

Solids of Revolution Formed by Rotating Functions

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Calculus II Volumes

In Calculus II, we learned how to calculate the volume of a solid formed by revolving a function around the x - or y -axis of a 2-dimensional graph. Take the function $R(x)$ in Figure 1. Revolving $R(x)$ around the x -axis, the solid formed has circular cross-sections. The radius of a circular cross-section is the distance from the x -axis to $R(x)$, so the area of the cross-sections is $A(x) = \pi R(x)^2$.

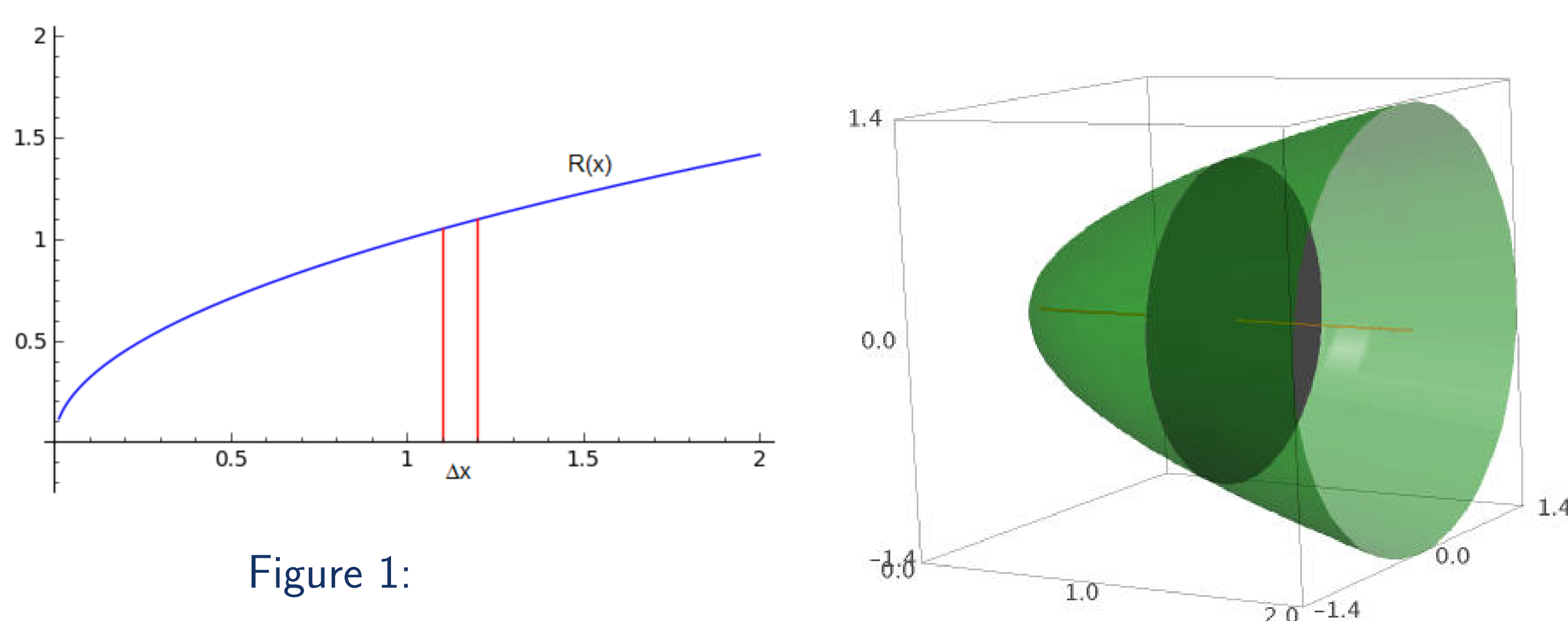


Figure 1:

The volume of this solid is the sum of infinitesimally small circles which "fill up" this solid. This is given by

$$\lim_{\Delta x \rightarrow 0} \sum_{k=1}^n \pi R(x_k)^2 \Delta x,$$

which translates to

$$\int_a^b \pi R(x)^2 dx.$$

Main Project

The idea for this research was to find the volume of a solid formed, not just by revolving a function about the x - or y -axis, but by revolving about an arbitrary line $y = mx + b$. We took two approaches to this project, namely a

- numerical technique and,
- develop an integral formula

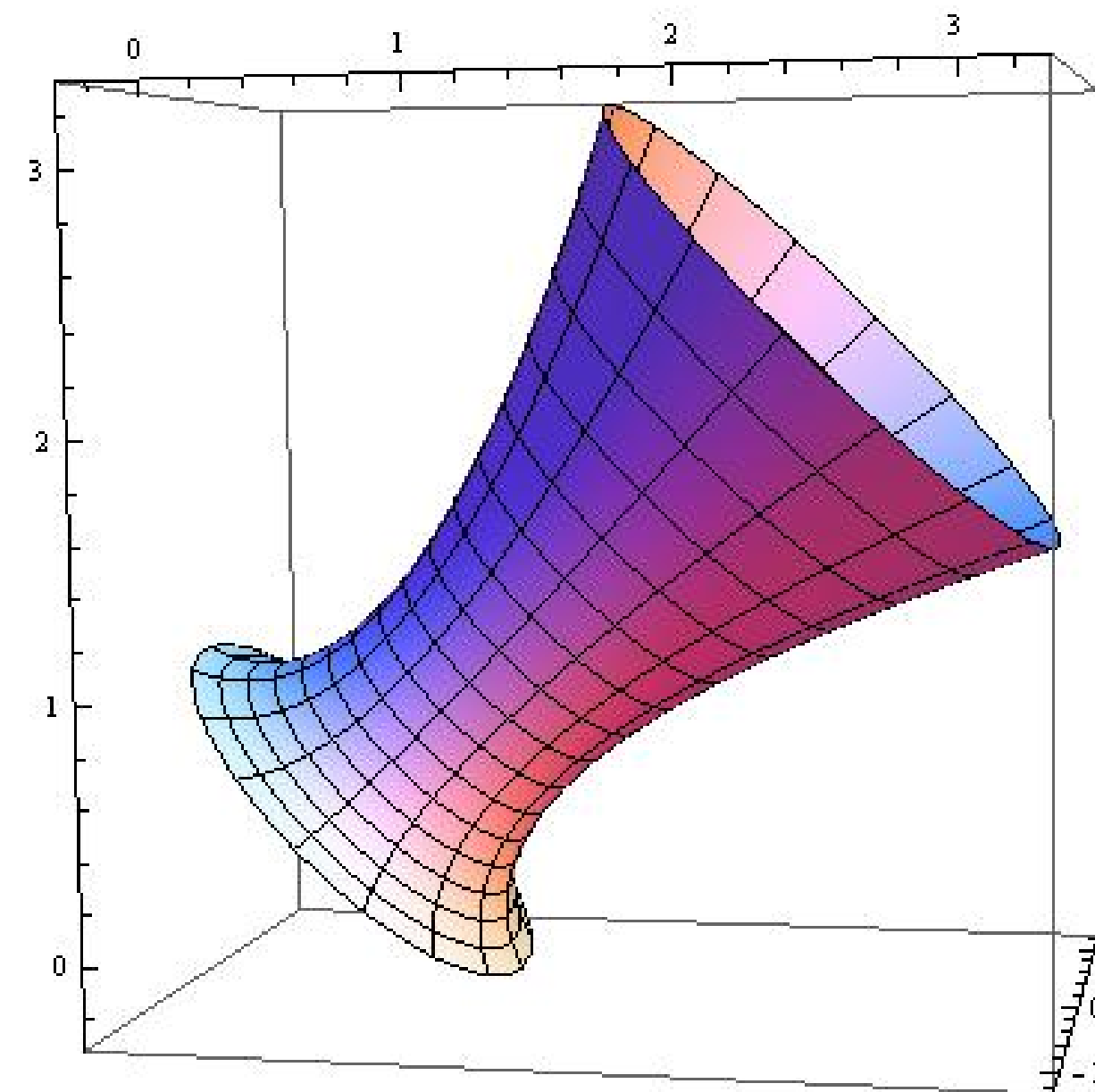


Figure 2: Revolved about slanted line

Numerical Technique

The idea here is to treat the line $y = mx + b$ "like" the x -axis. We call the line $y = mx + b$ the u -axis, and have our radius function in terms of u . Now we can sum the areas of our cross-sections in terms of u instead of x which we would normally do.

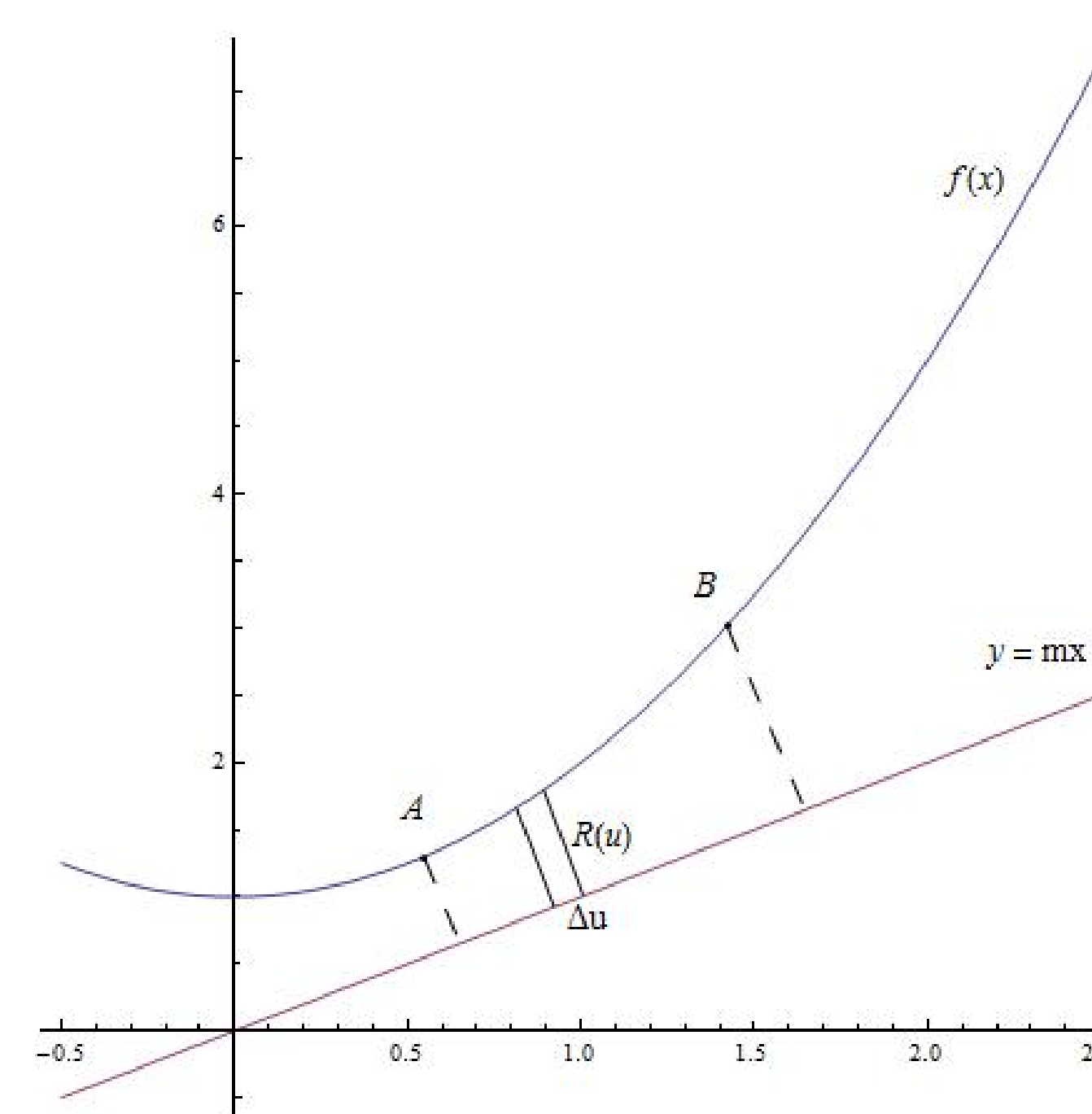


Figure 3:

and with some algebra we can write

$$\Delta u = \sqrt{1 + m^2} \Delta x$$

where m is the slope of the line $y = mx + b$.

Now we only want to rotate $f(x)$ in the interval $[A, B]$ along the u -axis, but we must project the interval $[A, B]$ from the u -axis onto the x -axis which allows us to sum our areas in the interval $[a, b]$ of the x -axis. We divide $[A, B]$ into n subintervals, thus we have n areas to sum. The radius, $R(u)$, of each subinterval is given by the length of the perpendicular line $l_k : y - mx_k = -\frac{1}{m}(x - x_k)$ between the u -axis and $f(x)$. So in terms of x our radius is $R(x_k)$. Now we can compute $\pi R(x_k)^2 \Delta u$ as described, and summing up these values we have our approximate volume

$$V = \sum_{k=1}^n \pi R(x_k)^2 \sqrt{1 + m^2} \Delta x$$

Integration Formula

In Figure 3, if we could find the angle between the x -axis and u -axis, call it θ , then we could use the axis rotation matrix,

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix},$$

to help us find an integral for the volume with respect to the u -axis. Using trigonometry we find $\theta = \tan^{-1}(\frac{\Delta y}{\Delta x}) = \tan^{-1}(m)$. Plugging θ into the rotation matrix we have

$$\begin{pmatrix} \cos(\tan^{-1}(m)) & \sin(\tan^{-1}(m)) \\ -\sin(\tan^{-1}(m)) & \cos(\tan^{-1}(m)) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1+m^2}} & \frac{m}{\sqrt{1+m^2}} \\ -\frac{m}{\sqrt{1+m^2}} & \frac{1}{\sqrt{1+m^2}} \end{pmatrix}.$$

Our integral will now be of the form

$$\int \pi R(u)^2 du.$$

Using the rotation matrix we find that

- $u = x \cos \theta + f(x) \sin \theta$
- $du = \cos \theta + \sin \theta f'(x) dx$
- $R(u) = v = -x \sin \theta + f(x) \cos \theta$

Now our approximation of volume can be obtained using the formula

$$\sum_{k=1}^n \pi R(u)^2 \Delta u.$$

Since our " u -axis" is not the same as the x -axis, then we need to know what u is in terms of x . Forming a right triangle with the u - and x -axes, from the Pythagorean Theorem we know $\Delta u^2 = \Delta y^2 + \Delta x^2$,

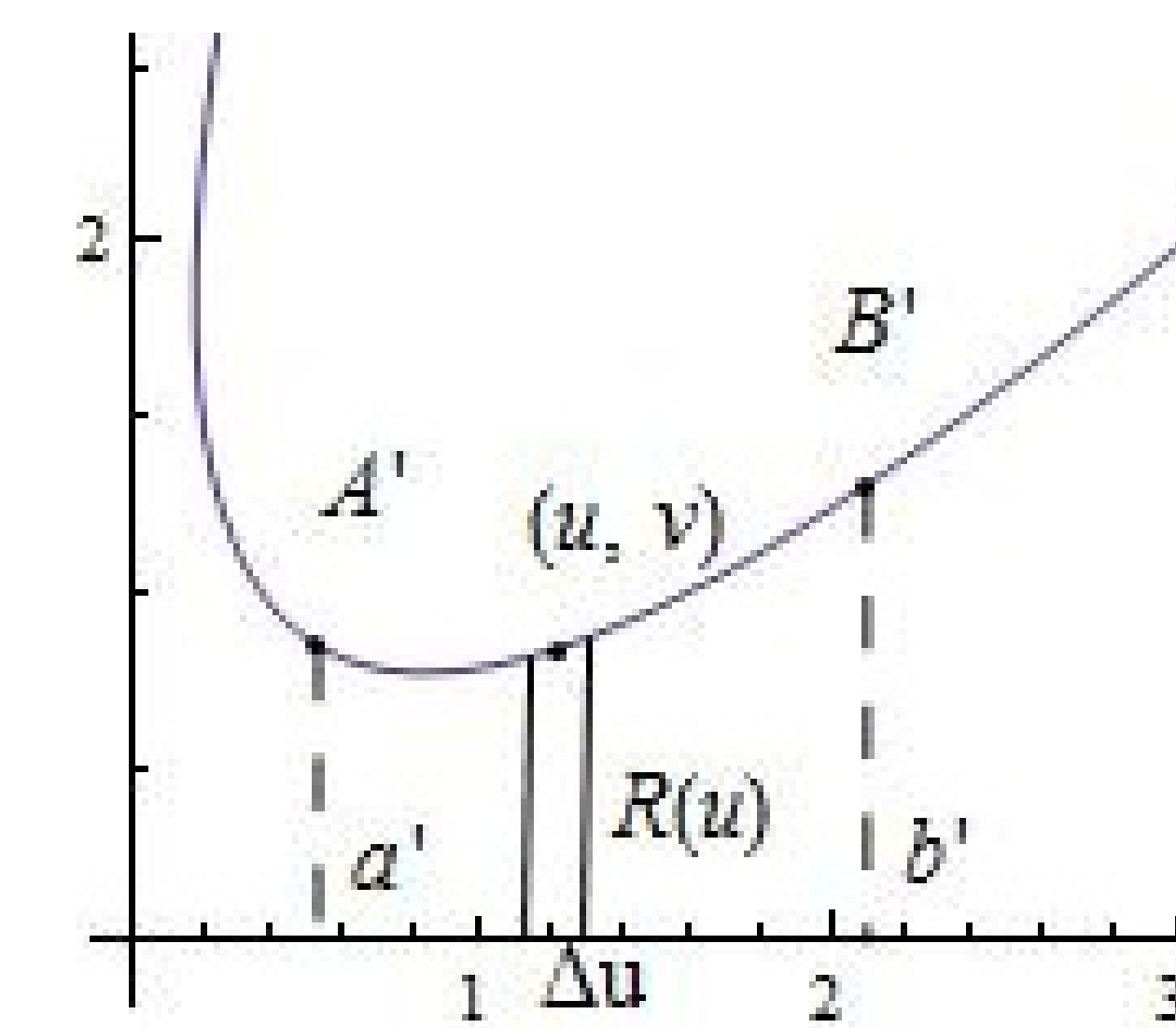


Figure 4: Rotation of Figure 3 can compute an exact volume using the integral

$$\pi \int_a^b (-\sin \theta x + \cos \theta f(x))^2 (\cos \theta + \sin \theta f'(x)) dx.$$

For this formula to work, we need the condition that $f \in C^1[a, b]$, in other words, f must have a continuous first derivative on the interval which we are rotating along. Testing our results have shown that our Numerical Technique is accurate to within several decimal places of the Integration Formula.

Extending the Project

Further research on this topic includes finding the volume of solids by rotating functions about other functions (not just straight lines,) such as in Figure 6 which

was generated by revolving the function $f(x) = -x^2 + 5$ about the function $g(x) = x^2 + x + 1$. Other ideas to extend this project include finding applications where finding the volume of irregular shapes may be useful. Perhaps some fields that could apply these methods may involve medical imaging analysis, atmospheric sciences, geology, or petrophysics to name a few.

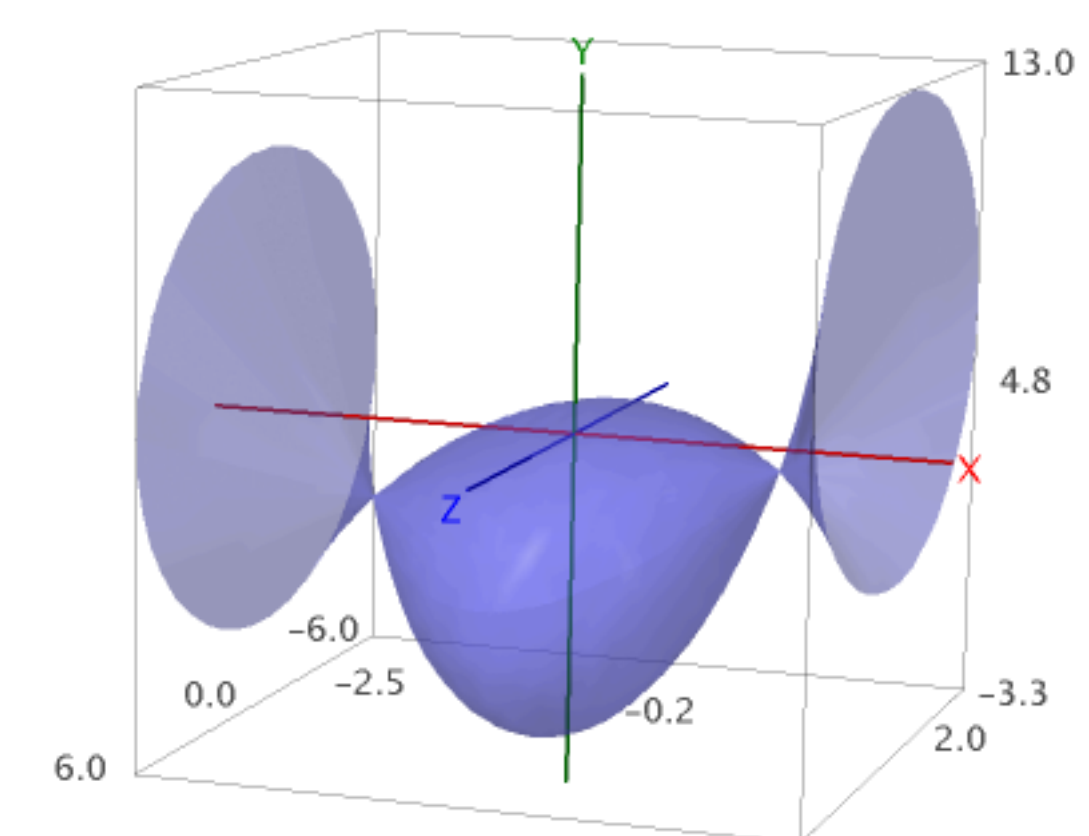


Figure 5: Candy Wrapper

References

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