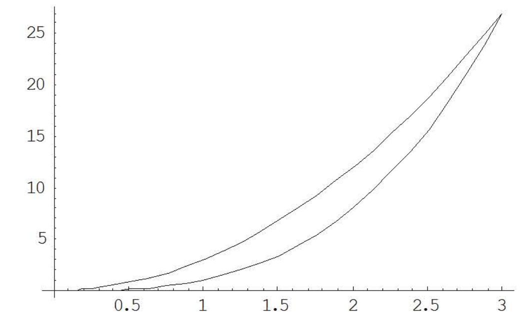
Finally we compare our algorithm to the answer found using techniques from Calculus II, and relize that the answers have a difference of:

N[OurAnswer-CalI, 10] = -5.9×10^{-9}

Future Work

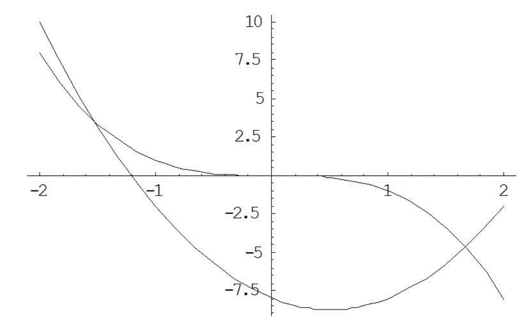
Through future studies using the Mathematica 5.1 program and studying the concepts of Calculus II, I should be able to come up with a technique in finding a formula that calculate the volume as one arbitrary curve is rotation around another arbitrary curve. This research will lead to the ability to revolve a curve such as $3x^2$ around the curve x^3. The problem that is presented is the changing direction of the radius along the length of the two curves.

Plot[{x³,3x²},{x,0,3}];



The ability to revolve a curve such as 3x2-3x-8 around the curve –3x3 on a closed interval, may present some issues. The problem that is presented here may involve dealing with negative space due to the elongated shape that the revolution may cause.

Plot[{-x³,3x²-3x-d8},{x,-2,2}];



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Products used

Mathematica 5.1